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OCTONIONS – HILBERT SPACES, FIBRATIONS AND ANALYSIS

By

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This is dedicated to the One I love

Mama's and Papa's

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LIST OF SYMBOLS

$\mathbb{R}$, the real numbers.
$\mathbb{C}$, the complex numbers.
$\mathbb{H}$, the quaternions.
$\mathbb{O}$, the octonions.
$\mathbb{C}^-$, the split-complex numbers.
$\mathbb{H}^-$, the split-quaternions.
$\mathbb{O}^-$, the split-octonions.
$\mathbb{Z}_2$, the field with 2 elements.
R, a commutative ring with unit.
k, a field.
$[a,b]$, the commutator, $ab-ba$.
$[a,c,b]$, the associator, $(ab)c-a(bc)$.
$\mathrm{Nuc}(A)$, the nucleus of A, pg. 5.
x^{-1}, the multiplicative inverse of x.
$N(x)$, a quadratic form, pg. 6.
$\mathrm{Cl}(V,Q)$, the Clifford algebra over a vector space V with quadratic form Q, pg. 10.
$\mathrm{Cl}(n,p,z)$, same for V over $\mathbb{R}$ with Q of signature (n,p,q).
$\mathrm{Cl}(n,p)$, $Cl(n,p,0)$.
$\mathrm{Cl}(n)$, $Cl(n,0,0)$.
$\mathrm{Cl}_{n,p}$, $Cl(n,p)$.
$\overline{x}$, the $*$-pre-algebra involution of x, pg. 11.
$\{e_1,e_2,e_3\}$, a triple of basis elements, pg. 13.
$q^\dagger$, the octonionic conjugate of q, pg. 15.
$\Re(q)$, the real part of q, $\frac{1}{2}(q+q^\dagger)$.
$\Im(q)$, the imaginary part of q, $\frac{1}{2}(q-q^\dagger)$.
$\overline{q}$, the complex conjugate of q, pg. 20.
$\overline{q}^\dagger$, the reversion of q.
$\mathfrak{P}(q)$, the proper part of q, $\frac{1}{2}(q+\overline{q})$.
$\mathfrak{S}(q)$, the split part of q, $\frac{1}{2}(q-\overline{q})$.
$\|q\|^2$, the Euclidean quadratic form of q.
$\mathrm{End}_R(X)$, the R-linear maps from X to itself, as an R-module, pg. 23.
$\overline{\varphi_A}$, the enveloping algebra of A, pg. 23.
$\mathrm{Span}_A(S)$, the space spanned S, as a pre-A-module.
$\mathrm{End}_{A|B}(X)$, the restriction of $\mathrm{End}_A(X)$ to coefficients in B, pg. 27.
$\langle p,q\rangle$, a Hermitian form, pg. 29.
$S^\perp$, the orthogonal complement of S, pg. 30.
$\overline{S}$, the orthogonal closure of S, pg. 30.
$\mathrm{proj}_x(y)$, the projection of y onto x, pg. 40.
l^2, the space of convergent sequences, pg. 42.
$\mathbb{R}^n$, an n dimensional real vector space.
S^n, an n-sphere.
$H^{n,m}$, a signature (n,m) hyperboloid, pg. 51.
Ω, the domain of an octonionic function, pg. 56.
$C^n(\Omega,\mathbb{R})$, the n-differentiable real functions.
$C^n(\Omega,\mathbb{O})$, the n-differentiable octonionic functions.
D, the octonionic Dirac operator, pg. 57.
Δ, the octonionic Laplace (wave if split) operator, pg. 57.

ABSTRACT

The goal of this thesis is to develop rigorous foundations for octonionic structures suitable for potential use in theoretical physics. I begin by defining the octonions and exploring some of their algebraic properties. In particular, they are endowed with a structure similar to a Hopf algebra, lacking only associativity.

Next I turn to octonionic linear algebra. We explore the differences between various notions of linear dependence and orthogonality and the subspaces generated as a result. We classify the octonionic spaces generated by a single element and show that all octonionic Hilbert spaces have an orthogonal basis. We define octonionic l^2 spaces and show they produce an orthomodular form with an infinite orthonormal sequence. Solèr's Theorem is then generalized to include these spaces.

My attention then shifts to the split signature composition algebras, starting with Hopf fibrations and the closely associated projective spaces. For the associative split composition algebras we get families of Hopf fibrations for each dimension. The octonionic cases fail to allow projective spaces beyond the plane.

I then consider analysis on octonionic functions, leading to Cauchy integral formula over both the split and proper algebras. This result over the split-octonions is novel, and completes the list of Cauchy integral formula over the composition algebras.

Finally, I consider a broad generalization of the octonions, discussed by Albuquerque and Majid, and highlight some of the issues that emerge once we extend past the composition algebras. In particular, I define doubling algebras and attempt to classify the doubling algebras of small dimension.

INTRODUCTION

Famously, in 1843 William Rowan Hamilton discovered the quaternions ($\mathbb{H}$) and immediately carved their multiplication table into the stones of a nearby bridge [12][15]. Hamilton also sent a letter to John T. Graves announcing the discovery. Graves wrote back asking “If with your alchemy you can make three pounds of gold, why should you stop there?” [31]

Within two months Graves had crafted the octonions ($\mathbb{O}$), falsely believing there would be an infinite family of doubled composition algebras. The octonions inspired Hamilton to create the term associative to describe a property this algebra lacked, while agreeing to help Graves publish his discovery. Unfortunately for Graves, Arthur Cayley published his independent discovery of the octonions in March 1845 before this was done [10].

Our modern vector notation quickly developed and supplanted the quaternions. The ability to expand vector notation to dimensions other than three made them applicable to decisively more situations. Products of vectors yielded several interpretations, from tensors to the geometric algebras of Grassmann and Clifford.

Hurwitz proved that the composition algebras over the real numbers are precisely the real numbers, complex numbers, quaternions and octonions, published posthumously in 1923 [33]. Zorn, known for his lemma, pioneered a split signature version of the octonions in 1933 [59]. His matrix multiplication remains a common alternative representation due to their computational efficiency. Goldstine and Horwitz described octonionic Hilbert spaces in 1962 [27]. After this, the quaternions, octonions and their split cousins fell into obscurity.

The quaternions have resurfaced in 3D graphics engines [53] and robotic controls [44] due to their high speed and stability. Applications for the octonions are only beginning to be explored, speculatively in theoretical high energy physics. In particular, Baez and Huerta have found applications of the octonions in describing the structures emerging in super symmetry [2][4][31]. Furey has generated an octonionic formulation for one generation of the Pati-Salam model [22]. Boyle and Farnsworth demonstrated that the symmetries found in the standard model do not emerge from the non-commutative associative geometry of Connes, but can be found if we allow non-associative geometry[5][20].

Quantum logic has a deep connection with infinite orthomodular spaces, so classifying such spaces has significant implications for quantum computing. Solèr proved in 1995 that these must be Hilbert spaces over the real numbers, complex numbers or quaternions [56]. The spaces of Goldstine and Horwitz [27] were excluded by Solèr, who assumed the algebra is associative. Ludkowski proved that octonionic l^2 and L^2 spaces form infinite orthomodular Hilbert spaces [40]. I generalize Solèr's theorem to this new setting.

Along the way I discuss the various notions of linear dependence and orthogonality that arise, and classify the orthogonal closures of a single element. This requires development of linear maps sufficient to construct elementary row operations to reduce arguments about higher dimensional spaces to a manageable number of dimensions.

In Chapter 3 we extend the concept of Hopf fibrations and the resulting projective spaces to the split signature composition algebras. For the octonionic cases we find projective planes, but the lack of associativity obstructs extending this to higher dimensions.

The split-octonions are the unique real composition algebra containging the $3 + 1$ Minkowski space. Gogberashvili has explored using the split-octonions analogous to the paravectors in Clifford algebras [24]. This includes a Dirac operator to express octonionic versions of Dirac's equation and Maxwell's equations [26][25]. The desire for a more intrinsic Dirac operator motivates Chapter 4, with a generalization of the Cauchy integral formula acting as a proof of concept.

Most pure mathematicians are motivated largely by the elegance of the structures in their own right. The utility of pure mathematics is the development of rigorous tools that researchers in other fields can pull off the shelf if and when the need arises.

Baez has observed that "Nobody has managed to develop a good theory of octonionic linear algebra [3]." This is based on the lack of a satisfactory definition of octonionic modules and linear maps between them.

This thesis aims to lay a foundation for future development of non-associative linear algebra in general, with a particular focus on the octonions. A definition of octonionic modules able to show that all octonionic Hilbert spaces have an orthogonal basis, producing well defined projections and allowing an extension of Solèr's theorem addresses the first issue. His second observation is addressed by restricting to real matrices acting on octonionic spaces as a subalgebra. This allows elementary row operations that streamline higher dimensional arguments to low dimensional spaces.

So far I have been following Schafer's convention of using algebra to mean a not necessarily associative algebra. For the remainder of this text I will use pre-algebra and pre-module when associativity is not assumed, reserving the terms algebra and module for the usual associative structures.

Results

In Theorem 1.5.1 I produce a diagonal strong involution algebra that is not the result of twisting a group algebra, as constructed by Albuquerque and Majid.

In Chapter 2 I develop a theory of pre-modules and octonionic Hilbert spaces. Proposition 2.3.5 shows how elementary row operations and automorphisms can be used to transform a vector to a standard form. Theorem 2.4.10 classifies the orthogonal closure of a single element. In Theorem 2.5.5 I show that if X is an inner product space over a pre-k-algebra, A, then A must be a diagonal strong involution algebra.

Theorem 2.6.11 shows that an octonionic Hilbert space has a basis. Next, Theorem 2.7.9 extends Solèr's theorem to allow division algebras, which adds only the octonionic Hilbert spaces as possibilities.

Finally, Theorem 2.8.1 extends Huo, Li and Ren's classification of alternative octonionic left pre-modules to a classification of alternative left pre-modules over diagonal strong involution pre-algebras.

In Chapter 3 Theorem 3.3.1 demonstrates that a standard construction of Hopf fibrations using the division pre-algebras generalize to the split signature composition pre-algebras.

In Chapter 4 Theorem 4.4.5 generalizes known Cauchy integral formula for the octonions and split-quaternions to the split-octonions.

In Chapter 5 I consider a subset of Albuquerque and Majid's diagonal strong involution pre-algebras that are closed under sub-twist pre-algebras and Cayley-Dickson doubling. Theorem 5.4.2 shows that any such pre-algebra larger than the octonions must contain a copy of the unique pre-algebra of this class of dimension 8 that is not the octonions.

CHAPTER 1

PRELIMINARIES

The primary goal of this chapter is to define the terms and notation used throughout the remainder of this thesis. Much of chapter can be found in Schafer's text on non-associative algebras [52]. A significant difference is that I reserve the term algebra for when the product is associative, adopting the term pre-algebra for the non-associative case.

A secondary goal is to enumerate and highlight the benefits of several competing notations for the octonions and provide explicit isomorphisms between the various representations found in literature.

1.1 Pre-Algebras

Let R be a commutative ring with identity.

Definition 1.1.1. *A* **left** R**-module** M *is an abelian group* $(M,+)$ *and a product* $\cdot : R \times M$ *such that for any* r *and* s *in* R *and* x *in* M*:*

$$r \cdot (x+y) = r \cdot x + r \cdot y, \tag{1.1}$$

$$(r+s) \cdot x = r \cdot x + s \cdot x, \tag{1.2}$$

$$(rs) \cdot x = r \cdot (s \cdot x), \text{ and} \tag{1.3}$$

$$1 \cdot x = x. \tag{1.4}$$

A **right** R**-module** is defined with the order of the product reversed. An R**-bimodule** is both a left and right R-module such that $(r \cdot x) \cdot s = r \cdot (x \cdot s)$. A **free** R-module has a linearly independent generating set, or **basis**.

Definition 1.1.2. *Let* X*,* Y *and* Z *be* R*-modules. A* **bilinear map***,* $B : X \times Y \to Z$*, is a function such that for all* x_1 *and* x_2 *in* X*,* y_1 *and* y_2 *in* Y *and* a *and* b *in* R*:*

$$B(ax_1 + bx_2, y_1) = aB(x_1, y_1) + bB(x_2, y_1), \text{ and} \tag{1.5}$$

$$B(x_1, ay_1 + by_2) = aB(x_1, y_1) + bB(x_1, y_2). \tag{1.6}$$

Definition 1.1.3. *A* **pre-R-algebra***, A, is an R-module equipped with a bilinear product $\cdot : A \times A \to A$ called multiplication.*

We can also define two useful operators to express the lack of commutativity and associativity:

$$\textbf{commutator} \qquad [x, y] = x \cdot y - y \cdot x, \tag{1.7}$$

$$\textbf{associator} \qquad [x, y, z] = (x \cdot y) \cdot z - x \cdot (y \cdot z). \tag{1.8}$$

An **R-algebra** is an associative pre-R-algebra, thus $[x, y, z] = 0$. A **unital** pre-R-algebra contains 1_A such that $1_A \cdot x = x = x \cdot 1_A$ for all x in A. A **commutative** pre-R-algebra satisfies $[x, y] = 0$. The **nucleus** of a pre-R-algabra, $\mathrm{Nuc}(A)$, is the set of all a in A that associate with any b and c in A. Explicitly, $[a, b, c] = [b, a, c] = [b, c, a] = 0$. This forms an associative sub-pre-R-algebra. The **center** of a pre-R-algabra is the subset of the nucleus that commutes with every element of A.

1.1.1 Algebras Parameterized by Rings

Often times we encounter families of algebras, A_R, generated by adjoining a product on the basis element to a given ring R. For example, if $\mathrm{char}(R) \neq 2$, $\mathbb{H}_R$ extends R linearly by adjoining i and j. We note that k can be defined as ij. Thus $\mathbb{H}_\mathbb{Z}$ are the quaternions restricted to integer coefficients, while $\mathbb{H}_\mathbb{C}$ are the quaternions extended to the complex numbers. When $\mathrm{char}(R) \neq 2$, we define $\mathbb{C}_R$ and $\mathbb{O}_R$ similarly, based on the complex numbers and the octonions. Often we consider the case where R is a field, k.

1.1.2 Weaker Notions of Associativity

A pre-R-algebra is **diassociative** if the sub-pre-algebra generated by any two element set is an R-algebra. A pre-R-algebra is **power associative** if the sub-pre-algebra generated by any one element set is an R-algebra.

A pre-R-algebra is **alternative** if $[x, x, y] = [x, y, x] = [y, x, x] = 0$ for all x and y in A. Actually, any two imply the third. Artin's theorem on alternative rings tells us this is equivalent to diassociativity. The name alternative comes from the fact that this implies that the associator is preserved by the alternative subgroup of the permutation group. Thus for any permutation of three elements σ, $[x_1, x_2, x_3] = \epsilon(\sigma)[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$, where ϵ is the sign of the permutation. If $\mathrm{char}(k) \neq 2$ this implication goes the other way as well.

Similarly, a pre-algebra is alternative iff it satisfies the **Moufang identities**:

$$z \cdot (x \cdot (z \cdot y)) = ((z \cdot x) \cdot z) \cdot y, \tag{1.9}$$

$$x \cdot (z \cdot (y \cdot z)) = ((x \cdot z) \cdot y) \cdot z, \tag{1.10}$$

$$(z \cdot x)(y \cdot z) = (z \cdot (x \cdot y)) \cdot z, \tag{1.11}$$

$$(z \cdot x)(y \cdot z) = z \cdot ((x \cdot y) \cdot z). \tag{1.12}$$

In Chapter 5 we consider loops, i.e. non-associative groups. For loops Moufang is stronger than alternative, thus the last two can be combined be removing the outermost parenthesis on the right hand side. Moufang's theorem states that in any loop satisfying the Moufang identities, any three elements which associate generate a group [45]. Bilinearity extends this to the statement that if $[x, y, z] = 0$ in a Moufang algebra, the sub-pre-algebra generated by x, y and z is an associative algebra.

Moufang-Lie algebras, sometimes called Malcev algebras, satisfy the following:

anti-symmetric $$x \cdot y = -y \cdot x, \tag{1.13}$$

Malcev $$(x \cdot y)(x \cdot z) = ((x \cdot y) \cdot z) \cdot x + ((y \cdot z) \cdot x) \cdot x + ((z \cdot x) \cdot x) \cdot y. \tag{1.14}$$

The Malcev identity is the required weakening of the Jordan identity needed for alternative algebras. Replacing the product of any alternative algebra with $a \circ b = a \cdot b - b \cdot a$ generates prototypical examples [43]. These are not to be confused with the Malcev Lie algebras found in rational homotopy theory, which are a special type of Lie algebra, and thus satisfy the Jordan identity [51].

1.1.3 Other Properties

All of these definitions are borrowed directly from the corresponding properties for algebras.

In a **division pre-algebra**, for any two non-zero elements x and y in A there are s and t such that $y = sx$ and $y = xt$. If $y = 1_A$ we call s and t **left and right inverses** respectively. If $s = t$, we call this a **two-sided inverse**, denoted x^{-1}. Unfortunately, without associativity it may be the case that $x^{-1}(xy) \neq y$. Alternativity is sufficient to guarantee this identity.

1.2 Composition Pre-Algebras

Let X be a free R-module and $x \in X$. A **quadratic form**, $N(x)$, is a homogeneous polynomial of degree 2 on X. A quadratic form is **diagonal** if $N(x) = \sum \lambda_i x_i^2$, and X with this form is denoted by $\langle \lambda_1, ..., \lambda_n \rangle$. A diagonal quadratic form is **degenerate** if any $\lambda_i = 0$. It is **positive definite** if R is an ordered ring and $\lambda_i > 0$ for all i.

If R is a field with $1 \neq -1$, then we can use orthogonal diagonalization to find a basis where $N(x)$ is diagonal. The **square classes** of a field are the equivalence classes under the relation $a \sim b$ iff $a = c^2 b$ for some $c \in k$. Sylvester's Law of Inertia generalizes to show that the multiset of the square classes of the λ_i is well defined relative to the choice of orthogonal diagonalization [57].

For $\mathbb{R}$ the square classes can be represented by 0 or ± 1, so the data in the multiset of λ_i can be expressed by a triple of integers called the **signature** of X. The prefix **split** is used to describe structures involving non-degenerate real quadratic forms whose signature includes non-zero values for both ± 1.

For $\mathbb{C}$ the generalization here reduces to the rank of quadratic form. This differs from the usual generalization restricting the non-singular maps produced by orthogonal diagonalization above to Hermitian matrices.

An n**-fold Pfister form** $\langle\langle \lambda_1, ..., \lambda_n \rangle\rangle$ is the form of a tensor product $\langle 1, \lambda_1 \rangle \otimes ... \otimes \langle 1, \lambda_n \rangle$ [49]. An example is $\langle\langle \lambda_1, \lambda_2 \rangle\rangle = \langle 1, \lambda_1, \lambda_2, \lambda_1 \lambda_2 \rangle$. A Pfister form is non-degenerate iff all $\lambda_i \neq 0$. An equivalence class can be defined on Pfister forms by identifying both forms whose λ_i are in the same square classes in k and forms that can be related by a choice of independent generators.

A **composition pre-algebra** is a pre-algebra A with a quadratic form such that for all x and y in A $N(xy) = N(x)N(y)$. In a **normed pre-algebra**, $N(x)$ is a norm and $N(xy) \leq N(x)N(y)$.

Theorem 1.2.1 (Hurwitz)**.** *The only unital normed pre-algebras over the real numbers are* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ *and* $\mathbb{O}$.

Proof. This is a sketch of the proof in [12].

In passing it is shown that a composition rule alone induces a bilinear product that becomes a semi-norm if the pre-algebra is positive definite.

$$\langle p, q \rangle = \frac{1}{2}(N(p+q) - N(p) - N(q)). \tag{1.15}$$

This in turn defines an anti-involution generally called conjugation,

$$q^* = 2\langle q, 1\rangle - q, \tag{1.16}$$

$$(q^*)^* = q, \tag{1.17}$$

$$(pq)^* = q^* p^*, \tag{1.18}$$

$$N(q) = qq^* = q^* q. \tag{1.19}$$

This is used to show that our pre-algebra A is the Cayley-Dickson doubling of an associative algebra B with an anti-involution. We can consider the double as a sum of B with a copy times a new imaginary unit. This gives us two versions of the Cayley-Dickson doubling rule, playfully known as "eyes-right" and "eyes-left", referencing the two options $q = a + bi$ or $q = a + ib$.

$$(a, b)^* = (a^*, -b), \tag{1.20}$$

$$(a, b)(c, d) = (ac - d^* b, da + bc^*), \qquad \text{eyes-right} \tag{1.21}$$

$$(a, b)(c, d) = (ac - db^*, cb + a^* d). \qquad \text{eyes-left} \tag{1.22}$$

These are isomorphic, with the explicit map preserving all basis with a sign change for $(0, 1)$. Unless specified otherwise, we use the "eyes-right" convention.

The Cayley-Dickson construction now requires that for B to be associative it must be a double of a commutative algebra C. Finally, for C to be commutative it must be a doubling of an algebra with trivial involution, which must be the field itself.

Thus we have the real numbers doubling three times through the complex numbers and quaternions before stalling out at the octonions. □

Definition 1.2.2. *The first application of the Cayley-Dickson construction over* $\mathbb{R}$ *produces the* **complex numbers**, $\mathbb{C}$. *The second application produces the* **quaternions**, $\mathbb{H}$. *The third application produces the* **octonions**, $\mathbb{O}$. *Over* $\mathbb{C}$ *we get the* **bicomplex numbers**, $\mathbb{C}_{\mathbb{C}}$, **complex quaternions**, $\mathbb{H}_{\mathbb{C}}$, *and* **complex octonions**, $\mathbb{O}_{\mathbb{C}}$.

We can continue to apply the Cayley-Dickson construction past the octonions to generate an infinite family of algebras, but the composition rule used to derive it breaks. Johnathan Smith has found doubling rules that preserve the composition rule, but require breaking left or right linearity [55]. This alternate product is also not continuous in the Euclidean subspace topology.

This can be generalized to a classification of composition algebras over fields with $1 \neq -1$ by the **generalized Cayley-Dickson construction**.

$$(a, b)^* = (a^*, -b) \tag{1.23}$$

$$(a, b)(c, d) = (ac - \lambda d^* b, da + bc^*), \tag{1.24}$$

$$N((a, b)) = N(a) + \lambda N(b), \tag{1.25}$$

where λ is any element of the field. Indeed, this construction extends to any commutative ring with unit but the classification becomes more complicated.

The generalized Cayley-Dickson construction produces Pfister forms with λ the signature of $(0, 1)$. This allows general composition algebras to be expressed as $\langle\langle\lambda_1, ..., \lambda_2\rangle\rangle$, encoding the λ_i used for the product at each step. The composition algebras within each Pfister equivalence class are isomorphic in a straightforward manner. This outlines the proof of the following standard results [52], though the classification of the quadratic forms using Pfister forms over general k seems to be new.

Theorem 1.2.3. *The only unital composition pre-algebras over k, $1 \neq -1$ are k, and Cayley-Dickson doublings of k for each equivalence class of non-degenerate Pfister 1-forms, 2-forms and 3-forms.*

Corollary 1.2.4. *The only unital composition pre-algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\mathbb{C}^-$, $\mathbb{H}^-$ and $\mathbb{O}^-$. Further, the only unital composition pre-algebras over $\mathbb{C}$ are $\mathbb{C}$, $\mathbb{C}_\mathbb{C}$, $\mathbb{H}_\mathbb{C}$ and $\mathbb{O}_\mathbb{C}$.*

Proof. For $k = \mathbb{R}$, a selection of independent generators shows that $\mathbb{H}^- = \langle\langle -1, -1\rangle\rangle \sim \langle\langle 1, -1\rangle\rangle$. Similarly $\mathbb{O}^- = \langle\langle -1, -1, -1\rangle\rangle \sim \langle\langle 1, -1, -1\rangle\rangle \sim \langle\langle 1, 1, -1\rangle\rangle$. Thus all of the indefinite cases of the same size are isomorphic.

For $k = \mathbb{C}$ the non-degeneracy of a composition pre-algebra forces all of the λ_i to be 1. □

Definition 1.2.5. *The* **split-complex numbers**, *$\mathbb{C}^-$,* **split-quaternions**, *$\mathbb{H}^-$, and* **split-octonions**, *$\mathbb{O}^-$, are produced by extending $\mathbb{R}$ the corresponding number of times using $\lambda = -1$ for at least one step.*

Corollary 1.2.6. *There are infinitely many unital composition algebras over $\mathbb{Q}$.*

Indeed there are infinitely many quadratic fields, representing only the equivalence classes of Pfister 1-forms. Degenerate Cayley-Dickson algebras (with one or more $\lambda_i = 0$) over $\mathbb{R}$ have found applications in automatic differentiation [30].

1.3 Clifford Algebras

Clifford algebras will appear multiple times in this work, so it is useful to define them. Much of this section is general knowledge, though more details can be found in [23].

Given a k-vector space V and a quadratic form $Q : V \to k$ a **Clifford algebra** $\mathrm{Cl}(V, Q)$ is the freest algebra, A, generated by V subject to $v^2 = Q(v)1_A$ for all $v \in V$. In modern treatments, this is expressed using a universal property.

Consider the category whose objects are pairs (A, i) where A is a unital associative algebra over k and i is a linear map $i : V \to A$ such that $i(v)^2 = Q(v)1_A$ for all $v \in V$, and whose morphisms are algebra homomorphisms. A Clifford algebra, $\mathrm{Cl}(V, Q)$, is an initial object of this category. Thus for any other such pair (B, j), there is a unique algebra homomorphism $f : \mathrm{Cl}(V, Q) \to B$ such that $f \circ i = j$.

An explicit construction can be generated by first finding the ideal, I_Q, in the tensor algebra over V, $T(V)$, generated by elements of the form $v \otimes v - Q(v)1_T$ for all $v \in V$. Then a Clifford algebra is the quotient algebra $\mathrm{Cl}(V, Q) = T(V)/I_Q$.

If $\mathrm{char}(k) \neq 2$, define a symmetric bilinear form on V as follows:

$$\langle u, v\rangle = \frac{1}{2}\left(Q(u+v) - Q(u) - Q(v)\right). \tag{1.26}$$

Now V has an orthogonal basis, e_i, which we identify with their images in $\mathrm{Cl}(V, Q)$. If $i \neq j$, we have the following relations:

$$e_i e_j = -e_j e_i, \tag{1.27}$$

$$e_i^2 = Q(e_i). \tag{1.28}$$

Now a basis for $\mathrm{Cl}(V, Q)$ can be identified with power sets of the basis of V. Thus if $\dim(V) = n$, then $\dim(\mathrm{Cl}(V, Q)) = 2^n$.

These become **Grassman algebras** if $Q(v) = 0$ for all $v \in V$. Clifford algebras are often called geometric algebras and Grassman algebras exterior algebras.

Since $\alpha : v \to -v$ preserves $Q(v)$, the universal property of Clifford algebras and α induce an automorphism on $\mathrm{Cl}(V, Q)$. This is called the grade involution, as it exhibits the Z_2 grading of the algebra.

The tensor product has a natural anti-automorphic involution that reverses the order of products. This induces an anti-automorphic involution called reversion. Clifford conjugation is the composition of these two involutions, resulting an another antiautomorphic involution.

Clifford algebras can be classified using the number of basis elements in each square class of k. For $\mathbb{R}$ these are ± 1 and 0. Thus $\mathrm{Cl}(n, p, z)$ is a Clifford algebra over a real vector space V, where $\dim(V) = n + p + z$ and n basis elements square to -1, p square to 1 and z to 0. The notation $\mathrm{Cl}(n, p)$ implies that $z = 0$, and $\mathrm{Cl}(n)$ implies that $p = z = 0$.

It should be noted that some authors swap the roles of n and p. Also, the notations $\mathrm{Cl}_{n,p}$ or $\mathrm{Cl}_{p,n}$ are common. The convention adopted here yields $\mathrm{Cl}(1) \cong \mathbb{C}$ and $\mathrm{Cl}(2) \cong \mathbb{H}$. We also have $\mathrm{Cl}(0, 1) \cong C^-$ and $\mathrm{Cl}(1, 1) \cong \mathrm{Cl}(0, 2) \cong H^-$.

Depending on sign conventions, $3 + 1$ Minkowski space yields either $\mathrm{Cl}(3, 1)$ or $\mathrm{Cl}(1, 3)$, which are distinct. Another convention is to use $\mathrm{Cl}(0, 3)$ and associate the unit with time, halving the dimensions needed. The combined scalar plus vector is called a paravector. The failure of $\mathrm{Cl}(0, 3)$ to be a composition algebra motivates my interest in the split-octonions.

1.4 $*$-Pre-Algebras

Definition 1.4.1. *A* **$*$-pre-algebra** *is a unital pre-algebra over a commutative ring with unit R with a linear map $\bar{\cdot} : A \to A$ satisfying:*

$$\overline{\overline{x}} = x, \tag{1.29}$$

$$\overline{x + y} = \overline{x} + \overline{y}, \tag{1.30}$$

$$\overline{xy} = \overline{y}\,\overline{x}, \tag{1.31}$$

$$\overline{1} = 1. \tag{1.32}$$

The composition pre-algebras with conjugation are prototypical examples. Matrices with transposition and Clifford algebras with one of their two anti-involutions provide more examples.

One consequence of this structure is that $A \cong A^{\mathrm{op}}$, where A^{op} is A as an R-module, but with the product reversed. Indeed, $\overline{x}$ yields the required algebra homomorphism.

1.5 Diagonal Strong Involution Pre-Algebras

Albuquerque and Majid considered pre-k-algebras resulting as a twisting of a finite group algebra [1]. Thus, if we have a k algebra over G and $F : G \oplus G \to k^*$, then the new product $a * b = F(a,b)ab$ extends linearly to all elements of the original group algebra kG, yielding a pre-K-algebra.

They define a **diagonal involution**, $\sigma(a)$ as one arising from a scaling of a basis of A by elements, possible different, in k. A **strong involution** pre-algebra is one having an involution such that both $a + \sigma(a)$ and $a\sigma(a)$ are in $k1_A$. They show this requires every element in the base group to have order 2, and thus the underlying groups are of the form $\mathbb{Z}_2^n$ for some n.

The entire family of algebras generated by repeated application of the generalized Cayley-Dickson construction, with $\sigma(a)$ being conjugation, produce this structure.

Theorem 1.5.1 (Prather)**.** *There are diagonal strong involution pre-algebras that are not the twisting of a group algebra.*

Proof. Consider the pre-algebra generated by extending the multiplication table for the basis given in Table 1.1 to the entire algebra. Let σ reflect all of the basis other than the unit. By inspection,

Table 1.1: Multiplication table for the basis elements of a strong involution algebra not arising from the twisting of a group.

1	A	B	C	D	E
A	-1	C	D	E	B
B	$-C$	-1	E	A	D
C	$-D$	$-E$	-1	B	A
D	$-E$	$-A$	$-B$	-1	C
E	$-B$	$-D$	$-A$	$-C$	-1

this is a diagonal strong involution pre-algebra under $\sigma(a)$.

Since the dimension is not a power of 2, it is not a twisting of $\mathbb{Z}_2^n$ for any n. $\square$

This algebra is the twisting of a loop algebra. Note that $(AA)B = -B \neq D = A(AB)$, so this pre-algebra is not even alternative.

1.6 Octonions

From Theorem 1.2.1, the octonions are the unique unital positive definite non-associative composition algebra over the real numbers. This forms an 8 dimensional algebra. There are several common notations used for this algebra. Much of the next two sections comes from Conway and Smith [12] or Dray and Manogue [15].

The most concise defines the octonions as a loop algebra over the identity e_0 and i_n, where n is in the cyclic group of order 7, under the following relations:

$$i_{\sigma(n)}i_{\sigma(n+1)} = \epsilon(\sigma)i_{\sigma(n+3)}, \text{ and} \qquad\qquad i_n^2 = -e_0, \tag{1.33}$$

where σ is a permutation on three elements and ϵ is the sign function. This highlights an order seven symmetry of the indices not visible in the Cayley-Dickson construction.

The multiplication derived from the Cayley-Dickson construction is given in Table 1.2. Here we rename the non identity elements e_i, so both notations are available as needed. One of many maps between the two are $e_1 = i_1$, $e_2 = i_2$, $e_3 = i_4$, $e_4 = -i_7$, $e_5 = i_3$, $e_6 = i_6$ and $e_7 = i_5$. This table highlights the twisted Z_2 grading of the algebra, if the indices are viewed in binary.

Table 1.2: Multiplication table for the basis elements of $\mathbb{O}$.

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Observe that the subtable generated by e_0 and any other basis spans a subalgebra identical to $\mathbb{C}$. Further, any two distinct basis other than e_0 multiply to a third, plus or minus, and these three plus e_0 span a copy of $\mathbb{H}$. Thus we can summarize the relevant product by expressing which basis form quaternionic triples, such as $\{e_1, e_2, e_3\}$,$\{e_1, e_4, e_5\}$,$\{e_1, e_7, e_6\}$, $\{e_2, e_4, e_6\}$, $\{e_2, e_5, e_7\}$,

$\{e_3, e_4, e_7\}$ and $\{e_3, e_6, e_5\}$. Further, these are ordered so that the first two multiply to the third, fixing the sign convention.

A directed Fano plane, as in Figure 1.1, can be used to graphically encode this. The vertices

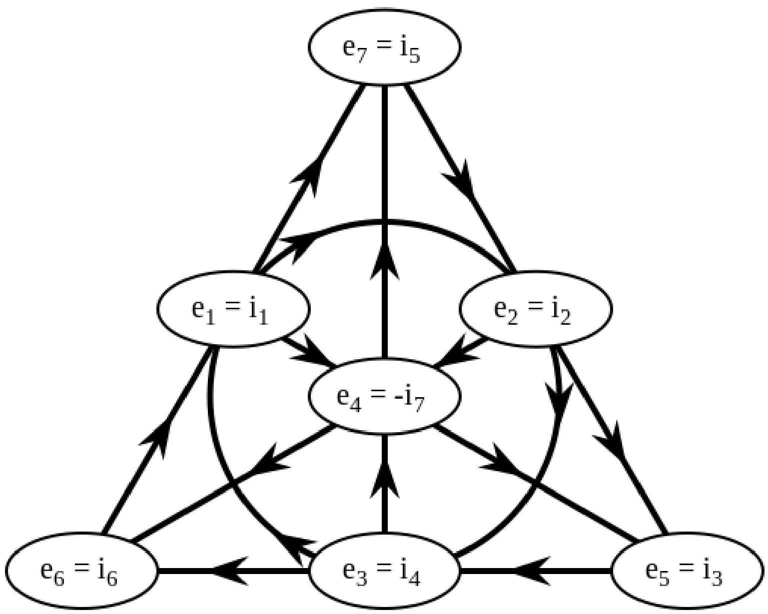

Figure 1.1: Fano Plane mnemonic for the octonionic multiplication table. The lines can be viewed as circles through the origin, with the omitted segment oriented in the same direction as the two shown.

represent the basis elements and the lines (including the circle in the middle) represent quaternionic subalgebras. The arrows indicate the orientation of the quaternionic algebras. Orienting the quaternionic algebras randomly yields an algebra isomorphic to the octonions one out of eight times. The other seven out of eight times you get a unique algebra, up to isomorphism. This pseudo-octonion algebra must appear in the sedenions, which will be discussed in Chapter 5. To get the octonions one can orient the lines of the outer triangle with the inner circle. The segments from the basis on the inner circle and the center must then either all face or all oppose the center. This Fano plane represents the "eyes-right" convention. The "eyes-left" convention would swap the direction of all lines through the central vertex. The notation $i = e_1$, $j = e_2$, $k = e_3$ and $l = e_4$ explains the motivation for this naming convention as the choice in sign for e_5, as $il = -li$.

We will need to consider these basis elements with complex coefficients, and have both the octonionic conjugation and complex conjugation. We reserve the over bar for complex conjugation. This motivates the following notation:

$$q = \sum_i q_i e_i, \qquad q^\dagger = q_0 e_0 - \sum_{i \neq 0} q_i e_i, \tag{1.34}$$

$$N(q) = \sum_i q_i^2 = qq^\dagger = q^\dagger q, \qquad q^{-1} = \frac{q^\dagger}{N(q)}, \quad N(q) \neq 0. \tag{1.35}$$

If any q_i is 0, then that term will be suppressed, and if all terms are 0, then $q = 0$. The restriction on the two sided inverse q^{-1} is included to allow straightforward generalizations to the complexified octonions and the split-octonions.

The functions $\Re(q)$ and $\Im(q)$ are the **scalar part** and **pure**, or **imaginary**, **part** of q respectively; preferring pure when we complexify $\mathbb{O}$.

$$2\Re(q) = q + q^\dagger = 2q_0 e_0, \tag{1.36}$$

$$2\Im(q) = q - q^\dagger = 2(q - q_0 e_0). \tag{1.37}$$

There is much more freedom than the 168 elements of the automorphism group of the Fano plane in the automorphism group of $\mathbb{O}$. Any pure unit octonion can be used for e_1, and any other orthogonal pure unit octonion can be used for e_2. This introduces $6 + 5 = 11$ degrees of freedom. The selection of e_1 and e_2 fixes e_3, leaving 3 degrees of freedom for the choice of e_4. The remaining basis elements are then fully determined. Thus the automorphism group has dimension 14, and is actually the closed real form[1] of the exceptional Lie group G_2 [2]. Proposition 2.3.5 gives an example of how $\mathrm{Aut}(\mathbb{O})$ is typically used.

From alternativity, we have the following relations:

$$[p, q, r] = [q, r, p] = -[q, p, r], \tag{1.38}$$

$$[p, q, r] = -[p^\dagger, q, r] = [p^\dagger, q^\dagger, r], \tag{1.39}$$

$$[p, p, q] = [p, q, q] = [p, q, p] = 0, \tag{1.40}$$

$$(pq)p^\dagger = p(qp^\dagger). \tag{1.41}$$

[1] The notation G_2 is used to describe a family of three real Lie algebras with similar structure, one of which is the unique complex Lie algebra G_2. The other two can be distinguished by compactness. We will see the others shortly.

The last line shows that conjugation by an octonion (in the sense of group theory) is well defined, since p^{-1} and $N(p)$ are in the algebra generated by p.

The multiplication of non-zero elements in $\mathbb{O}$ is a prototypical Moufang loop. This adds the following identities, that trivially also work for $\mathbb{O}$:

$$(pqp)r = p(q(pr)),$$

$$p(qrq) = ((pq)r)q,$$

$$p(qr)p = (pq)(rp).$$

The nucleus and center of $\mathbb{O}$ is the embedded copy of $\mathbb{R}$.

1.7 Indefinite Cousins

Much of this section is from [15].

The complex octonions, $\mathbb{O}_{\mathbb{C}} \cong \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$, are simply the basis elements of $\mathbb{O}$ in Table 1.2 with complex, rather than real, coefficients. Taking the real components of a quaternionic subalgebra and the complex components of the remaining components produces another closed real pre-algebra isomorphic to the split-octonions, $\mathbb{O}^-$. In particular, using e_n with n even as the quaternionic subalgebra will be used below. This contrasts with the typical choice using the first four basis elements as the quaternionic subalgebra. This is so the Hodge dual defined in Chapter 4 coincides with an intrinsic involution.

Both of these algebras have zero divisors, namely any element with $N(q) = 0$. A prototypical example is $(1 + ie_1)(1 - ie_1) = 0$. They also have nilpotent elements, such as $(ie_1 + e_2)$, that square to 0.

For $\mathbb{O}^-$ we redefine the basis to absorb the complex root. The resulting multiplication table is given in Table 1.3. The split signature of $N(q)$ is where the name split-octonions comes from:

$$N(q) = qq^\dagger = \sum_i (-1)^i q_i^2.$$

Unit split-octonions form a hyperboloid. This is often extended to include the manifold of solutions to $|N(q)| = 1$, particularly for the complex octonions.

The automorphism groups of $\mathbb{O}_{\mathbb{C}}$ and $\mathbb{O}^-$ are the complex and split form of G_2 respectively. A consequence of Corollary 1.2.4 is that $\mathbb{O}^- \otimes \mathbb{C} \cong \mathbb{O}_{\mathbb{C}}$.

Table 1.3: Multiplication table for the basis elements of $\mathbb{O}^-$.

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_0	e_3	e_2	e_5	e_4	$-e_7$	$-e_6$
e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	$-e_2$	$-e_1$	e_0	e_7	e_6	e_5	e_4
e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	$-e_4$	$-e_7$	$-e_6$	$-e_1$	e_0	$-e_3$	$-e_2$
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_6	e_5	$-e_4$	$-e_3$	e_2	e_1	e_0

Vector Notation. The split-quaternions can be used to represent the rotations and boosts in Minkowski $2+1$ space as a paravector from Clifford algebras. I like to use j_x and j_y to denote the spatial basis and i_θ for the single axial vector when using this interpretation.

The split-octonions can be similarly viewed as the sum of a scalar, pseudo-scalar, (polar) proper vector and (axial) pseudo-vector, analogous to paravectors over the Clifford algebra $\mathrm{Cl}(0,3)$.

The scalar and proper vector form a Minkowski $3+1$ space. Further, the scalar and pseudo-vector form a subalgebra isomorphic to $\mathbb{C}^-$. Hamilton originally defined a quaternion as the quotient of two vectors. The split-octonions highlight the distinction between the original polar vectors and the resulting axial vector.

This motivates the use of $\vec{I}$ and $\vec{J}$ for the pseudo-vector and proper vector respectively. It is then intuitive to choose K to represent the pseudo-scalar. This notation was introduced to the literature by Merab Gogberashvili [24]. The vectors may be represented by their components, or in polar notation in terms of unit vectors $\hat{u}$ and $\hat{v}$.

$$q = s + \vec{I} + \vec{J} + pK \tag{1.42}$$

$$= s + \rho I_{\hat{u}} + r J_{\hat{v}} + pK \tag{1.43}$$

$$= (s; \vec{I}_x, \vec{I}_y, \vec{I}_z; \vec{J}_x, \vec{J}_y, \vec{J}_z; p) \tag{1.44}$$

Note that $\vec{J}_x$ represents the x component of a proper vector while $J_{\hat{x}}$ represents a unit proper vector in the x direction.

The product can then be represented using familiar scalar, dot and cross products of the components, as in Table 1.4. One map between this table and the one above is given by $I_{\hat{x}} = e_2$, $I_{\hat{y}} = e_4$, $I_{\hat{z}} = e_6$, $J_{\hat{x}} = e_7$, $J_{\hat{y}} = e_1$, $J_{\hat{z}} = e_3$ and $K = e_5$.

Table 1.4: Multiplication table for the vector representation of $\mathbb{O}^-$.

e_0	$\vec{I}$	$\vec{J}$	K
$\vec{I}$	$\cdot : -1$ $\times : \vec{I}$	$\cdot : -K$ $\times : -\vec{J}$	$\vec{J}$
$\vec{J}$	$\cdot : K$ $\times : -\vec{J}$	$\cdot : 1$ $\times : \vec{I}$	$\vec{I}$
K	$-\vec{J}$	$-\vec{I}$	1

Zorn Matrices. Beyond the personal appeal, vector notation also makes the isomorphism with Zorn matrices easier to express. Zorn matrices were first described by Maxwell Zorn in 1933 [59]. In deriving the expressions one finds a choice of sign, and both conventions can be found in literature:

$$a = s + p \qquad \vec{u} = \vec{J} \pm \vec{I} \tag{1.45}$$

$$b = s - p \qquad \vec{v} = \vec{J} \mp \vec{I} \tag{1.46}$$

$$\begin{pmatrix} a & \vec{u} \\ \vec{v} & b \end{pmatrix} \begin{pmatrix} a' & \vec{u}' \\ \vec{v}' & b' \end{pmatrix} = \begin{pmatrix} aa' + \vec{u} \cdot \vec{v}' & a\vec{u}' + \vec{u}b' \\ \vec{v}a' + b\vec{v}' & \vec{v} \cdot \vec{u}' + bb' \end{pmatrix} \pm \begin{pmatrix} 0 & \vec{v} \times \vec{v}' \\ -\vec{u} \times \vec{u}' & 0 \end{pmatrix} \tag{1.47}$$

$$\det \begin{pmatrix} a & \vec{u} \\ \vec{v} & b \end{pmatrix} = ab - \vec{v} \cdot \vec{u} = N(q) \tag{1.48}$$

It is well known that an idempotent basis can be used to show that $\mathbb{C}^-$ and $\mathbb{H}^-$ are isomorphic to $\mathbb{R} \oplus \mathbb{R}$ and $M_2(\mathbb{R})$ respectively. Zorn matrices represent the analogous idempotent basis for $\mathbb{O}^-$. These representations halve the computation cost of multiplication since half of the Cayley table becomes 0.

1.7.1 Reversion

This section borrows notation and language from Clifford algebras for the split-octonions to highlight the analogy between $\mathbb{O}^-$ and $\mathrm{Cl}(0, 3)$.

Conjugation of the imaginary unit introduces another involution on $\mathbb{O}_{\mathbb{C}}$, $\overline{q}$, akin to the **grade involution** of Clifford algebras. Since $\mathbb{C}$ is commutative this is an automorphism. For $\mathbb{O}$ this is simply the identity, but for $\mathbb{O}^{-}$ this changes the sign of the split roots. Composing this with octonionic conjugation yields an anti-automorphism, akin to the **reversion** of Clifford algebras. The composition of any two of these involutions yields the third:

$$\begin{aligned}
\overline{q} &= \sum_i (-1)^i q_i e_i, \\
\overline{q}^{\dagger} &= q_0 e_0 - \sum_{i \neq 0} (-1)^i q_i e_i = (\overline{q})^{\dagger} = \overline{(q^{\dagger})}, \\
\overline{(pq)} &= \overline{p}\,\overline{q}, \\
\overline{(pq)}^{\dagger} &= \overline{q}^{\dagger} \overline{p}^{\dagger}, \\
N(q) &= N(q^{\dagger}) = N(\overline{q}) = N(\overline{q}^{\dagger}).
\end{aligned}$$

These new involutions can be used to express the Euclidean norm. They can also be used to define the **proper** and **split parts** of a split-octonion:

$$\begin{aligned}
2\|q\|^2 &= 2\Re(q\overline{q}^{\dagger}) = q\overline{q}^{\dagger} + \overline{q}q^{\dagger}, \\
2\mathfrak{P}(q) &= q + \overline{q}, \\
2\mathfrak{S}(q) &= q - \overline{q}.
\end{aligned}$$

I will make use of the observation that $\overline{q}$ is an orientation preserving isometry of $\mathbb{R}^8$ and thus any sphere centered at the origin in Chapter 4. Conversely, $\overline{q}^{\dagger}$ and $q^{\dagger}$ are orientation reversing isometries of $\mathbb{R}^8$.

1.8 Dual Quasi-Hopf Algebra

This section largely summarizes the result of Albuquerque and Majid [1]. The goal is to emphasis the explicit structure of there result, without the distraction of the formal tools needed to generalize them beyond the octonions.

The axioms of a **coalgebra** over a field, k, are dual to those of a unital algebra, after interpreting them as appropriate maps. Let A be a k-vector space. A **unit** can be viewed as map $\eta : k \to A$

taking $a \to a1_A$. A **counit** is then a map $\epsilon : A \to k$. A **coproduct** is a map from $\Delta : A \to A \otimes_k A$. These are required to be compatible as follows:

$$(id \otimes_k \Delta) \circ \Delta = (\Delta \otimes_k id) \circ \Delta, \tag{1.49}$$

$$(id \otimes_k \epsilon) \circ \Delta = (\epsilon \otimes_k id) \circ \Delta \cong id. \tag{1.50}$$

A prototypical example is $A = k^S$, the set of functions from k to S, for any finite set S with $\Delta : s \to s \otimes_k s$ and $\epsilon : s \to 1$ for all $s \in S$ extended linearly to all of A.

A **bialgebra** is both an algebra and coalgebra, where the associated functions are compatible as follows:

$$\Delta \circ \cdot = (\cdot \otimes_k \cdot) \circ (id \otimes_k \tau \otimes_k id) \circ \Delta \otimes_k \Delta, \tag{1.51}$$

$$\epsilon \circ \cdot \cong \epsilon \otimes_k \epsilon, \tag{1.52}$$

$$\eta \otimes_k \eta \cong \Delta \circ \eta, \text{ and} \tag{1.53}$$

$$\epsilon \circ \eta = id, \tag{1.54}$$

where $\tau : A \otimes_k A$ is the map $a \otimes_k b \to b \otimes_k a$ extended linearly to all of $A \otimes_k A$. Group algebras with the above coproduct and counit are typical examples.

Hopf algebras are bialgebras with a k-linear map $S : A \to A$ called the **antipode** that is compatible with the bialgebra structure as follows:

$$\cdot \circ (S \otimes_k id) \circ \Delta = \epsilon \circ \eta = \cdot \circ (id \otimes_k S) \circ \Delta \tag{1.55}$$

Group algebras with the antipode extended linearly from the map $S(g) = g^{-1}$ are prototypical examples. The octonions cannot form a Hopf algebra, since they are not associative.

The term quantum groups is ambiguously used for several related concepts. The strictest sense of the term refers to Hopf algebras with a quasi-triangular property. Often, the term is used for generalizations of this structure. Majid's text on quantum group theory provides much more context and details [42].

Drinfeld introduced a method of twisting Hopf algebras that required weakening coassociativity to produce quasi-Hopf algebras [16]. Albuquerque and Majid dualized Drinfeld twists, producing dual quasi-Hopf algebras [1]. This allows them to twist group algebras into well behaving non-associative algebras, and transfer nice properties from the group algebras.

Albuquerque and Majid constructed the Cayley-Dickson algebras in this manner. The resulting counit, coproduct and antipode used to create a Hopf algebra from the appropriate group algebra creates dual quasi-Hopf algebras for the Cayley-Dickson algebras. Indeed, this construction yields dual quasi-Hopf algebras for the more general class of diagonal strong involution algebras resulting from the twisting of a group.

In particular, there is a dual quasi-Hopf structure over the octonions worthy of inclusion in the discussion of quantum groups. It is identical to using the octonionic basis in the construction above for a group algebra. Specifically, $\Delta(e_i) = e_i \otimes_k e_i$, $\epsilon(e_i) = 1$ and $S(e_i) = e_i^\dagger$ are extended linearly to all of $\mathbb{O}$.

CHAPTER 2

NON-ASSOCIATIVE LINEAR ALGEBRA

This chapter aims to define a workable notion of octonionic spaces as a first step towards linear algebra. The first obstacle is to find appropriate notions of modules and actions. We can then use the properties of an inner product to establish constraints on the algebra itself.

With this groundwork, definitions for octonionic Hilbert spaces are straightforward. Indeed, octonionic l^2 spaces emerge that are infinite and orthomodular. Solèr's theorem states that any orthomodular form over a division ring having an infinite orthonormal sequence is a Hilbert space over the real, complex or quaternionic numbers. This is not a contradiction, since the octonions are not a ring. I generalize this result to add precisely the Hilbert spaces over the four real division pre-algebras.

2.1 Introduction

The definition of spaces, much less orthomodular forms, are complicated if we allow non-associative algebras. Goldstine and Horwitz explicitly defined octonionic Hilbert spaces analogous to free modules with left multiplication [27]. Huo, Li and Ren used a similar definition when classifying alternative left modules over the octonions [32]. This definition does not address general octonionic modules, nor modules over other non-associative algebras. Albuquerque and Majid constructed comodules and corepresentations over non-associative algebras by twisting group algebras [1]. This does not, however, give us octonionic modules.

Considering left multiplication of an algebra on itself as a prototypical module suggests replacing multiplication by the composition of the linear maps on the underlying spaces. For an associative algebra acting on itself this becomes the standard definition of a module.

Another issue is the definition of an inner product. Complex inner products generalize linearity with conjugate linearity. Quaternions require one to distinguish between left and right linear. Octonions require linearity to be up to an associativity constraint.

This inner product is enough to generalize projections onto subspaces, though the subspace structure is nuanced. Requiring completeness yields an octonionic Hilbert space. The octonionic sequence space, l^2, produces an infinite orthonormal octonionic Hilbert space.

Baez observed that no one has yet to produce a satisfactory linear algebra over $\mathbb{O}$ [3]. Here we have the ability to construct octonionic infinite dimensional orthonormal Hilbert spaces, define a projection operator and prove that any octonionic module has an orthogonal basis. Indeed, we have linear maps capable of performing elementary row operations.

We close the chapter with the recent classification of alternative left octonionic modules by Huo, Li and Ren[32].

Going forward we introduce the term pre-module to distinguish when associativity is not assumed.

2.2 Pre-Modules

We desire a notion of an octonionic module. We also expect pre-algebras to be modules over themselves under left multiplication. The usual axioms of a module amount to an algebra homomorphism between the algebra and a subspace of the linear transformations between the module. The latter are associative, so a non-associative modules must be something else, a pre-module.

The definitions from this section are generalizations of the usual definitions over associative modules, or borrowed from similar terms used with non-associative algebras in Schafer [52].

2.2.1 Left Multiplication as Action

Let k be a field and A be a pre-k-algebra. Let $\mathrm{End}_R(A)$ be the linear maps from A to itself as a k-vector space. If A is n dimensional and R is a field, then $\mathrm{End}_R(A)$ is isomorphic to the $n \times n$ matrices over R. Left multiplication defines $\varphi : A \to \mathrm{End}_R(A)$ taking a to $\varphi_a(x) = ax$. Unfortunately there is a significant snag.

Since A is non-associative while the $\varphi_a \in \mathrm{End}_R(A)$ are associative, clearly this can't be an algebra homomorphism. Further, if a and b fail to associate with some c in A, then $\varphi_a \circ \varphi_b$ yields an element of $\mathrm{End}_R(A)$ that may not φ_d for any d. Thus the set of all φ_a, φ_A, is not algebraically closed.

Given a subset S of an algebra, A, the algebra **generated** by S, $\overline{S}$, is the smallest subalgebra of A containing S. The **enveloping algebra** of A is then $\overline{\varphi_A}$. In some sense what makes enveloping algebras interesting is their codimension in $\mathrm{End}_R(A)$. For the octonions this is a minimal 0, while for associative algebras of finite dimension n this is a maximal $n^2 - n$.

2.2.2 Pre-Modules

Let R be a commutative ring with identity. A **module**, X, over an R-algebra A is an R-module with a ring homomorphism, φ, from A to $\mathrm{End}_R(X)$. As set functions, $\mathrm{End}_R(X)$ is associative, so if A acts non-associatively, no such homomorphisms exist. Given a linear map $\varphi : A \to \mathrm{End}_R(X)$ we can define a bilinear product $\cdot : A \times X \to X$ as $a \cdot x = \varphi_a(x)$, and vice versa.

Definition 2.2.1. *A* **left pre-module***, X, over a pre-R-algebra, A, is an R-module and a bilinear map linear map $\cdot : A \times X \to X$.*

Define the **enveloping algebra** of A on X as $\overline{\varphi_A}$ in $\mathrm{End}_R(X)$. A pre-A-module is an $\overline{\varphi_A}$-module.

As a notation, for $r \in \overline{\varphi_A}$ and $x \in X$ we can write $rx = r \cdot x = r(x)$. It is useful to express these properties more concretely. For all a, b in A, r, s in the enveloping algebra of A and x, y in X we have:

$$r \cdot (x + y) = r \cdot x + r \cdot y, \tag{2.1}$$

$$(r + s) \cdot x = r \cdot x + s \cdot x, \text{ and} \tag{2.2}$$

$$r(s(x)) = (r \circ s)(x). \tag{2.3}$$

The first two axioms state the enveloping algebra is left and right distributive. The third emphasizing that we are merely using the function composition in $\overline{\varphi_A}$.

If A is unital, then an A-pre-module is **unital** if $1_A x = x$. If A has two-sided inverses we further require $a^{-1} \cdot (a \cdot x) = a \cdot (a^{-1} \cdot x) = x$. An A-pre-module is **non-degenerate** if $ax = 0$ implies either $a = 0$ or $x = 0$.

It is useful to define an associator to describe the lack of associativity of A.

$$[a, b, x] = (ab) \cdot x - a \cdot (b \cdot x). \tag{2.4}$$

A module is an associative pre-module, i.e. $[a, b, x] = 0$. For an alternative algebra, an **alternative** pre-module requires $[a, b, x] = -[b, a, x]$.

A **right pre-A-module** is defined in the obvious way. Note that the composition in (2.3) is then in $A^{\mathrm{op}} \to \mathrm{End}_R(X)$.

Octonionic spaces in literature are alternative pre-modules [27][40][32].

2.2.3 Examples

The enveloping algebra of a pre-algebra on itself by left multiplication, as in Section 2.2.1, is a motivating action.

A finite direct sum of n copies of an algebra, A^n, with multiplication defined pointwise, is another. Indeed, we can use pointwise multiplication for the functions from a given set to A, A^S. For the octonions $\overline{\varphi_{\mathbb{O}}}$ is then isomorphic to $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^8)$, with $\mathrm{End}_{\mathbb{R}}(\mathbb{O}^n)$ being diagonal with each diagonal entry being identical.

The quaternions acting on the split-octonions by left multiplication as a subalgebra provide an example of an action by an associative algebra that is a pre-module but not a module. The quaternions act associatively with themselves, and map the split signature subspace to itself. Thus they separate $\mathbb{O}^-$ into two four dimensional subspaces. On the first, the enveloping algebra is isomorphic to $\mathbb{H}$. On the second the enveloping algebra is isomorphic to $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^4)$. Thus on all of $\mathbb{O}^-$, $\overline{\varphi_{\mathbb{H}}} \cong \mathbb{H} \oplus_{\mathbb{R}} \mathrm{End}_{\mathbb{R}}(\mathbb{R}^4)$. Two instructive elements are $1 \oplus_{\mathbb{R}} 0 = \frac{1}{2}(x - i \cdot (j \cdot (k \cdot x)))$ and $0 \oplus_{\mathbb{R}} I = \frac{1}{2}(x + i \cdot (j \cdot (k \cdot x)))$. This results in a 20 dimensional subspace of the 64 dimensional $\mathrm{End}_{\mathbb{R}}(\mathbb{O}^-)$, for a codimension of 44.

2.2.4 Pre-Bimodules

Definition 2.2.2. *A* **pre-AB-bimodule** *X is both a left pre-A-module with $\cdot : A \times X \to X$ and a right pre-B-module with $\cdot : X \times B \to X$.*

If we let $\varphi_a = a \cdot x$ and $\psi_b = x \cdot b$, the enveloping algebra of a bimodule is $\overline{\varphi_A \cup \psi_B}$.

Even for associative algebras acting on themselves as a bimodule, the left and right enveloping algebras may map to separate segments of $\mathrm{End}_R(A)$. For example, the enveloping algebra of $\mathbb{H}$ on itself is isomorphic to $\mathbb{H}$ as a left or right module, since $\mathbb{H}$ is a $*$-pre-algebra so $\mathbb{H} \cong \mathbb{H}^{\mathrm{op}}$. However, $\overline{\varphi_{\mathbb{H}} \cup \psi_{\mathbb{H}}} = \mathrm{End}_{\mathbb{R}}(\mathbb{H})$.

This introduces a new associator,

$$[a, x, b] = (a \cdot x) \cdot b - a \cdot (x \cdot b). \tag{2.5}$$

A pre-A-bimodule is **alternative** if for all a and b in A and x in X,

$$[a,b,x]=[b,x,a]=[x,a,b]=-[b,a,x]=-[a,x,b]=-[x,b,a]. \tag{2.6}$$

2.3 Linear Dependence

Definition 2.3.1. *Let the* **span** *of S, $\mathrm{Span}_A(S)$, be the set of all $x\in X$ such that there is a finite sum $x=\sum_i a_i s_i$ with $a_i\in A$ and $s_i\in S$.*

For non-associative pre-algebras, $\mathrm{Span}_A^2(S)=\mathrm{Span}_A(\mathrm{Span}_A(S))\neq \mathrm{Span}_A(S)$ in general. Thus $\mathrm{Span}_A(S)$ is not closed.

In Subsection 2.8.1 we explicitly show that $\overline{\varphi_{\mathbb{O}}}$ is Cl(6). Let $x=(1,e_1,e_2,e_3,e_4,e_5,e_6,e_7)$ in $\mathbb{O}^8$. Then $\mathrm{Span}_{\mathbb{O}}^5(\{x\})\subsetneq \mathrm{Span}_{\mathbb{O}}^6(\{x\})=\overline{\varphi_A}(\{x\})=\mathbb{O}^8$.

Huo, Li and Ren call $S\in X$ linearly dependent if there is a finite sum $\sum_i a_i s_i=0$ with $a_i\in A$ and $s_i\in S$ [32]. Argument analogous to the above shows this is not closed under linear relations. Extending the definition of Huo, Li and Ren to allow coefficients in $\overline{\varphi_A}$ runs into issues, since the enveloping algebra may have non-invertible elements.

For $\mathbb{O}$ there are elements, α, of $\overline{\varphi_{\mathbb{O}}}$ that annihilate the real component. Then $\alpha\cdot(1,0)+\alpha\cdot(0,1)=0$, showing that $(1,0)$ and $(0,1)$ would be linearly dependent. Rather than expanding the coefficients to $\overline{\varphi_{\mathbb{O}}}$, we could have restricted them to $\mathbb{R}$ to get a subset that is closed under the allowed linear relations. These considerations motivate the following definitions.

Definition 2.3.2. *A vector, x, is* $\overline{\varphi_A}$**-linearly dependent** *on a subset S of X if $x\in \mathrm{Span}_{\overline{\varphi_A}}(S)$. It is* $\mathrm{Nuc}(a)$**-linearly dependent** *on a subset S if $x\in\mathrm{Span}_{\mathrm{Nuc}(A)}(S)$.*

We say $\mathrm{Span}_{\overline{\varphi_A}}(S)$ is the subset $\overline{\varphi_A}$**-generated** by S, and $\mathrm{Span}_{\mathrm{Nuc}(A)}(S)$ is the subset $\mathrm{Nuc}(A)$**-generated** by S. While $\mathrm{Span}_{\overline{\varphi_A}}(S)$ yields a pre-A-module, $\mathrm{Span}_{\mathrm{Nuc}(A)}(S)$ only yields a pre-$\mathrm{Nuc}(A)$-module. Orthogonality will introduce another concept of a subset generated by S somewhere in between.

A subset B is $\overline{\varphi_A}$**-linearly independent** or $\mathrm{Nuc}(A)$**-linearly independent** if no element $b\in B$ is in $\mathrm{Span}_{\overline{\varphi_A}}(B-\{b\})$ or $\mathrm{Span}_{\mathrm{Nuc}(A)}(B-\{b\})$ respectively. A subset is $\overline{\varphi_A}$**-free** or $\mathrm{Nuc}(A)$**-free** if it is generated by a linearly independent set, called a $\overline{\varphi_A}$**-basis** or $\mathrm{Nuc}(A)$**-basis** respectively. The prefixes can be suppressed if the type of linear dependence is clear from the context.

Proposition 2.3.3. *The pre-A-modules A^n are $\overline{\varphi_A}$-free.*

Proof. Consider the set B consisting of $b_i = (0, ..., 1, ..., 0)$, where the i_{th} component is 1. These are $\overline{\varphi_A}$-linearly independent and $A^n = \mathrm{Span}_{\overline{\varphi_A}}(B)$. Hence B is a $\overline{\varphi_A}$-basis for A^n, and A^n is $\overline{\varphi_A}$-free. $\square$

2.3.1 Linear Maps

Let A be a unital pre-algebra. We turn our attention to linear maps from A^n to A^m. Without associativity, matrices with entries in A are not associative, so these maps do not compose properly.

Even for $\mathrm{End}_A(A^n)$ this raises issues in the definitions of elementary matrices from $A^n \to A^n$. Swapping rows works and is idempotent, as usual. Scaling a row by an element of A breaks, in general. If A is alternative and a has an inverse a^{-1} in A, then scaling by a and a^{-1} are at least inverses. Adding a scaled multiple of one row to another likewise has the familiar inverse, but do not compose in general.

Let B be a subalgebra of A. Define $\mathrm{End}_{A|B}(A^n)$ to be the **restriction** of $\mathrm{End}_A(A^n)$ to matrices with coefficients in B. When $B = \mathrm{Nuc}(A)$, then composition in $\mathrm{End}_{A|\,\mathrm{Nuc}(A)}(A^n)$ will be associative due to the associativity of $\mathrm{Nuc}(A)$ and its action on A. In particular, the elementary row operations in $\mathrm{End}_{A|\,\mathrm{Nuc}(A)}(A^n)$ work as expected on A^n.

For $\mathbb{O}$, $\mathrm{Nuc}(\mathbb{O}) = \mathbb{R}$ and $\mathrm{End}_{\mathbb{O}|\mathbb{R}}(\mathbb{O}^n) \cong \mathrm{End}_{\mathbb{R}}(\mathbb{R}^n)$.

2.3.2 Example

First, let us consider $\mathbb{O}^2$ with $\overline{\varphi_{\mathbb{O}}}$-linearly dependence. Clearly $B_1 = \{(1, 0), (0, 1)\}$ generates $\mathbb{O}^2$. Further, $B_2 = \{(1, e_1)\}$ is linearly independent, as a singleton, hence is a basis for some free space.

Now consider the following, recalling Table 1.2:

$$e_2 x = (e_2, -e_3), \tag{2.7}$$

$$e_4 x = (e_4, -e_5), \tag{2.8}$$

$$e_5(e_2 x) = (-e_7, e_6), \tag{2.9}$$

$$e_3(e_4 x) = (e_7, e_6), \tag{2.10}$$

$$e_3(e_4 x) - e_5(e_2 x) = (2e_7, 0), \text{ and} \tag{2.11}$$

$$e_3(e_4 x) + e_5(e_2 x) = (0, 2e_6) \tag{2.12}$$

We can then left multiply the last two by the inverse of the non-zero term to find that x generates the vectors $(1, 0)$ and $(0, 1)$. Thus B_2 also generates $\mathbb{O}^2$. This demonstrates the following:

Proposition 2.3.4. *A subspace of $\mathbb{O}^n$ may have $\overline{\varphi_{\mathbb{O}}}$-bases with different cardinality.*

Any ordered pair of octonions whose two components are not scaled by a real number will generate $\mathbb{O}^2$. The only proper subspaces are generated by a basis of the form $\{(a, b)\}$, where $a, b \in \mathbb{R}$.

Next we consider $\mathbb{O}^2$ with a $\mathrm{Nuc}(\mathbb{O})$-linear dependence. Since $\mathrm{Nuc}(\mathbb{O}) = \mathbb{R}$, we get precisely the subspaces generated on the underlying $\mathbb{R}$-module.

Finally we consider elementary row operations in $\mathbb{O}^n$.

Proposition 2.3.5 (Prather)**.** *Let $X = \mathbb{O}^n$, and $x \in X$. Using elementary row operations from $\mathrm{End}_{\mathbb{O}|\mathbb{R}}(\mathbb{O}^n)$ we can transform x to one with at most 8 non-zero components. Automorphisms of $\mathbb{O}$ further reduces the number of cases to initial portions of the following forms:*

- $(\cos\theta + \sin\theta e_1, e_2, \cos\phi e_3 + \sin\phi e_4, ...)$,
- $(\cos\theta + \sin\theta e_1, e_2, e_3, e_4, ...)$, *and*
- $(1, e_1, e_2, \cos\phi e_3 + \sin\phi e_4, ...)$,
- $(1, e_1, e_2, e_3, e_4, ...)$,

where $\theta, \phi \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ and the remaining components of x are in $\mathrm{Span}_{\mathbb{R}}(\{e_5, e_6, e_7\})$.

Proof. The components of x span some real subspace in $\mathbb{O}$. Let d be the dimension of this subspace. Since $\mathrm{Dim}_{\mathbb{R}}(\mathbb{O}) = 8$, $d \leq 8$.

Using elementary row operations in $\mathrm{End}_{\mathbb{O}|\mathbb{R}}(\mathbb{O}^n)$ we can transform x to a vector with linearly independent values in the first d components and 0 elsewhere. Further, we can eliminate the e_0 coefficient from all components except x_1. We can then use the familiar Gram-Schmidt process to guarantee the components of x are orthonormal in $\mathbb{O}$ as a real vector space. Saving x_1 for last will preserve the e_0 coefficients.

Suppose sufficient x_i are non-zero.

Now we can use the automorphisms of $\mathbb{O}$ to use the first few x_i to set e_1, e_2. Only x_1 contains a non-zero real value. If x_1 is not real use the pure value in the complex space spanned by x_1 to fix e_1. Since x_2 is orthogonal to this complex algebra, and is pure, it can be used to set e_2. If x_1 is real, we can use x_2 to set e_1 and x_3 to set e_2. Either way, this fixes e_3.

We can now use elementary row operations to isolate e_3, as we did above for the real part. Now either the next component of x is simply e_3, or it contains a component orthogonal to the quaternion subalgebra generated by e_1 and e_2 that we can use to set e_4. This now fixes the automorphism, hence the remaining basis.

The first form listed in the statement of the proposition represents the general case, with the remaining forms dealing with the fact that e_1 or e_4 are undefined when θ or ϕ are 0 respectively.

The remaining components of x are in $\mathrm{Span}_{\mathbb{R}}(\{e_5, e_6, e_7\})$, since they must be orthogonal to the x_i given, or would have been used to generate an earlier form in the list.

If we run out of non-zero x_i we simply end with the initial portion of one of the forms given. □

Corollary 2.3.6. *If $d = 2$, the only form needed is* $(\cos\theta + \sin\theta e_1, e_2, 0, ...)$.

Proof. For $(1, e_1, 0, ...)$ we could have selected an automorphism to produce $(1, e_2, 0, ...)$ instead. □

2.4 Non-Associative Hermitian Spaces

Orthomodular forms and Hilbert spaces require orthogonality. This motivates the more general Hermitian spaces. The definitions here are straightforward generalizations from the associative setting, or common operations from non-associative algebras.

The results in this section are a rephrasing of the results by Ludkowski [40].

2.4.1 Hermitian Spaces

Definition 2.4.1. *Let A be a $*$-pre-algebra over R. A* **Hermitian space** *is a left pre-A-module with a* **Hermitian form** $\langle\cdot,\cdot\rangle : X \times X \to A$ *such that for all u, v and w in X and a, b in R:*

$$\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle, \tag{2.13}$$

$$\langle u, v\rangle = \overline{\langle v, u\rangle}. \tag{2.14}$$

Often we consider the case where a and b are in A. If $A = \mathbb{R}$ this makes no difference. If $A = \mathbb{C}$ we must consider conjugate linearity. If $A = \mathbb{H}$ we must consider left and right conjugate linearity, namely $\langle au, bv\rangle = a\langle u, v\rangle \bar{b}$. If $A = \mathbb{O}$ we must introduce associators.

$$[a, u, v] = \langle au, v\rangle - a\langle u, v\rangle\,, \text{ and} \tag{2.15}$$

$$[u, v, b] = \langle u, v\rangle b - \left\langle u, \bar{b}v\right\rangle. \tag{2.16}$$

These are related by conjugation:

$$\overline{[a, u, v]} = \overline{\langle au, v\rangle} - \overline{a\langle u, v\rangle} = \langle v, au\rangle - \overline{\langle u, v\rangle}\,\overline{a} = \langle v, au\rangle - \langle v, u\rangle\,\overline{a} = -[v, u, \overline{a}]. \tag{2.17}$$

A Hermitian form is **degenerate** if there is some $u \neq 0$ such that $\langle u, v\rangle = 0$ for all v, and **non-degenerate** otherwise.

A Hermitian form is **alternative** if $[a, u, u] = [a, u, au] = 0$. This allows us to conclude that

$$\langle au, au\rangle = a\langle u, au\rangle = a(\langle u, u\rangle\,\overline{a}), \tag{2.18}$$

$$[u, u, a] = -\overline{[\overline{a}, u, u]} = 0 \text{ and} \tag{2.19}$$

$$[au, u, \overline{a}] = -\overline{[a, u, au]} = 0. \tag{2.20}$$

A **subspace** of X is a subset that is closed under linear combinations with coefficients in the nucleus of A. Orthogonality for octonionic modules has two definitions in the literature. We say u and v are **strongly orthogonal** if $\langle u, v\rangle = 0$, and **weakly orthogonal** if $\Re(\langle u, v\rangle) = 0$. Ludkowski uses the former[40], while Goldstine and Horwitz use the latter[27]. Strong orthogonality is well defined for general algebras, while weak orthogonality only makes sense for diagonal strong involution algebras.

Many standard result for Hermitian spaces are topological or reduce to arguments on the underlying R-module structure of A and X, and extend to the non-associative setting in either with little difficulty. We will assume strong orthogonality unless stated otherwise.

Let X be a Hermitian space and $S \subset X$. Define $S^{\perp} = \{x \in X | \forall s \in S, \langle s, x\rangle = 0\}$ as the **orthogonal complement** of S. Two subsets S and T are **orthogonal** if every $s \in S$ is orthogonal to every $t \in T$. A subset S is **closed** when $S = S^{\perp\perp}$. These can be modified to indicate the sense of orthogonality used, as needed.

Proposition 2.4.2. *For any subset S of X, $S \subset S^{\perp\perp}$ [40].*

Proof. Let u be any element of S and v any element of $S^{\perp}$. By the definition of $S^{\perp}$, $\langle u, v\rangle = 0$. Since $\overline{0} = 0$, $\langle v, u\rangle = 0$. Since u and v were arbitrary, $u \in (S^{\perp})^{\perp} = S^{\perp\perp}$, and $S \subset S^{\perp\perp}$. $\square$

Proposition 2.4.3. *If $T \subset S$ then $S^{\perp} \subset T^{\perp}$ and $T^{\perp\perp} \subset S^{\perp\perp}$ [40].*

Proof. Suppose $u \in S^{\perp}$. Then $\langle u, v\rangle = 0$ for any v in S. But any v in T is also in S. Thus $\langle u, v\rangle = 0$ for any v in T. Now $u \in T^{\perp}$. Since u was arbitrary, $S^{\perp} \subset T^{\perp}$.

Applying this to $S^{\perp} \subset T^{\perp}$ yields $T^{\perp\perp} \subset S^{\perp\perp}$. □

Proposition 2.4.4. *For any subset S, $S^{\perp}$ is closed [40].*

Proof. From Proposition 2.4.2 we have $S \subset S^{\perp\perp}$ and $S^{\perp} \subset (S^{\perp})^{\perp\perp}$. From the former and Proposition 2.4.3 we have $(S^{\perp\perp})^{\perp} \subset S^{\perp}$. But then $S^{\perp} = S^{\perp\perp\perp}$, and $S^{\perp}$ is closed. □

In particular $S^{\perp\perp}$ is closed for any S, and is the **orthogonal closure** of S. If the elements of S are orthogonal, S is an **orthogonal basis** for $S^{\perp\perp}$.

Proposition 2.4.5. *If S and T are orthogonal, then so are $S^{\perp\perp}$ and $T^{\perp\perp}$ [40].*

Proof. By the definition of orthogonal space we have $T \subset S^{\perp}$. Thus by Proposition 2.4.3 $S^{\perp\perp} \subset T^{\perp}$ and $T^{\perp\perp} \subset S^{\perp\perp\perp}$. But then every vector of $S^{\perp\perp}$ is orthogonal to every vector of $T^{\perp\perp}$. □

Given two Hermitian spaces S and T over A the **direct sum** $S \oplus T$ is the set of ordered pairs (s_i, t_i), where $s \in S$ and $t \in T$, with the action defined pointwise and the Hermitian form $\langle (s_1, t_1), (s_2, t_2)\rangle = \langle s_1, s_2\rangle + \langle t_1, t_2\rangle$. If S and T are subspaces of X we can define $S + T$ as the subspace of the form $s + t$, with $s \in S$ and $t \in T$.

Proposition 2.4.6. *Let S and T be orthogonal subspaces of X. Then $S + T \cong S \oplus T$ [40].*

Proof. By definition, $\langle s + t, s + t\rangle = \langle s, s\rangle + \langle s, t\rangle + \langle t, s\rangle + \langle t, t\rangle$. From orthogonality this becomes $\langle s + t, s + t\rangle = \langle s, s\rangle + \langle t, t\rangle$. But this is the defining relation for $S \oplus T$. □

Proposition 2.4.7. *Let S_i be a finite set of mutually orthogonal subspaces of X. Then*

$$\left\{ \sum_{s_i \in S_i} s_i \right\} \cong \bigoplus_i S_i$$

[40].

Proof. This follows by recursively applying Proposition 2.4.6. □

These theorems demonstrate how an orthogonal basis play an important role, particularly for a finite Hermitian spaces. Similarly, the orthogonal complement of a single element can't be broken down any further. A **cyclic** subspace, or **cycle**, is the orthogonal closure of a single non-zero element.

Unfortunately, Corollary 2.4.11 shows that $\bigoplus_i S_i$ may not be closed, even if the S_i are closed.

A Hermitian space X is called **orthomodular** when for any closed subset S, $X = S \oplus S^\perp$.

Proposition 2.4.8. *The Hermitian form of an orthomodular space must be non-degenerate [40].*

Proof. If the Hermitian form is degenerate, then there is some $u \neq 0$ such that $\langle u, v\rangle = 0$ for all $v \in X$. But then u must be in $S^\perp$ for any S, including $S^\perp$. Hence u must be included in any closed subspace. If S is closed, u is then in both S and $S^\perp$. Since $u \neq 0$, $S \oplus S^\perp$ can't be X. Thus X is not orthomodular. □

Two map ϕ and ψ in $\mathrm{End}_{A|\,\mathrm{Nuc}(A)}(A^n)$ are **Hermitian adjoints** if $\langle \phi(x), y\rangle = \langle x, \psi(y)\rangle$.

2.4.2 Examples

The primary motivation comes from A^n. Let $p, q \in A^n$. Now A^n forms a Hermitian space with the Hermitian form

$$\langle p, q\rangle = \sum_{i=1}^{n} p_i \overline{q_i}. \tag{2.21}$$

The proof that this is a Hermitian form follows directly from the linearity of the product in A and conjugation being an anti-automorphism.

If we restrict our attention to $\mathrm{End}_{\mathbb{O}|\mathbb{R}}(\mathbb{O}^n)$, the adjoints in this subspace are the usual transpose as a real matrix.

For infinite Hermitian spaces one must consider whether convergence issues allow the Hermitian form to be well defined. If only finitely many values are non-zero then there will only be finitely many non-zero entries to add, avoiding convergence issues. When R is $\mathbb{R}$ or $\mathbb{C}$ and the set is countable one gets l^2 sequence spaces. If the set is measurable this extends to L^p spaces in the usual way.

Proposition 2.4.9. *The transformations of Proposition 2.3.5 satisfy* $\{\phi(S)\}^{\perp\perp} = \phi(\{S^{\perp\perp}\})$. *For* $\psi \in \mathrm{End}_{\mathbb{O}|\mathbb{R}}(\mathbb{O}^n)$, $\{\psi(S)\}^\perp = \psi^{-1}(\{S^\perp\})$ *while for* $\alpha \in \mathrm{Aut}\,\mathbb{O}$, $\{\alpha(S)\}^\perp = \alpha(\{S^\perp\})$.

Proof. The transformations, ϕ, used to reduce x in Proposition 2.3.5 are a composition of an invertible linear map, $\psi \in \mathrm{End}_{\mathbb{O}|\mathbb{R}}(\mathbb{O}^n)$ and $\alpha \in \mathrm{Aut}(\mathbb{O})$.

Now ψ can be represented as a matrix, A. Further, $A^\dagger = A$, since its coefficients are in $\mathbb{R}$. Thus $\langle \psi x, \psi^{-1} y \rangle = (xA)(yA^{-1})^\dagger = (xA)(A^{-1}y^\dagger)$. Since the coefficients of A are in the nucleus of $\mathbb{O}$, we can move the parenthesis to cancel A and A^{-1} and $\langle \psi x, \psi^{-1} y \rangle = \langle x, y \rangle$. Thus $x \in \psi^{-1}(\{S^\perp\})$ implies $x \in \{\psi(S)\}^\perp$. Since ψ is invertible, $\{\psi(S)\}^\perp = \psi^{-1}(\{S^\perp\})$.

Applying this twice we have $\{\psi(S)\}^{\perp\perp} = \psi(\{S^{\perp\perp}\})$.

An automorphism, α, of $\mathbb{O}$ preserves all formula involving conjugation, sums and products of octonionic values by definition. In particular, this preserves $\langle x, y \rangle$. Thus $\langle \alpha(x), \alpha(y) \rangle = \alpha(\langle x, y \rangle)$. If $\langle x, y \rangle \in \mathbb{R}$, then it is invariant under automorphisms of $\mathbb{O}$, since 1 is fixed. In particular, this is true if $\langle x, y \rangle = 0$. Thus $\{\alpha(S)\}^\perp = \alpha(\{S^\perp\})$, and $\{\alpha(S)\}^{\perp\perp} = \alpha(\{S^{\perp\perp}\})$.

Composing these two partial results yields $\{\phi(S)\}^{\perp\perp} = \phi(\{S^{\perp\perp}\})$. $\square$

It is useful to classify the cyclic subspaces of $\mathbb{O}^n$ with the action defined point-wise and Hermitian form above.

Theorem 2.4.10 (Prather)**.** *Using strong orthogonality, the cycles in $\mathbb{O}^n$, $\{x\}^{\perp\perp}$, are the set $\{qx | q \in A\}$, where*

- *$A = \mathbb{O}$ if $d = 1$,*
- *$A \cong \mathbb{C}$, where A is generated by $\{x_i^\dagger x_j\}$ if $d = 2$, or*
- *$A = \mathbb{R}$ if $d > 2$,*

where d is defined as in Propositions 2.3.5.

Using weak orthogonality, the cycles in $\mathbb{O}^n$ are all of the form $\{qx | q \in \mathbb{R}\}$.

Proof. Let's consider strong orthogonality first.

Further, we restrict our attention to the reduced forms of Propositions 2.3.5.

Let $x = (x_1, ..., x_d, 0, ..., 0)$. For $y \in \{x\}^\perp$ and $z \in \{x\}^{\perp\perp}$, let $y = (y_1, ..., y_n)$ and $z = (z_1, ..., z_n)$.

Note that for $i > d$, y, with $y_i = 1$ and $y_j = 0$ for $j \neq i$, is in $\{x\}^\perp$. Then $\langle z, y \rangle = z_i = 0$. Thus $z = (z_1, ..., z_d, 0, ..., 0)$ for all $z \in \{x\}^{\perp\perp}$.

Similarly, if $d = 1$ then $y_1 = 0$ for all $y \in \{x\}^\perp$. But then z_1 is free to be any element of $\mathbb{O}$, while the rest must be 0. Since $\mathbb{O}$ is a division algebra, $z_1 = qx_1$ for some $q \in \mathbb{O}$. Since $x_i = z_i = 0$ for $i > 1$, $z = qx$. But this completes the $d = 1$ case.

If $d = 2$, then $x_1 y_1^\dagger + x_2 y_2^\dagger = 0$, so $y_1^\dagger = -\frac{1}{N(x_1)} x_1^\dagger (x_2 y_2^\dagger)$. Similarly, $z_1 y_1^\dagger + z_2 y_2^\dagger = 0$ so $z_2 = -\frac{1}{N(y_2)}(z_1 y_1^\dagger) y_2$. Hence,

$$z_2 = \frac{1}{N(x_1)N(y_2)}(z_1(x_1^\dagger(x_2 y_2^\dagger)))y_2. \tag{2.22}$$

Since x is given, (2.22) imposes restrictions on z_1. Specifically, z_2 must remain constant for any $y_2 \in \mathbb{O} - \{0\}$.

By Corollary 2.3.6, $x = (\cos\theta + \sin\theta e_1, e_2, 0, ...)$ is the only form we need to consider. Choosing $y_2 = 1$ we can find $z_2 = z_1(x_1^\dagger x_2) = z_1(\cos\theta e_2 - \sin\theta e_3)$. Now choosing $y_2 = e_2$ yields,

$$\begin{align} z_2 &= -(z_1((\cos\theta + \sin\theta e_1)^\dagger(e_2 e_2)))e_2 = (z_1(\cos\theta - \sin\theta e_1))e_2 \tag{2.23}\\ &= [z_1, \cos\theta - \sin\theta e_1, e_2] + z_1((\cos\theta - \sin\theta e_1)e_2) \tag{2.24}\\ &= [z_1, \cos\theta - \sin\theta e_1, e_2] + z_1(\cos\theta e_2 - \sin\theta e_3) = [z_1, \cos\theta - \sin\theta e_1, e_2] + z_2 \tag{2.25}\\ 0 &= [z_1, \cos\theta - \sin\theta e_1, e_2] = \cos\theta[z_1, 1, e_2] - \sin\theta[z_1, e_1, e_2] = -\sin\theta[z_1, e_1, e_2]. \tag{2.26} \end{align}$$

Thus we have two cases, (1) $\sin(\theta) = 0$ or (2) $[z_1, e_1, e_2] = 0$.

Case (1): If $\sin(\theta) = 0$, $\theta = 0$ since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus $x = (1, e_2)$. Now compute z_2 with $y_2 = e_4$.

$$\begin{align} z_2 &= -(z_1((1)^\dagger(e_2 e_4)))e_4 = -(z_1 e_6)e_4 = -[z_1, e_6, e_4] - z_1(e_6 e_4) = -[z_1, e_6, e_4] + z_1 e_2 \tag{2.27}\\ &= [z_1, e_6, e_4] + z_2. \tag{2.28} \end{align}$$

Thus z_1 must be in the subalgebra generated by e_6 and e_4. Repeating this with $y_2 = e_1$ shows that z_1 must be in the subalgebra generated by e_3 and e_1. The intersection of these algebras is the complex subalgebra generated by $x_1^\dagger x_2 = e_2$.

In this complex subalgebra the expression on the right of (2.22) is generated by e_2 and e_4, and is associative. Thus $z = (z_1, z_1 e_2) = z_1 x$, and $\{x\}^{\perp\perp} = \{z_1 x | z_1 \in A\}$, where A is the complex subalgebra generated by $x_1^\dagger x_2$.

Case (2): If $[z_1, e_1, e_2] = 0$, then z_1 is in the subalgebra generated by e_1 and e_2. Then for some real numbers a_i we have $z_1 = a_0 + a_1e_1 + a_2e_2 + a_3e_3$. Further, letting $y_2 = e_4$ we have

$$z_2 = -(z_1((\cos\theta + \sin\theta e_1)^\dagger(e_2e_4)))e_4 = -(z_1(\cos\theta e_6 + \sin\theta e_7))e_4 \tag{2.29}$$

$$= -[z_1, \cos\theta e_6 + \sin\theta e_7, e_4] - z_1((\cos\theta e_6 + \sin\theta e_7)e_4) \tag{2.30}$$

$$= -[z_1, \cos\theta e_6 + \sin\theta e_7, e_4] + z_1(\cos\theta e_2 + \sin\theta e_3) \tag{2.31}$$

$$= -[z_1, \cos\theta e_6 + \sin\theta e_7, e_4] + z_2 + 2z_1 \sin\theta e_3 \tag{2.32}$$

$$2z_1 \sin\theta e_3 = [z_1, \cos\theta e_6 + \sin\theta e_7, e_4]. \tag{2.33}$$

We can use this to generate dependencies among the a_i.

$$2z_1 \sin\theta e_3 = 2\sin\theta(a_0e_3 - a_1e_2 + a_2e_1 - a_3) = -2\sin\theta(a_3 - a_2e_1 + a_1e_2 - a_0e_3) \tag{2.34}$$

$$= [a_0 + a_1e_1 + a_2e_2 + a_3e_3, \cos\theta e_6 + \sin\theta e_7, e_4] \tag{2.35}$$

$$= a_1[e_1, \cos\theta e_6, e_4] + a_1[e_1, \sin\theta e_7, e_4] + a_2[e_2, \cos\theta e_6, e_4] + a_2[e_2, \sin\theta e_7, e_4] \tag{2.36}$$

$$+ a_3[e_3, \cos\theta e_6, e_4] + a_3[e_3, \sin\theta e_7, e_4] \tag{2.37}$$

$$= a_1 \cos\theta[e_1, e_6, e_4] + a_1 \sin\theta[e_1, e_7, e_4] + a_2 \sin\theta[e_2, e_7, e_4] + a_3 \cos\theta[e_3, e_6, e_4] \tag{2.38}$$

$$[e_1, e_6, e_4] = (e_1e_6)e_4 - e_1(e_6e_4) = -e_7e_4 + e_1e_2 = 2e_3 \tag{2.39}$$

$$[e_1, e_7, e_4] = (e_1e_7)e_4 - e_1(e_7e_4) = e_6e_4 + e_1e_3 = -2e_2 \tag{2.40}$$

$$[e_2, e_7, e_4] = (e_2e_7)e_4 - e_2(e_7e_4) = -e_5e_4 + e_2e_3 = 2e_1 \tag{2.41}$$

$$[e_3, e_6, e_4] = (e_3e_6)e_4 - e_3(e_6e_4) = e_5e_4 + e_3e_2 = -2e_1 \tag{2.42}$$

$$2z_1 \sin\theta e_3 = 2a_1 \cos\theta e_3 - 2a_1 \sin\theta e_2 + 2a_2 \sin\theta e_1 - 2a_3 \cos\theta e_1. \tag{2.43}$$

Now comparing the four coefficients of z_1 we have the following linear relations:

$$-2a_3 \sin\theta = 0 \tag{2.44}$$

$$2a_2 \sin\theta = 2a_2 \sin\theta - a_3 \cos\theta \tag{2.45}$$

$$-2a_1 \sin\theta = -2a_1 \sin\theta \tag{2.46}$$

$$2a_0 \sin\theta = 2a_1 \cos\theta. \tag{2.47}$$

This shows that $a_3 = 0$, a_1 and a_2 are unrestrained, while a_0 depends on a_1. Now z_1 is in the space spanned by $e_2 = x_2 = x_1(x_1^\dagger x_2)$ and $\cos\theta + \sin\theta e_1 = x_1$. Thus z_1 is in the space spanned by $\{x_1, x_2\}$.

Let $z_1 = qx_1$, where q is in the subalgebra generated by $x_1^\dagger x_2$. Now $z = (z_1, z_1(x_1^\dagger x_2)) = (qx_1, (qx_1)(x_1^\dagger x_2))$. This last expression associates due to alternativity. Thus $\{x\}^{\perp\perp} = \{qx | q \in A\}$, where A is the complex subalgebra generated by $x_1^\dagger x_2$.

This completes the case for $d = 2$.

For $d > 2$ we get that z_1 must be in the subalgebra generated by $x_1^\dagger x_2$ and the subalgebra generated by $x_1^\dagger x_3$. Since x_2 and x_3 are distinct orthogonal pure unit octonions these are distinct complex subalgebras. Hence the intersection must be the real axis. Thus $\{x\}^{\perp\perp} = \{qx | q \in \mathbb{R}\}$.

This completes all cases for the classification of cycles under strong orthogonality for the reduced forms. Proposition 2.4.9 assures that the rank of the cyclic subspaces is preserved under transformations in $\mathrm{End}_{\mathbb{O}|\mathbb{R}}(\mathbb{O})$. All that remains is to consider which complex subalgebra appears in the $d = 2$ case.

Since $d = 2$, there must be an x_i and x_j such that $x_i^\dagger x_j$ is not real, defining a complex subalgebra as above. Further, if two such pairs exist, $x_i' = a_0 x_i + a_1 x_j$ and $x_j' = b_0 x_i + b_1 x_j$, so $x_i'^\dagger x_j' = (a_0 x_i + a_1 x_j)^\dagger (b_0 x_i + b_1 x_j) = a_0 b_0 N(x_i) + a_1 b_1 N(x_j) + a_0 b_1 x_i^\dagger x_j + a_1 b_0 (x_i^\dagger x_j)^\dagger$, which is in the same complex subalgebra. Thus $\mathrm{Span}_{\mathbb{R}}(x_i^\dagger x_j | 1 \le i, j \le n) \cong \mathbb{C} \subset \mathbb{O}$.

With weak orthogonality the dependence on y_j weakens to $\Re(y_j x_j^\dagger) = -\sum_{i \ne j} \Re(y_i x_i^\dagger)$. This allows y_1 to be a 7 dimensional slice in $\mathbb{O}$. But now the possible choices in y_1 alone force q to be real. Thus all cycles are of the form $\{qx | q \in \mathbb{R}\}$. $\square$

This proof works just as well for to the octonionic l^2 spaces below. This also shows that the spaces generated by weak orthogonality generates the same spaces as weak linear dependence. Only the spaces generated by strong orthogonality over a vector with real coefficients produce pre-$\mathbb{O}$-modules.

As a final example, we produce the counter example mentioned above.

Corollary 2.4.11 (Prather)**.** *If S and T are closed subspaces, $S + T$ may not be.*

Proof. Consider $x = (1, e_1, e_2, e_3)$ and $y = (e_1, 1, e_3, e_2)$. Now $\langle x, y \rangle = 0$, so they are orthogonal. By Theorem 2.4.10, $\mathrm{Dim}_{\mathbb{R}}(\{x\}^{\perp\perp}) = \mathrm{Dim}_{\mathbb{R}}(\{y\}^{\perp\perp}) = 1$. Now $x + y = (1 + e_1, 1 + e_1, e_2 + e_3, e_2 + e_3)$ and $\{x + y\}^{\perp\perp} = \{q(x - y) | q = a + be_3\}$, where a and b are real. Thus $e_3(x + y) = (e_3 + e_2, e_3 + e_2, -e_1 - 1, -e_1 - 1) \in \{x + y\}^{\perp\perp} \subset \{x, y\}^{\perp\perp}$. But $e_3(x + y) \notin \mathrm{Span}_{\mathbb{R}}(\{x, y\})$.

This shows that $\mathrm{Dim}_{\mathbb{R}}(\{x, y\}^{\perp\perp}) \ge 3 > 2 = \mathrm{Dim}_{\mathbb{R}}(\{x\}^{\perp\perp}) + \mathrm{Dim}_{\mathbb{R}}(\{y\}^{\perp\perp})$. Thus the sets in Proposition 2.4.7 may not be closed, even if the S_i are. $\square$

2.5 Non-Associative Inner Product Spaces

Definition 2.5.1. *An* **inner product space***, X, over a pre-R-algebra, A, is a subspace of an alternative Hermitian space whose Hermitian form is positive definite.*

This requires R to be ordered, and thus $\text{char}(R) = 0$. Explicitly:

$$\|u\|^2 = \langle u, u \rangle \in R1_A, \tag{2.48}$$

$$\|u\|^2 \geq 0, \tag{2.49}$$

$$\|u\|^2 = 0 \text{ iff } u = 0. \tag{2.50}$$

This induces an inner product on X as an R-module. It is well known that an inner product defines a distance function $d^2 = \|x - y\|^2$ that satisfies the properties of a metric, and produces the resulting metric topology.

2.5.1 Algebra Constraints

When R is a field, an inner product requires additional constraints beyond that of a $*$-algebra.

Proposition 2.5.2 (Prather)**.** *Let X be a non-degenerate inner product space over a pre-k-algebra A. Then $N(a) = a\overline{a}$ is a norm on A as a k-vector space.*

Proof. Since X is non-degenerate we can find a u such that $\|u\|^2 \neq 0$. Let $a \in A$.

Using the inner product on X, particularly alternativity and $\langle u, u \rangle \in k1_A$ and the unital property of A we have:

$$\|au\|^2 = a \langle u, u \rangle \overline{a} = \|u\|^2 a\overline{a}. \tag{2.51}$$

Since u was arbitrary this must be true for any non-zero $u \in X$. Thus we can define $N_A(a)$ as follows:

$$N_A(a) = a\overline{a} = \frac{\langle au, au \rangle}{\langle u, u \rangle} \in k1_A \tag{2.52}$$

Indeed, $N_A(a) \geq 0$, as a ratio of positive values. Observe that if $N_A(a) = 0$ then $\|au\|^2 = 0$ for all $u \neq 0$, so $au = 0$ for any u. But then $a = 0$, since X is non-degenerate.

Further, $\frac{\overline{a}}{N_A(a)}$ is a right inverse of a. By repeating the above with $\overline{a}$ we find $\frac{\overline{a}}{N_A(\overline{a})}$ is a left inverse of a. These coincide iff $N_A(a) = N_A(\overline{a})$. $\square$

Note that the entire family of Cayley-Dickson algebras satisfy this condition.

Since $\overline{a}$ is a linear map, the product is bilinear, and $N_A(a) = a\overline{a}$ is a quadratic form. From the above it is positive definite. From a positive definite quadratic form we can define an inner product on A as a k-vector space,

$$\langle p, q \rangle_A = \frac{1}{2} \left(N_A(p+q) - N_A(p) - N_A(q) \right). \tag{2.53}$$

Note that, as a k-vector space, we have the usual projection operations.

Proposition 2.5.3 (Prather)**.** *For all $a \in A$, $\overline{a} = 2\langle a, 1 \rangle_A - a$ and $N_A(a) = N_A(\overline{a})$.*

Proof. First, we find the conjugate of elements orthogonal to the unit in A.

$$2\langle u, 1 \rangle_A = 0 = N_A(u+1) - N_A(u) - N_A(1) = (u+1)(\overline{u+1}) - u\overline{u} - 1 \tag{2.54}$$

$$= u\overline{u} + u + \overline{u} + 1 - u\overline{u} - 1 = u + \overline{u} \tag{2.55}$$

$$\overline{u} = -u. \tag{2.56}$$

Thus $N_A(u) = u(-u) = -u^2$.

Using projection we can decompose $a = b + u$, where $b \in k1_A$ and $\langle u, 1 \rangle_A = 0$. Here we use the fact that the identity in A commutes with any element of A.

$$\overline{a} = \overline{b+u} = b - u = 2b - b - u = 2\langle a, 1 \rangle_A - a. \tag{2.57}$$

$$N_A(\overline{a}) = \overline{a}a = 2\langle a, 1 \rangle_A a - a^2 = 2a\langle a, 1 \rangle_A - a^2 = a\overline{a} = N_A(a). \tag{2.58}$$

□

This shows that conjugation negates the space orthogonal to the unit and that the left and right inverses are equal.

Proposition 2.5.4 (Prather)**.** *If a and b are orthogonal to the unit and each other, then $ab = -ba$.*

Proof.

$$\langle a, b \rangle = 0 = 2(N_A(a+b) - N_A(a) - N_A(b)) = (a+b)(\overline{a+b}) - a\overline{a} - b\overline{b} \tag{2.59}$$

$$= a\overline{a} + a\overline{b} + b\overline{a} + b\overline{b} - a\overline{a} - b\overline{b} = -ba - ab. \tag{2.60}$$

$$ab = -ba. \tag{2.61}$$

Note that we used orthogonality with the unit to assert that $\overline{a} = -a$ and $\overline{b} = -b$. □

Since the unit is always in the center, we can't hope to extend this anti-symmetry further. This property is called **anti-commutative**.

Theorem 2.5.5 (Prather)**.** *If X is an inner product space over a pre-k-algebra, A, then A must be a diagonal strong involution pre-algebra.*

Proof. The conjugation from the $*$-pre-algebra structure provides the needed involution, which is diagonal by inspection. From 2.5.3 we have $a + \overline{a} = 2\langle a, 1\rangle$, which is in $k1_A$ by construction. Further $a\overline{a} = N(a) \in k1_A$. But these are precisely the requirements of a diagonal strong involution pre-algebra. ☐

If the inner product is of the typical form A must be a division algebra.

Theorem 2.5.6 (Prather)**.** *If a pre-k-algebra A is an inner product space over itself, then A is a division pre-k-algebra.*

Proof. From the alternativity of an inner product we have $\langle ab, ab\rangle = N_A(a)\langle b, b\rangle$. Letting $b = 1$ and $\lambda = \langle 1, 1\rangle$ we have $\langle a, a\rangle = N_A(a)\lambda$. Thus $N_A(ab)\lambda = N_A(a)N_A(b)\lambda$.

By positive definiteness $\lambda \neq 0$ and A must be a composition pre-k-algebra. But a positive definite composition pre-k-algebras is a division pre-k-algebra. ☐

If A is a split composition pre-algebra, then $\langle p, q\rangle = \sum_{i=1}^{n} p_i\overline{q_i}$ still behaves nicely, but fails to be an inner product due to the lack of positive definiteness. This is usually what one has in mind if they mention an inner product space over these pre-algebras, though pseudo-inner product space would be more accurate.

2.6 Non-Associative Hilbert Spaces

We now generalize Hilbert spaces and verify that many standard results still hold in this setting.

Recall that an inner product space X requires R to be an ordered ring, which has the open interval topology.

If R is possibly non-Archimedian we need to choose from the several competing notions of completeness. In particular, the convergence of Cauchy sequences does not equal the upper bound property in the non-Archimedian setting. Restricting to $\mathbb{R}$ allows us to use the standard definition of completeness.

Definition 2.6.1. *A* **Hilbert space** *is an inner product space over a pre-$\mathbb{R}$-algebra that is complete.*

Most of the results in this section are standard in the associative setting, using mostly topology with little to no mention of the structure it is over. This trivializes their generalization to the non-associative case.

Consider a topologically closed subspace of a Hilbert space. This subspace has an inner product from the Hilbert space. Further, this space contains all limits, since it is topologically closed. Thus any topologically closed subspace of a Hilbert space is also a Hilbert space.

2.6.1 Properties

When discussing orthomodular spaces we defined a subspace such that $S = S^{\perp\perp}$ as closed. We now have a definition of a closed subspace arising from the topology of the space. It would be nice to show these coincide.

These results are well known in the classical setting, as demonstrated by the lecture material of DuChateau [17]. The proofs are largely topological, so the algebra does not play much of a role.

Note that any $u \in S \cap S^{\perp}$ implies $\langle u, u\rangle = 0$, hence $u = 0$. Thus $S \cap S^{\perp} = \{0\}$.

Theorem 2.6.2. *If X is a Hilbert space and S is a topologically closed subspace of X, then for any $x \in X$ there is a unique s in S minimizing $\|x - s\|^2$ [17].*

Definition 2.6.3. *The* **projection** *of x onto S, $\mathrm{proj}_S(x)$, is this unique s.*

Proof. This proof is paraphrased from Franz Luef [41].

Observe that $\|x - s\|^2$ is real and bounded below by 0. Since $\mathbb{R}$ is complete the infimum, d^2, exists. But then there are $s \in S$ such that $\|x - s\|^2$ is arbitrarily close to d^2. Since S is topologically closed, there must be a value in S with $\|x - s\|^2 = d^2$.

We will make use of the following identity.

$$\|a \pm b\|^2 = \langle a \pm b, a \pm b\rangle = \langle a, a\rangle \pm \langle a, b\rangle \pm \langle b, a\rangle + \langle b, b\rangle \tag{2.62}$$

$$= \|a\|^2 + \|b\|^2 \pm \langle a, b\rangle \pm \langle b, a\rangle \tag{2.63}$$

Suppose there are two values s_1 and s_2 such that $\|x - s_i\|^2 = d^2$. Note that $t = \frac{1}{2}(s_1 + s_2)$ is in S, so $\|t - x\|^2 \geq d^2$.

$$\|s_1 - s_2\|^2 = \|s_1 - x\|^2 + \|s_2 - x\|^2 - \langle s_1 - x, s_2 - x\rangle - \langle s_2 - x, s_1 - x\rangle \tag{2.64}$$

$$= 2\|s_1 - x\|^2 + 2\|s_2 - x\|^2 - \|s_1 + s_2 - 2x\|^2 \tag{2.65}$$

$$= 2\|s_1 - x\|^2 + 2\|s_2 - x\|^2 - 4\|t - x\|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0, \tag{2.66}$$

However, $\|s_1 - s_2\|^2 \geq 0$, so $\|s_1 - s_2\|^2 = 0$. Thus $s_1 - s_2 = 0$ and $s_1 = s_2$.

This shows $\mathrm{proj}_S(x)$ is unique. $\square$

For any vector s define $\mathrm{proj}_s(x)$ to be $\mathrm{proj}_S(x)$, where S is the smallest topologically closed subspace containing s. Consider $z = \frac{\langle x,y\rangle}{\langle x,x\rangle}x$. In general $\langle x, y\rangle$ is an element of A. Theorem 2.6.11 tells us that in general z is not in $x^{\perp\perp}$, however we still have $z \in \mathrm{Span}_{\overline{\varphi_A}}(S)$.

Proposition 2.6.4. *If S is topologically closed in X then $X = S \oplus S^\perp$ [17].*

Proof. Let x be any element of X, s an arbitrary element of S and $u = \mathrm{proj}_S\, x$.

Let $w = u - \mathrm{proj}_s(x - u)$. Note that $w \in S$. Now $x - w = x - u + \mathrm{proj}_s(x - u)$. By the triangle inequality we have $\|x - w\|^2 \leq \|x - u\|^2 + \|\mathrm{proj}_s(x - u)\|^2$. But u minimizes $\|x - t\|^2$ for $t \in S$ by definition, and $\|\mathrm{proj}_s(x - u)\|^2 \geq 0$. Thus $w = u$ and $\|\mathrm{proj}_s(x - u)\|^2 = 0$. Since s was arbitrary, this is true for any $s \in S$ and $x - u \in S^\perp$.

But now $x = u + (x - u) \in S \oplus S^\perp$. Since x was arbitrary in X, $X \subset S \oplus S^\perp$. $S \oplus S^\perp \subset X$ trivially, so $X = S \oplus S^\perp$. $\square$

The above standard propositions allow us to prove the following result, which is often stated without proof.

Theorem 2.6.5. *A subspace S of a Hilbert space X is topologically closed iff $S = S^{\perp\perp}$ [17].*

Proof. If S is not a topologically closed subspace of X then there there is a limit point in X not in S. Further, there is a sequence of s_i approaching the limit point. The inner product between any point in $S^\perp$ and any s_i is 0 by definition. But the inner product is continuous, so it must be 0 at the limit point. Thus the limit point must be in $S^{\perp\perp}$, and $S \neq S^{\perp\perp}$.

Conversely, suppose S is topologically closed. Then $x \in S^{\perp\perp} = s + t$, where $s \in S$ and $t \in S^\perp$. Now $0 = \langle x, t\rangle = \langle s, t\rangle + \langle t, t\rangle = \|t\|^2$. Thus $t = 0$ and $x \in S$. $\square$

This justified the common use of the name closed for subspaces of a Hermitian spaces satisfying $S = S^{\perp\perp}$, regardless of whether there is any other topology on the space.

2.6.2 Infinite Dimensional Octonionic Hilbert Spaces

The l^2 sequence space seems a natural place to start looking for an infinite dimensional octonionic Hilbert space. Ludkowski has studied these spaces, though the definition of octonionic module was implied[40]. The l^2 **sequence space** is the set of sequences where the sum of the norms converges absolutely. Since the norm is positive definite, this can be expressed as

$$\sum_{n\in\mathbb{N}} N(x_n) < \infty. \tag{2.67}$$

Theorem 2.6.6 (Ludkowski)**.** *The octonionic l^2 sequence space is an infinite dimensional Hilbert space.*

Proof. Let v_i be the element of l^2 which is 1 in the ith spot and 0 elsewhere. Each v_i has unit norm. If $i \neq j$, $\langle v_i, v_j\rangle$ is 0. Thus this pre-module contains an infinite orthonormal set.

The proof of completeness follows the reasoning typical for handling complex measures.

To be complete every Cauchy sequence in l^2 must have a limit that is also in l^2. Consider the sum $\sum_{n\in\mathbb{N}} |x - x_n|^2$. In the underlying real vector space we must have absolute convergence and order does not matter. Thus this is $\sum_i \sum_{n\in\mathbb{N}} |(x-x_n)_i|^2$. But in $\mathbb{R}$ l^2 is complete, so we can find an x_i and N_i such that $\sum_{n\in\mathbb{N}} |x_i-(x_n)_i|^2$ can be made less than $\epsilon/8$ for all $n > N_i$. Let $N = \max(N_i)$. Now $\sum_i \sum_{n\in\mathbb{N}} |x - x_n|^2 < \epsilon$ for all $n > N$. Thus x_n converges to x in l^2, so l^2 is complete.

Since x_n was arbitrary, X is a Hilbert space. □

Extending this proof to the octonionic L^2 function space over a measurable manifold is straightforward in a standard way.

Theorem 2.6.7 (Ludkowski)**.** *The octonionic l^2 sequence space is orthomodular.*

Proof. Let M be a closed subspace of l^2. Let $x \in l^2$. Let $u = \mathrm{proj}_M(x)$. Thus $x - u \in M^\perp$, and $x \in M + M^\perp$. But x was arbitrary in l^2, so $l^2 \subset M + M^\perp$ for any closed subspace M. Since we always have $M + M^\perp \subset l^2$, $M + M^\perp = l^2$. Since M was arbitrary, l^2 is orthomodular. □

Corollary 2.6.8 (Ludkowski)**.** *The octonionic l^2 sequence space is an orthomodular infinite dimensional Hilbert space.*

A topological space, X, is called **separable** if it has a dense countable subset. In the classical setting this is equivalent to having a countable basis, showing $X \cong l^2$.

The following results are essentially modified standard results from quaternionic Hilbert spaces. Goldstine and Horwitz included separability as a postulate for Hilbert spaces [27]. Ludkowski shows that the l^2 space over $\mathbb{O}_{\mathbb{Q}}$ is dense in the octonionic l^2 space, with no mention of separability [40].

Proposition 2.6.9. *The octonions are separable [40].*

Proof. Recall that $\mathbb{Q}$ is countable dense in $\mathbb{R}$. Thus for any $\epsilon > 0$, and octonion $q = \sum_i q_i e_i$ we can find q_i' in the octonions over the rationals, $\mathbb{O}_{\mathbb{Q}}$, such that $|q_i - q_i'| < \frac{\sqrt{\epsilon}}{3}$. But then $N(q-q') < \frac{8}{9}\epsilon < \epsilon$, and $\mathbb{O}_{\mathbb{Q}}$ is dense in $\mathbb{O}$. As a finite Cartesian product of countable sets, being the Cartesian product of eight copies of $\mathbb{Q}$, $\mathbb{O}_{\mathbb{Q}}$ is countable.

Thus $\mathbb{O}_{\mathbb{Q}}$ is a countable dense subset of $\mathbb{O}$. $\square$

Proposition 2.6.10. *The octonionic l^2 space is separable [40].*

Proof. Let $x \in l^2$, with $x = (x_1, x_2, ...)$. For any $\epsilon > 0$ we can fine x_i' in $\mathbb{O}_{\mathbb{Q}}$ such that $|x_i - x_i'| < \sqrt{\frac{\epsilon}{2^i}}$. But now $N(x_i - x_i') < \sum_i \frac{1}{2^i}\epsilon = \epsilon$.

The set $\mathbb{O}_{\mathbb{Q}} \times \mathbb{N}$ is countable, as a Cartesian product of two countable sets, and the vectors x' are countable as the subset of $\mathbb{O}_{\mathbb{Q}} \times \mathbb{N}$ with convergent norm.

Thus restricting the octonionic l^2 sequence space to those with coefficients in $\mathbb{O}_{\mathbb{Q}}$ yields a countable dense subset. $\square$

The standard proof that every octonionic vector space has a $\overline{\varphi_{\mathbb{O}}}$-basis hits a snag due to the zero divisors in the enveloping algebra. Orthogonality allows us to overcome this issue.

Theorem 2.6.11 (Prather)**.** *Every octonionic Hilbert space, X, has an orthogonal basis.*

Proof. Let F be the family of mutually orthogonal subsets of X, pre-ordered by set inclusion. Let C be any chain in F. Let U be the union of the elements of C.

If U is not mutually orthogonal, then there are u_i nd u_j such that $\langle u_i, u_j \rangle \neq 0$. But u_i is appears in C_i and u_j appears in C_j. Since C is a chain, we can assume $C_i \subset C_j$. But then u_i and u_j are both in C_j and must be orthogonal, a contradiction. Thus U must be mutually orthogonal.

Since U is mutually orthogonal, $U \in F$, and maximal in C. Since C was arbitrary, Zorn's Lemma asserts that there is a maximal M in F. Let S be $M^{\perp\perp}$. Note that S is closed.

Suppose $S \neq X$. Clearly $S \subset X$, so we must have some $x \in X$ where $x \notin S$. Since S is closed, $X = S \oplus S^{\perp}$. Thus $x = s + t$ where $s \in S$ and $t \in S^{\perp}$. Since $x \notin S$ we have $t \neq 0$. But then t is orthogonal to every subset of S, in particular M. Thus $M \cup \{t\}$ is mutually orthogonal, and in F by definition. This contradicts M being maximal. Thus $S = X$.

Now M is a mutually orthogonal set that generates X, hence an orthogonal basis. □

If X is separable, this basis must be countable. Unfortunately this does not guarantee that $X \cong l^2$ because the subspaces can be any form from Theorem 2.4.10.

2.7 Solèr's Theorem

Theorem 2.7.1 (Solèr)**.** *If X is an orthomodular Hermitian space over a skew-field K having an infinite orthonormal sequence, then K is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and X is a Hilbert space [56].*

The octonionic l^2 space does not violate Solèr's Theorem because $\mathbb{O}$ is non-associative, and hence not a skew field. Further, the definition of Hermitian form used by Solèr would require the inner product to be compatible with octonionic scalars, and l^2 only has compatibility up to an associativity constraint.

Most of the results are from Solèr's paper, though several were known before her work [56].

Proposition 2.7.2. *If X is an orthomodular Hermitian space over a pre-k-algebra with an infinite orthonormal sequence, then the characteristic of k is 0. [56]*

Proof. Suppose the characteristic of R is some finite n. Let e_i for $1 \leq i \leq n$ be distinct elements of the infinite orthonormal sequence. Let $x = \sum_{i=1}^{n} e_i$. Now

$$\langle x, x \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle e_i, e_j \rangle = \sum_{i=1}^{n} \langle e_i, e_i \rangle = n = 0. \tag{2.68}$$

But now x is degenerate, and any orthomodular Hermitian space must be non-degenerate.

Thus the characteristic of k can't be finite, and must be 0. □

We now work wo show $k = \mathbb{R}$.

Proposition 2.7.3. *If X is an orthomodular Hermitian space over a pre-k-algebra with an infinite orthonormal sequence, then $\mathbb{R} \subset k$ [56].*

Proof. Since the characteristic of k is 0 we know that $\mathbb{Q} \subset k$. We need only produce an element x of X with rational coefficients such that $\langle x, x\rangle = a$ for any real number a.

We can express a in base 4 as $\sum_{i=1}^{n} a_i 4^{-\alpha_i}$, where α_i are integers such that $\alpha_i < \alpha_{i+1}$ and $a_i \in 0, 1, 2, 3$. Now let $x_{3i} = 2^{-\alpha_i}$ if $a_i \in 1, 3$ and 0 otherwise. Similarly define $x_{3i-1} = x_{3i-2} = 2^{-\alpha_i}$ if $a_i \in 2, 3$ and 0 otherwise. Let e_i be an orthonormal sequence and $x = \sum_{i=1}^{n} x_i e_i$. By construction, $\langle x, x\rangle$ yields the base 4 expansion of a, so a must be in k.

Since a was arbitrary, $\mathbb{R} \subset k$. □

Proposition 2.7.4. *If X is an orthomodular Hermitian space over a pre-k-algebra with an infinite orthonormal sequence, then k must be an ordered field [56].*

Proof. A standard Zorn's lemma proof shows that any field extension of $\mathbb{R}$ must be K or $K[i]$ for some ordered field K [56].

Suppose $i \in k$, and e_i is an orthonormal sequence in X. Let $x = e_0 + ie_1$. Now $\langle x, x\rangle = 1 - 1 = 0$. But this contradicts the fact that X is non-degenerate. Thus $i \notin k$.

But then k must be an ordered field. □

We begin with two useful propositions.

Proposition 2.7.5. *If X is an orthomodular Hermitian space over a pre-k-algebra with an infinite orthonormal sequence, there is no vector of the form $\sum_{i=1}^{\infty} e_i$ in X [56].*

Proof. Suppose $x = \sum_{i=1}^{\infty} e_i$. Then $x = y + z$ where $y = \sum_{i=1}^{\infty} e_{2i-1}$ and $z = \sum_{i=1}^{\infty} e_{2i}$.

If $x \in X$, then $\langle x, x\rangle \in k$. By construction, $\langle x, x\rangle = \langle y, y\rangle = \langle z, z\rangle = a$. However,

$$a = \langle x, x\rangle = \langle y + z, y + z\rangle = \langle y, y\rangle + \langle y, z\rangle + \langle z, y\rangle + \langle z, z\rangle \tag{2.69}$$

$$= a + 0 + 0 + a = 2a. \tag{2.70}$$

Since k is a field, this requires $a = 0$, which would violate X being non-degenerate.

Thus $x \notin X$. □

Here we encounter the ingenuity of Solèr, which I will only outline.

Proposition 2.7.6 (Solèr)**.** *If X is an orthomodular Hermitian space over a pre-k-algebra with an infinite orthonormal sequence, e_i, and $a \in k - 0, \pm 1$ then precisely one of the following are in X:*

$$u = \sum_{i=0}^{n} a^i e_i \qquad \textit{with } \langle u, u \rangle = \frac{1}{1-a^2} \textit{ or} \tag{2.71}$$

$$v = \sum_{i=0}^{n} a^{-i} e_i \qquad \textit{with } \langle v, v \rangle = \frac{1}{1-a^{-2}}. \tag{2.72}$$

Proof. The full proof in [56] is very intricate, but the idea is straightforward.

Let $\langle u, u \rangle = (1-a^2) + b$, for some $b \in k$. Either $b = 0$ and we are done, or we can use u and b to construct a vector in X of the form excluded by Proposition 2.7.5. But then we can demonstrate that v must exist.

Further, by an argument similar to Proposition 2.7.5 we can show that $\langle u, v \rangle \notin k$, so at most one can be in X.

The actual proof uses a more involved perturbation of $\langle u, u \rangle$ to help streamline the production of the excluded vector. □

Proposition 2.7.7 (Solèr)**.** *If X is an orthomodular Hermitian space over a pre-k-algebra with an infinite orthonormal sequence, then $k = \mathbb{R}$.*

Proof. Any proper ordered field extension of $\mathbb{R}$ must contain a positive infinitesimal value δ, such that $0 < n\delta < 1$ for all $n \in \mathbb{N}$.

Now for any $a = n\delta$ the vector v above leads to a contradiction, so the vector u must exist. But using this family of vectors we can construct a vector in X excluded by Proposition 2.7.5.

Thus k must not contain any infinitesimal elements. Since $\mathbb{R} \in k$, this requires $k = \mathbb{R}$. □

The non-associativity only comes into play at this point, though the outline remains the same.

Proposition 2.7.8 (Prather)**.** *If X is an orthomodular Hermitian space over a division pre-algebra, A, with an infinite orthonormal sequence, then A is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. From the previous result, A must be an algebra over $\mathbb{R}$. Theorem 1.2.1 can be modified to the only four division $*$-algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. □

Theorem 2.7.9 (Prather)**.** *If X is an orthomodular Hermitian space over a division pre-algebra having an infinite orthonormal sequence, e_i, then X is a Hilbert space over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. An orthomodular Hermitian space over a pre-$\mathbb{R}$-algebra must be an inner product space. All that is left to show is that X is topologically closed.

Let x_i be a Cauchy sequence in X, and $F = \{x_i\}^{\perp\perp}$. Solèr showed that for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ strong orthogonality implies that F must be the respective l^2 space. This is not true for $\mathbb{O}$.

However, F is a real vector space containing x_i with a countable basis. Cauchy in X implies Cauchy in F as an octonionic subspace. Further, Cauchy in F as an octonionic inner product space is equivalent to Cauchy in F as a real vector space. Since F has a countable basis, F is a real l^2 space, and x_i converges in F to x.

Since $F \subset X$ by construction, $x \in X$. Since the x_i was an arbitrary Cauchy sequence, X is topologically closed. Hence X is a Hilbert space. □

2.8 Alternative Pre-Modules

Huo, Li and Ren discuss several equivalent statements of the alternative property that have been used in literature before classifying octonionic pre-modules [32]. Their proof extends directly to alternative pre-modules over diagonal strong involution pre-algebras.

Theorem 2.8.1 (Prather)**.** *An alternative left pre-module over a diagonal strong involution pre-algebra with dimension n, as a k-vector space, is a* $\mathrm{Cl}(n-1)$ *module.*

Proof. Recall that a diagonal strong involution pre-algebra has a norm $a\overline{a}$ that induces an inner product. Let e_i be an orthogonal basis relative to this inner product with e_0 the identity. Further, let i and j be distinct non-zero indices. As a notation, let $L_i = \varphi(e_i)$, and $L_{ij} = L_i L_j = L_i \circ L_j$. Extend this in the obvious way to L_α, where α is a string of indices. By the linearly of A and X, $\overline{\varphi_A}$ is generated as a ring by L_i. Clearly, L_0 is the identity, so $\overline{\varphi_A}$ is generated by the L_i, $i \neq 0$.

From the alternative identity, $[e_i, e_i, x] = -[e_i, e_i, x] = 0$. Thus $L_i \circ L_i(x) = (e_i e_i)x = -x$ for all i. From the anti-commutativity of a diagonal strong involution pre-algebra, $e_i e_j + e_j e_i = 0$. Now,

$$[e_i, e_j, x] = -[e_j, e_i, x], \tag{2.73}$$

$$(e_i e_j)x - e_i(e_j x) = -(e_j e_i)x + e_j(e_i x), \tag{2.74}$$

$$(L_{ij} + L_{ji})(x) = (e_i e_j + e_j e_i)x = 0. \tag{2.75}$$

Thus $L_i L_j = -L_j L_i$. But these are precisely the relations for $\mathrm{Cl}(n-1)$. □

Corollary 2.8.2 (Huo, Li and Ren)**.** *An alternative left pre-module over* $\mathbb{O}$ *is a* Cl(7) *module.*

For the octonions this is $\mathrm{Cl}(7) \cong M(8) \oplus M(8)$, where $M(8)$ are the 8×8 real matrices. Huo, Li and Ren demonstrate that indeed there are two cases, depending on whether $L_{1234567} = \pm 1$. They observe that, given a particular multiplication, this corresponds to left multiplication by q or $q^\dagger$ respectively. Alternatively, this is whether an eyes-right (-1) or eyes-left $(+1)$ convention is used. Either way, the octonions act on themselves as $M(8) \cong \mathrm{Cl}(6)$.

2.8.1 Octonionic Enveloping Algebra as Cl(6)

Cohl Furey described how compositions of left multiplication generate Cl(6) [22]. The $L_i = \varphi(e_i)$ are sparse 8×8 matrices. We can extend this to L_α, where α is a string of indices, using $L_{\alpha\beta} = L_\alpha \circ L_\beta$ recursively.

Using the isomorphism between the split-quaternions and 2×2 matrices,

$$1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad I = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \qquad J = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \qquad K = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \tag{2.76}$$

we can represent these using 4×4 matrices of split-quaternions.

We could have defined R_a using right multiplication instead of left multiplication. Examining the element of $M_{\mathbb{O}}$ that is 0, except for $a_{01} = 1$ helps to visualize how all of $\mathrm{End}_{\mathbb{R}}(\mathbb{O})$ can be generated. The eyes-right convention then yields the following matrices:

$$L_1 = \begin{bmatrix}I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & -I\end{bmatrix} \qquad L_2 = \begin{bmatrix}0 & J & 0 & 0\\ -J & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix} \qquad L_3 = \begin{bmatrix}0 & -K & 0 & 0\\ K & 0 & 0 & 0\\ 0 & 0 & 0 & I\\ 0 & 0 & I & 0\end{bmatrix} \tag{2.77}$$

$$L_4 = \begin{bmatrix}0 & 0 & -J & 0\\ 0 & 0 & 0 & 1\\ J & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix} \qquad L_5 = \begin{bmatrix}0 & 0 & -K & 0\\ 0 & 0 & 0 & -I\\ K & 0 & 0 & 0\\ 0 & -I & 0 & 0\end{bmatrix} \qquad L_6 = \begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & -J & 0\\ 0 & J & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix} \tag{2.78}$$

$$L_7 = \begin{bmatrix}0 & 0 & 0 & I\\ 0 & 0 & -K & 0\\ 0 & K & 0 & 0\\ I & 0 & 0 & 0\end{bmatrix} \qquad R_1 = \begin{bmatrix}I & 0 & 0 & 0\\ 0 & -I & 0 & 0\\ 0 & 0 & -I & 0\\ 0 & 0 & 0 & I\end{bmatrix} \qquad a_{01} = \begin{bmatrix}\frac{1}{2}(I+K) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{2.79}$$

The following results are now straightforward:

$$L_{1234567} = -1, \tag{2.80}$$

$$R_1 = \frac{1}{2}\left(L_{23} + L_{45} + L_{76} - L_1\right) = \frac{1}{2}\left(L_{23} + L_{45} - L_1 - L_{12345}\right), \tag{2.81}$$

$$a_{01} = \frac{1}{8}\left(2L_1 + 2L_{23} + 2L_{45} + 2L_{76} + 3L_{247} + L_{256} + L_{346} + L_{357}\right). \tag{2.82}$$

Using the relation $L_{1234567} = -1$, choosing the eyes-right convention, we can reduce any extended basis to one with at most three distinct indices This gives us $\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} = 64$ distinct extended basis, so these are in $1-1$ correspondence with the basis elements of Cl(6).

An example of how this arithmetic works is instructive.

$$L_{12}L_{34} = -L_{1234567}L_{1234} = -L_{12345671234} = (-1)^{19}L_{11223344567} = (-1)^5 L_{567} = -L_{567}. \tag{2.83}$$

To make the structure resemble Cl(6) we must select an index, usually 7, to represent the 6-volume form. Let a, b and c be distinct indices other than 0 or 7. The 1-forms are the six L_a, 2-forms the 15 L_{ab}, 3-forms the 20 L_{abc}, 4-forms the 15 L_{ab7}, 5-forms the 6 L_{a7}. This breaks the symmetry of the octonions, but in a manner reminiscent of how E_8 lattices are found to yield maximal orders (indeed, having a type of unique factorization) within the octonions [12].

CHAPTER 3

SPLIT SIGNATURE HOPF FIBRATIONS

This chapter is largely adopted from a paper of the same name by the author and Nolder [47].

3.1 Introduction

The construction of Hopf fibrations using the four real division pre-algebras is well known. Here we extend the construct of Dray and Manogue to the split composition pre-algebras [15]. This results in Hopf fibrations between hyperboloids rather than spheres. These hyperboloids are homeomorphic to direct products of spheres and Euclidean spaces. This results in the following fibrations:

$$S^0 \times \mathbb{R} \to S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}, \tag{3.1}$$

$$S^1 \times \mathbb{R}^2 \to S^3 \times \mathbb{R}^4 \to S^2 \times \mathbb{R}^2, \tag{3.2}$$

$$S^3 \times \mathbb{R}^4 \to S^7 \times \mathbb{R}^8 \to S^4 \times \mathbb{R}^4. \tag{3.3}$$

These maps have been previously described by Hasebe [29]. His construction uses separate geometric constructions for each, rather than the unified algebraic construction provided here.

3.2 Preliminaries

Let F, E and B be topological spaces and $\pi : E \to B$. In a **fiber bundle** E is locally homeomorphic to $F \times B$, with π being the projection map. We denote spaces having this relationship using $F \to E \to B$.

The topology of B is needed to define locality. Then we can say that for any $b \in B$ there is some neighborhood U containing b such that there is a homeomorphism between $\pi^{-1}(U)$ and $F \times U$.

The fiber bundle $E = F \times B$ with the projection map $\pi : E \to B$ yields a trivial fiber bundle. The Möbius strip is a fiber bundle with F being intervals and B a circle. For any $b \in B$ is contained in an open semi-circle U centered on b, and the preimage of U is homeomorphic to $U \times F$. Changing F to a circle yields a similar argument for the Klein bottle as a fiber bundle.

Hopf found that S^3 allows a fiber bundle with fibers S^1 and base S^2. An alternative construction of this map will be expanded on below.

3.2.1 Hyperboloids

Define a signature $(n+1, m)$ **regular hyperboloid**, $H^{n,m}$ as the $(n+m)$-manifold embedded in $(n+m+1)$-dimensional Euclidean space as $x_0^2 + x_1^2 + ... + x_n^2 - x_{n+1}^2 - ... - x_{n+m}^2 = 1$. Note that indices of the hyperboloid sum to the dimension of the manifold, while the signature sums to the space it is embedded in.

This yields three 2D hyperboloids, the sphere $H^{2,0} \cong S^2$, a hyperboloid of one sheet $H^{1,1} \cong S_1 \times \mathbb{R}$ and a hyperboloid of two sheets $H^{0,2} \cong S^0 \times \mathbb{R}^2$.

In general, $H^{n,m} \cong S^n \times R^m$. An explicit isotopy within the embedded space above can be found by observing that the radius of the positive coordinates must be non-zero. The isotopy is then the identity on the negative coordinates, and scaling the positive definite subspace radially onto the unit sphere.

$$\rho^2 = x_0^2 + ... + x_n^2 = 1 + x_{n+1}^2 + ... + x_{n+m}^2 > 0, \tag{3.4}$$

$$\alpha = 1 + \left(\frac{1}{\rho} - 1\right) t, \tag{3.5}$$

$$(x_0, ..., x_n, x_{n+1}, ...x_{n+m}) \to (\alpha x_0, ..., \alpha x_n, x_{n+1}, ..., x_{n+m}). \tag{3.6}$$

3.3 Hopf Fibrations

This section closely follows the construction of the Hopf fibrations in section 12.1 of Dray and Manogue [15].

Theorem 3.3.1 (Nolder and Prather)**.** *The construction of the Hopf fibrations from the division pre-algebras generalizes to the composition pre-algebras.*

Proof. Let A be a composition pre-algebra over $\mathbb{R}$ with signature (n, m). From Corollary 1.2.4, we know this is either $(2^n, 0)$ for $0 \le n \le 3$ or $(2^n, 2^n)$ for $0 \le n \le 2$.

Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$ be a two dimensional Hermitian space over A and $v^\dagger = (\overline{a}, \overline{b})$ be its Hermitian transpose. Now we can multiply v and $v^\dagger$ in two ways.

$$v^\dagger v = N(a) + N(b), \text{ and} \tag{3.7}$$

$$vv^\dagger = \begin{pmatrix} N(a) & a\overline{b} \\ b\overline{a} & N(b) \end{pmatrix}. \tag{3.8}$$

To avoid issues coming from non-positive norms, we will restrict our attention to v where $v^\dagger v = 1$. This subset of v lies on $H^{2n-1,2m}$.

Let's look at $\det(vv^\dagger)$ in two ways. First, we show it must be 0. Then we identify it with a hyperboloid by letting $t = \frac{1}{2}(N(a) + N(b)) = \frac{1}{2}$, $x = \frac{1}{2}(N(a) - N(b))$ and $q = b\overline{a}$.

$$\det(vv^\dagger) = N(a)N(b) - (b\overline{a})(a\overline{b}) = N(a)N(\overline{b}) - N(a\overline{b}) = 0 \tag{3.9}$$

$$= \det \begin{pmatrix} t+x & \overline{q} \\ q & t-x \end{pmatrix} = t^2 - x^2 - N(q). \tag{3.10}$$

$$t^2 = x^2 + N(q) = \frac{1}{4}. \tag{3.11}$$

This last expression identifies $vv^\dagger$ as $H^{n,m}$. Thus $\pi : v \to vv^\dagger$ is a map $H^{2n-1,2m} \to H^{n,m}$. All that remains is to identify the equivalency classes of this map.

The octonionic pre-algebras require care here, due to the loss of associativity. First, we map v to a member of its equivalence class with a real coefficient. We also must show that the two u when $N(a) = N(b) = \frac{1}{2}$ are in the same class.

$$u = \begin{cases} v\frac{\overline{a}}{|a|} & N(a) \geq \frac{1}{2} \\ v\frac{\overline{b}}{|b|} & N(b) \geq \frac{1}{2} \end{cases}. \tag{3.12}$$

This map makes one of the two components of u be a real number.

Now let $u(A)$ be the elements of A such that $N(q) = 1$, and $\xi \in u(A)$. Consider $w = u\xi$. In particular, u and ξ are defined using only real numbers and two elements of A. The alternativity of the composition pre-algebras ensures this generates an associative sub-pre-algebra. Thus expressions using only u and ξ do associate.

$$ww^\dagger = (u\xi)(u\xi)^\dagger = (u\xi)(\overline{\xi}u^\dagger) = u((\xi\overline{\xi})u^\dagger) = u(N(\xi)u^\dagger) = uu^\dagger. \tag{3.13}$$

Thus the w form an equivalence class of u with shape $H^{n-1,m}$. Since we selected $N(a)$ and $N(b)$ to be positive, we can recover v using either $\xi_a = \frac{a}{|a|}$ or $\xi_b = \frac{b}{|b|}$, depending on which map was used to create u. If $N(a) = N(b) = \frac{1}{2}$, both are invertible and $u_a = u_b\xi_b\xi_a^{-1}$.

All that remains is to show these maps are locally homeomorphic to a direct product. Consider U_a consisting of all v such that $N(a) > 0$, and similarly for U_b. Since $N(a)$ is continuous and the image, $(0, \infty)$, is open these are open. Further U_a and U_b cover B. But now we have a continuous map $\pi^{-1}((U_a) \to U \times u(A)$ given by (u, ξ_a). This map is a bijection, with multiplication yielding a continuous inverse. Thus this is a homeomorphism, and $\pi^{-1}(U_a)$ is homeomorphic to $U_a \times u(A)$. Likewise for U_b.

Thus $u(A) \to v \to vv^\dagger$ yields the fibrations listed in Table 3.1 $\square$

Table 3.1: Hopf fibrations for all composition pre-algebras over $\mathbb{R}$.

Pre-Algebra	Fibration	Alias
$\mathbb{R}$	$S^0 \to S^1 \to S^1$	
$\mathbb{C}$	$S^1 \to S^3 \to S^2$	
$\mathbb{C}^-$	$H^{0,1} \to H^{1,2} \to H^{1,1}$	$S^0 \times \mathbb{R} \to S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}$
$\mathbb{H}$	$S^3 \to S^7 \to S^4$	
$\mathbb{H}^-$	$H^{1,2} \to H^{3,4} \to H^{2,2}$	$S^1 \times \mathbb{R}^2 \to S^3 \times \mathbb{R}^4 \to S^2 \times \mathbb{R}^2$
$\mathbb{O}$	$S^7 \to S^{15} \to S^8$	
$\mathbb{O}^-$	$H^{3,4} \to H^{7,8} \to H^{4,4}$	$S^3 \times \mathbb{R}^4 \to S^7 \times \mathbb{R}^8 \to S^4 \times \mathbb{R}^4$

Johannes Wallner extended a similar construction to $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ [58]. The resulting spaces are also more intricate than the spheres and hyperboloids considered here.

3.3.1 Direct Product

The aliases of the split fibrations raise the possibility that they are direct products of the usual Hopf fibrations with the trivial fibrations of Euclidean spaces. Indeed, since $\mathbb{R}^n$ is contractible, they are homotopic to the usual Hopf fibrations.

The fiber bundle $S^1 \times \mathbb{R}^2$ can be modeled by the region between the planes $z = 0$ and $z = 4\pi$, identifying the two planes. The helicoid with $2n$ rotations between the planes is a double cover of the base $S^1 \times \mathbb{R}$, identifying points n rotation apart. The fibers would then be the projection of the normal at a given point to the plane through that point with constant z, identifying the parallel fibers through the points identified in the base to get $S^0 \times \mathbb{R}$.

By construction the base is a cylinder with n full twists. There is a clear homeomorphism between any two of these fibrations. However, the respective bases are not isotopic as embeddings of $S^1 \times \mathbb{R}^2$. Thus they are knotted.

Valentin Ovsienko and Serge Tabachnikov constructed skew affine fibrations using projections of the quaternionic and octonionic Hopf fibrations [48]. They define affine spaces to be skew if all pairs of lines, one from each space, are skew. They then classify all such affine fibrations, ruling out the cases here. We can weaken the definition of skew to non-parallel if any pair of lines are skew. The split signature fibrations may yield families of non-parallel affine fibrations parameterized by the usual Hopf fibrations.

While the split fibrations are likely homeomorphic to a direct product, it is not clear they are isotopic to the usual definition of a direct product as an embedding in the Euclidean spaces used to define them. Indeed, I conjecture that the fibrations are knotted in the sense above, with the fibration for $\mathbb{C}^-$ isotopic in $\mathbb{R}^4$ to the helicoid fibrations above with $n = 1$.

3.4 Higher Dimensions

This construction is related to the definitions of projective spaces, as shown in Table 3.2. Thus a review of what is known about such spaces seems fitting here.

Let A be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; $u(A)$ be the norm 1 elements of A, $S(A^n)$ the unit sphere in the Euclidean space A^{n+1} and $A\mathbb{P}^n$ the n-dimensional A projective space. The projective spaces are then defined by fibrations of the form $u(A) \to S(A^n) \to A\mathbb{P}^n$, constructed analogous to the Hopf fibrations above.

Table 3.2: Hopf fibrations for associative composition pre-algebras over $\mathbb{R}$ in higher dimensions.

Algebra	Fibrations
$\mathbb{R}$	$S^0 \to S^n \to \mathbb{RP}^n$
$\mathbb{C}$	$S^1 \to S^{2n+1} \to \mathbb{CP}^n$
$\mathbb{H}$	$S^3 \to S^{4n+3} \to \mathbb{HP}^n$

For the octonions there are no projective spaces for $n > 2$. Even for $n = 2$ the space is not all of $S(\mathbb{O}^2)$, but rather the subspace where the three components generate an associative subalgebra.

For the split pre-algebras the zero divisors mean we need to consider what a projective space should even be. The definition for $u(A)$ should be as above, and $S(A^n)$ generalize intuitively to hyperboloids, $H(A^n)$. Indeed, this covers the composition pre-algebras over $\mathbb{C}$.

For the positive definite pre-algebras, $A\mathbb{P}^n$ is defined as the quotient of $A^{n+1} - \{0\}$ with $A - \{0\}$. Indeed, using the composition properties of these algebras allows us to remove the radius in each to restrict to the unit spheres of our construction. For the split cases we remove not only 0 but also the zero divisors of A^{n+1} and A.

We can always multiply a vector by a split root, if needed, to get a vector with positive signature. Thus, while the two spaces with opposite signature are not connected, the quotient is. Thus, for all of the associative composition algebras over $\mathbb{R}$ and $\mathbb{C}$, we have $u(A) \to H(A^n) \to A\mathbb{P}^n$, and for $n = 1$ we get the octonionic pre-algebras.

CHAPTER 4

SPLIT-OCTONIONIC CAUCHY INTEGRAL FORMULA

This chapter explores analysis on the composition pre-algebras. The analogue of these results exist for the positive definite and associative pre-algebras. This work completes the list by including the split-octonions and has previously been published [50]. The techniques used here have been used by Libine and Sandine to develop analysis on Clifford algebras with indefinite signature [39]. Indeed, only the sections eliminating the associator present issues for the doubling pre-algebras of Chapter 5.

Kraußhar has developed analogous results for octonionic Bergman and Szegö kernels, Hardy spaces and Kerzman-Stein operators [35][34].

The definitions, notation and outline here are mostly generalized from Li and Peng's result on the octonions [36]. The modifications required for the split signature follow Libine's result on the split-quaternions [38].

4.1 Octonionic Functions

Let $\mathbb{O}^-$ and $\mathbb{O}_\mathbb{C}$ be the split and complex octonions, as defined in section 1.7. Identify $\mathbb{O}$ and $\mathbb{O}^-$ with $\mathbb{R}^8$ with the usual topology. Let Ω be an open connected set in $\mathbb{R}^8$, M a compact subset of Ω with smooth boundary ∂M. Define an **octonionic function** $f : \Omega \to \mathbb{O}$ as a sum of basis elements times real valued functions f_i:

$$f = \sum_i e_i f_i.$$

Further define f to be $C^n(\Omega, \mathbb{O})$ **smooth** iff each f_i is in $C^n(\Omega, \mathbb{R})$.

4.1.1 Topology

If f is continuous we get two useful theorems from basic topology.

Theorem 4.1.1. *The functions $\|f\|^2$ and $N(f)$ are bounded on M [46].*

Proof. Both $\|q\|^2$ and $N(q)$ are continuous and the composition of two continuous functions is continuous. But M is compact, so its image in $\mathbb{R}$ is compact. Thus the image of M is bounded. $\square$

Theorem 4.1.2. *Let q_0 be in Ω. For any $\epsilon > 0$ there is some $r > 0$ such that for any q in a ball of radius r, centered at q_0, $\|f(q) - f(q_0)\|^2 < \epsilon$ [46].*

Proof. Note $\|f(q) - f(q_0)\|^2$ is continuous, and the interval $(-\epsilon, \epsilon)$ is open. Thus its preimage, U, must be open. Now $\Omega \cap U$ must also be open. Thus there must be some δ-neighborhood of q_0 in $\Omega \cap U$. Set r to any such δ. $\square$

4.1.2 Regular Functions

For the complex numbers the classes of functions for which the difference quotient exists, that satisfy the Cauchy-Riemann equations, or have a convergent Taylor series in some neighborhood of any point are equivalent.

Over the quaternionic and octonionic pre-algebras these are distinct concepts. Indeed, the difference quotient only exists for linear functions [6]. Reserving the name analytic for the class of functions with locally convergent Taylor series, we follow Emanuello and Nolder in calling functions satisfying generalized Cauchy-Riemann equations regular [19].

The identification of $\mathbb{O}$ and $\mathbb{O}^-$ with $\mathbb{R}^8$ also induces a **Dirac operator** and its involutions:

$$\begin{aligned} D &= \sum_i e_i \frac{\partial}{\partial x_i}, \\ D^\dagger &= \sum_i e_i^\dagger \frac{\partial}{\partial x_i}, \\ \overline{D} &= \sum_i \overline{e_i} \frac{\partial}{\partial x_i}, \\ \overline{D}^\dagger &= \sum_i \overline{e_i}^\dagger \frac{\partial}{\partial x_i}. \end{aligned}$$

Note that $DD^\dagger = \sum_i \sum_j e_i e_j^\dagger \frac{\partial^2}{\partial x_i \partial x_j}$. Thanks to anti-commutativity canceling terms off the diagonal, this defines a **Laplace operator**. For $\mathbb{O}^-$ this becomes a wave equation:

$$\Delta = DD^\dagger = D^\dagger D = \overline{D}^\dagger \overline{D} = \overline{D}\,\overline{D}^\dagger = \sum_i N(e_i) \frac{\partial^2}{\partial x_i^2}.$$

While anti-commutativity is convenient when defining a Laplace operator, it forces us to consider which side of f we apply D. The following definitions apply equally to $\mathbb{O}^-$.

Definition 4.1.3. *A function $f \in C^1(\Omega, \mathbb{O})$ is* **left (right) $\mathbb{O}$-regular** *on U if:*

$$Df = \sum_i e_i \frac{\partial f}{\partial x_i} = 0 \qquad \left(fD = \sum_i \frac{\partial f}{\partial x_i} e_i = 0 \right).$$

Definition 4.1.4. *A function $f \in C^2(\Omega, \mathbb{O})$ is* **$\mathbb{O}$-harmonic** *on U iff $\Delta f = 0$.*

Using the alternative identity, $D^\dagger(Df) = (D^\dagger D)f = \Delta f$, so any left $\mathbb{O}$-regular function is harmonic. Further, since $\Delta f = D(D^\dagger f)$, if f is harmonic, then $D^\dagger f$ is left $\mathbb{O}$-regular. In this case $D^\dagger f$ is the **gradient** of f. Similarly for right $\mathbb{O}$-regular functions.

We can extend Li and Peng's proof of the following result for the octonions to include the split-octonions as well.

Theorem 4.1.5. $\Gamma = \frac{\alpha}{N(q)^3}$ *is harmonic for both $\mathbb{O}$ and $\mathbb{O}^-$, when $N(q) \neq 0$ [36].*

Proof.

$$\begin{aligned}
\Delta\Gamma &= \sum_i N(e_i) \frac{\partial^2}{\partial q_i^2} \frac{\alpha}{N(q)^3} = \sum_i \frac{\partial}{\partial q_i} \frac{-6\alpha q_i}{N(q)^4} = -6\alpha \sum_i \left(\frac{-8N(e_i) q_i^2}{N(q)^5} + \frac{1}{N(q)^4} \right) \\
&= 48\alpha \frac{N(q)}{N(q)^5} - \frac{48\alpha}{N(q)^4} = 0.
\end{aligned}$$

□

4.2 Exterior Algebras and Derivatives

We now shift our focus to $\mathbb{O}_{\mathbb{C}}$ which is identified with $\mathbb{C}^8$. The remaining results then apply to $\mathbb{O}$ and $\mathbb{O}^-$ by taking an appropriate subalgebra of $\mathbb{O}_{\mathbb{C}}$.

Using the identification of $\mathbb{O}_{\mathbb{C}}$ with $\mathbb{C}^8$ we can establish an exterior algebra $\Lambda(\mathbb{O}_{\mathbb{C}})$ generated by dz^i for $0 \leq i \leq 7$. Let multiplication in $\Lambda(\mathbb{O}_{\mathbb{C}})$ be denoted by $\wedge$.

We are mostly concerned with the Hodge star dual relationship between 1-forms and 7-forms, which we identify with elements of $\mathbb{O}_{\mathbb{C}}$. The volume 8-form also is needed, and we use the following notation:

$$\begin{aligned}
dq &= \sum_i e_i dq_i && \in \Lambda^1(\mathbb{O}_{\mathbb{C}}), \\
\star(dq) &= \sum_i (-1)^i e_i \bigwedge_{j \neq i} dq_j && \in \Lambda^7(\mathbb{O}_{\mathbb{C}}), \\
dV &= \bigwedge_i dq_i = dq \wedge (\star(dq)).
\end{aligned}$$

In particular, we chose to use the even indices for the quaternionic subalgebra so the sign changes of the Hodge dual would coincide with grade inversion for $\mathbb{O}^-$.

4.3 Octonionic Stokes' Theorem

We quote this standard result from differential geometry, before exploring its consequences on octonionic functions.

Theorem 4.3.1 (Generalized Stokes Theorem)**.** *Let Ω be an open connected set in $\mathbb{R}^n$, M a compact subset of Ω with smooth boundary ∂M. Let ω be a smooth $(n-1)$-form on Ω. Then:*

$$\int_{\partial M} \omega = \int_M d\omega.$$

If f and g are smooth real valued functions on Ω, then $\omega = (-1)^n f g \star dq_i$ is a smooth $(n-1)$-form on Ω. Further $d\omega = \left(\frac{df}{dq_i} g + f \frac{dg}{dq_i}\right) dV$, yielding:

$$\int_{\partial M} f g n_i dS = \int_M \frac{df}{dq_i} g + f \frac{dg}{dq_i} dV.$$

Let $\phi = \sum_s \phi_s e_s$ and $\psi = \sum_t \psi_t e_t$, where ϕ_s and ψ_t are real valued functions. Then for each s, t and i we have:

$$\int_{\partial M} \phi_s \psi_t n_i dS = \int_M \phi_s \frac{\partial \psi_t}{\partial q_i} + \frac{\partial \phi_s}{\partial q_i} \psi_t dV.$$

Multiplying by e_i, summing over i, noting that everything is real, and using the definition of D we have:

$$\int_{\partial M} \phi_s n \psi_t dS = \int_M \phi_s (D\psi_t) + (\phi_s D)\psi_t dV,$$

where $n = \sum_i n_i e_i$. Note that we put D in the middle to avoid introducing a commutator in the next step.

Multiplying by e_t on the right, summing over t and noting the octonions are alternative yields:

$$\int_{\partial M} \phi_s n \psi dS = \int_M \phi_s (D\psi) + (\phi_s D)\psi dV.$$

Multiplying by e_s on the right, recalling the definition of $[e_s, \phi_s D, \psi]$ and summing over s gives us:

$$\int_{\partial M} \phi(n\psi) dS = \int_M \phi(D\psi) + \sum_s e_s((\phi_s D)\psi) dV \tag{4.1}$$

$$= \int_M \phi(D\psi) + (\phi D)\psi - \sum_s [e_s, \phi_s D, \psi] dV \tag{4.2}$$

$$\int_{\partial M} (\phi n)\psi dS = \int_M \phi(D\psi) + (\phi D)\psi + \sum_t [\phi, D\psi_t, e_t] dV, \tag{4.3}$$

where the third line is found by combining e_s before e_t as above. Note that the roles of ϕ and ψ are interchangeable.

4.3.1 Vanishing Associator

We now look for a condition that assures the associator in (4.2) goes to 0.

Theorem 4.3.2 (Li and Peng)**.** *Let Γ be scalar valued and $\Phi = D^\dagger \Gamma = \sum_s \phi_s e_s$. Then whenever $\phi_s D$ is defined, $\sum_s [e_s, \phi_s D, \Psi] = 0$.*

Proof. Expand D to give us $\sum_s [e_s, \phi_s D, \Psi] = \sum_{s,i} [e_s, \frac{\partial \phi_s}{\partial x_i} e_i, \Psi]$. Observe that if s or i are 0 then the first or second argument is real and thus the associator is 0. Since octonionic algebras are alternative, if $s = i$ then the associator is 0. This leaves terms where $s \neq i$, neither of which are 0.

These can be grouped into pairs $\sum_{s,i} [e_s, \frac{\partial \phi_s}{\partial x_i} e_i, \Psi] + [e_i, \frac{\partial \phi_i}{\partial x_s} e_s, \Psi]$ where $s < i$. But now $\frac{\partial \phi_s}{\partial x_i} = -\frac{\partial^2 \Gamma}{\partial x_s \partial x_i} = \frac{\partial \phi_i}{\partial x_s}$ is scalar valued and can be factored out of the sum. Now $[e_s, e_i, \Psi] + [e_i, e_s, \Psi] = 0$ since octonionic pre-algebras are alternative. □

4.4 Cauchy Integral Formula

The general outline of the proof of the Cauchy integral formula is to find a $\Phi(q)$ defined almost everywhere that makes this outline work.

Let Ω and M be defined as above. Let $1_M(q)$ be 0 on the exterior of M and 1 on the interior of M. Let $q_0 \in \Omega$ and B_r be the interior of a solid ball centered at q_0 with radius r.

If $1_M(q_0) = 0$ then we can proceed without bothering to remove a ball about q_0 to get 0. Assume $1_M(q_0) = 1$. Since f is continuous we can find an r such that $\|f(q) - f(q_0)\| < \delta$ for any δ and $q \in B_r$ with $B_r \subset \mathrm{int}(M)$.

Now apply (4.2) to M/B_r, recall that f is left $\mathbb{O}^-$-regular, so $Df = 0$. If Γ is a scalar valued harmonic function, then $\Phi = \Gamma D^\dagger$ makes the associator term vanish and makes Φ right regular.

$$\begin{aligned}\int_{\partial M/B_r} \Phi(\star(dq)f) &= \int_{M/B_r} \left(\Phi(Df) + (\Phi D)f - \sum_s [e_s, \phi_s D, f]\right) dV = 0 \\ &= \int_{\partial M} \Phi(\star(dq)f) - \int_{\partial B_r} \Phi(\star(dq)f), \text{ so} \\ \int_{\partial M} \Phi(\star(dq)f) &= \lim_{r\to 0} \int_{\partial B_r} \Phi(\star(dq)f) = \lim_{r\to 0} \int_{\partial B_r} \Phi\left(\frac{q}{r} f(q_0)\right) dS.\end{aligned}$$

Thus we need only find a scalar valued Γ that makes the right hand side $f(q_0)$ for small r. By considering powers of r, $\Gamma = \frac{\alpha}{\|q-q_0\|^6}$ is a natural guess for $\mathbb{O}^-$. Here α is a constant that will need to be determined. Unfortunately this yields a limit of 0 for the split composition pre-algebras. For $\mathbb{O}$ this is equivalent to $\frac{\alpha}{N(q-q_0)^3}$. With care taken near the cone $N(q-q_0) = 0$ this works for $\mathbb{O}^-$ as well.

This gives us $\Phi = \frac{-6\alpha \overline{q}^\dagger}{N(q-q_0)^4}$. We may assume $q_0 = 0$ by translation.

4.4.1 Octonions, Euclidean Norm

Recall that $N(q) = \|q\|^2 = r^2$ on ∂B_r, $\overline{q}^\dagger = q^\dagger$ and $[q^\dagger, q, f] = 0$ for $\mathbb{O}$. Thus:

$$\begin{aligned}\lim_{r\to 0} \int_{\partial B_r} \Phi\left(\frac{q}{r} f(0)\right) dS &= -6\alpha \lim_{r\to 0} \int_{\partial B_r} \frac{q^\dagger}{\|q\|^8} \left(\frac{q}{r} f(0)\right) dS \\ &= -6\alpha \lim_{r\to 0} \int_{\partial B_r} \frac{N(q)}{r^9} dS f(0) \\ &= -6\alpha \lim_{r\to 0} \frac{1}{r^7} \int_{\partial B_r} dS f(0) = -6\alpha \frac{\pi^4}{3} f(0).\end{aligned}$$

Now $-6\alpha = \frac{3}{\pi^4}$ yields an octonionic Cauchy integral formula.

Theorem 4.4.1 (Octonionic Cauchy Integral Formula, Dentoni and Sce)**.** *Let Ω be an open connected region in $\mathbb{O}$. Let $M \subset \Omega$ be compact with smooth boundary ∂M. Let f be left $\mathbb{O}$-regular on Ω. Define $\Phi(q) = \frac{3}{\pi^4} \frac{q^\dagger}{N(q)^4}$. Then:*

$$\int_{\partial M} \Phi(q-q_0)(\star(dq)f(q)) = 1_M(q_0) f(q_0).$$

Dentoni and Sce expressed this using $\Phi = \frac{1}{768\pi^4}\Delta^3\frac{1}{q-q_0}$ [14], though they misplaced a factor of 16. Li and Peng found the form produced here [36]. This is comparable to Fueter's result for the quaternions [21].

Theorem 4.4.2 (Quaternionic Cauchy Integral Formula, Fueter)**.** *Let Ω be an open connected region in $\mathbb{H}$. Let $M \subset \Omega$ be compact with smooth boundary ∂M. Let f be left $\mathbb{H}$-regular on Ω. Define $\Phi(q) = \frac{1}{2\pi^2}\frac{q^\dagger}{N(q)^2}$. Then:*

$$\int_{\partial M} \Phi(q - q_0)(\star(dq)f(q)) = 1_M(q_0)f(q_0).$$

The classical result over $\mathbb{C}$ looks a bit different in this notation. Noting that for any curve γ, $\star(dz)f(z) = -if(z)dz$ yields the traditional form of this theorem.

Theorem 4.4.3 (Cauchy Integral Formula, Cauchy)**.** *Let Ω be an open connected region in $\mathbb{C}$. Let $M \subset \Omega$ be compact and simply connected with smooth boundary ∂M. Let f be $\mathbb{C}$-regular on Ω. Let γ be a degree 1 embedding of S_1 into ∂M. Define $\Phi(z) = \frac{1}{2\pi}\frac{z^\dagger}{N(z)}$. Then:*

$$\int_{\partial M} \Phi(z - z_0)(\star(dz)f(z)) = 1_M(z_0)f(z_0),$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}dz = 1_M(z_0)f(z_0).$$

4.4.2 Split-Octonions, Split Norm

Note that $\Phi(q)$ is not defined when $N(q) = 0$. In $\mathbb{O}^-$ this must happen in M/B_r, ∂M and ∂B_r. Thus the integrals in our outline are improper. We follow Libine and find functions in $\mathbb{O}_\mathbb{C}$ that converge point-wise to the desired function but whose integrals converge to a real value. Such limits have meaning as distributions.

Perturbing Φ. We amend the basic outline by defining a perturbed q_ε such that $N(q_\varepsilon)$ is a scalar value in $\mathbb{O}_\mathbb{C}$ and use this in our definitions of Γ_ε and Φ_ε. We define $\|q\|_\varepsilon^2$ for convenience.

$$
\begin{aligned}
q_\varepsilon &= \frac{1}{2}\left((1+\sqrt{1-\varepsilon^2})q + i(1-\sqrt{1-\varepsilon^2})\overline{q}\right),\\
q_\varepsilon^\dagger &= \frac{1}{2}\left((1+\sqrt{1-\varepsilon^2})q^\dagger + i(1-\sqrt{1-\varepsilon^2})\overline{q}^\dagger\right),\\
N_\varepsilon(q) &= N(q_\varepsilon) = N(q) + i\varepsilon\|q\|^2,\\
\|q\|_\varepsilon^2 &= \|q\|^2 + i\varepsilon N(q) \neq \frac{1}{2}\left(q_\varepsilon\overline{q_\varepsilon}^\dagger + \overline{q_\varepsilon}q_\varepsilon^\dagger\right) = \|q_\varepsilon\|^2,\\
\Gamma_\varepsilon &= \frac{\alpha}{N_\varepsilon(q)^3},\\
\Phi_\varepsilon &= D^\dagger\Gamma_\varepsilon = \frac{-6\alpha(\overline{q}^\dagger + i\varepsilon q^\dagger)}{(N(q)+i\varepsilon\|q\|^2)^4}.
\end{aligned}
$$

Since Γ_ε is scalar valued, $\Phi_\varepsilon D = (D^\dagger\Gamma_\varepsilon)D = (\Gamma_\varepsilon D^\dagger)D = \Delta\Gamma_\varepsilon$. Following the calculation of Theorem 4.1.5 we have $\Phi_\varepsilon D = -48i\varepsilon\alpha\frac{\|q\|_\varepsilon^2}{N_\varepsilon(q)^5} \neq 0$. Thus we need to reconsider the ΦD term in the integral of our outline.

Polar coordinates. We will also have use for polar coordinates. Let $0 \leq \rho$ be the radius, $0 \leq \theta \leq \frac{\pi}{2}$ the angle between $\mathfrak{P}(q)$ and $\mathfrak{S}(q)$, a copy of S_3 for the spherical angle of $\mathfrak{P}(q)$ and another for $\mathfrak{S}(q)$. Let dV be the volume form of $\mathbb{O}^-$ and dS the volume form of the unit sphere.

$$
\begin{aligned}
dV &= \frac{1}{8}\rho^7\sin^3(2\theta)d\rho dS_3^{\mathfrak{S}}dS_3^{\mathfrak{P}}d\theta,\\
dS &= \frac{1}{8}\sin^3(2\theta)dS_3^{\mathfrak{S}}dS_3^{\mathfrak{P}}d\theta,\\
\|q\| &= \rho, \qquad N(q) = \rho^2\cos(2\theta).
\end{aligned}
$$

The cone $N(q) = 0$ corresponds to $\theta = \frac{\pi}{4}$. It is useful to change the θ coordinate to $u = -\cos(2\theta)$, with $-1 \leq u \leq 1$ so the cone $N(q) = 0$ is at $u = 0$. Note this substitution is orientation preserving.

$$
\begin{aligned}
dV &= \frac{1}{16}\rho^7(1-u^2)d\rho dS_3^{\mathfrak{S}}dS_3^{\mathfrak{P}}du,\\
dS &= \frac{1}{16}(1-u^2)dS_3^{\mathfrak{S}}dS_3^{\mathfrak{P}}du,\\
\|q\| &= \rho, \qquad N(q) = -\rho^2 u.
\end{aligned}
$$

Evaluating $\int_{M/B_r}(\Phi_\varepsilon D)fdV$.

$$\begin{aligned}\int_{M/B_r}(\Phi_\varepsilon D)fdV &= -48i\varepsilon\alpha\int_{M/B_r}\frac{\|q\|_\varepsilon^2}{N_\varepsilon(q)^5}fdV\\ &= -48i\varepsilon\alpha\int_{M/B_r}\frac{\|q\|^2+i\varepsilon N(q)}{(N(q)+i\varepsilon\|q\|^2)^5}fdV\\ &= -48i\varepsilon\alpha\int_{M/B_r}\frac{\rho^2-i\varepsilon u\rho^2}{(-u\rho^2+i\varepsilon\rho^2)^5}f\frac{1-u^2}{16\rho}d\rho dS_3^{\mathfrak{S}}dS_3^{\mathfrak{P}}du\\ &= 3i\varepsilon\alpha\int_{M/B_r}\frac{1-i\varepsilon u}{(u+i\varepsilon)^5}\frac{1-u^2}{\rho^8}fd\rho dS_3^{\mathfrak{S}}dS_3^{\mathfrak{P}}du.\end{aligned}$$

Since f is smooth and $\rho \neq 0$ in M/B_r we can integrate over ρ, $dS_3^{\mathfrak{S}}$ and $dS_3^{\mathfrak{P}}$ to get $h(u)$. If $\partial M/B_r$ intersects the cone $u = 0$ transversely h will be as smooth as f on some interval $(-\delta, \delta)$. Now $g = (1 - i\varepsilon u - u^2 + i\varepsilon u^3)h$ is equally smooth on M/B_r, as the product of smooth functions.

$$\lim_{\varepsilon\to 0}\left\|\int_{M/B_r}(\Phi_\varepsilon D)f\right\| \leq 3|\alpha|\lim_{\varepsilon\to 0}\varepsilon\left\|\int_{-\delta}^{\delta}\frac{gdu}{(u+i\varepsilon)^5}\right\|.$$

But this integral is finite by the following standard result from the theory of distributions, so the limit is 0.

Theorem 4.4.4 (Convergence of Limits)**.** *Let $f \in C^n$. Then the following limit is finite [38].*

$$\lim_{\varepsilon\to 0}\int_{-\delta}^{\delta}\frac{fdx}{(x-i\varepsilon)^n}.$$

Proof. Integrating by parts letting $u = f$ and $dv = (u - i\varepsilon)^{-n}du$ yields a function that may be evaluated to a finite value and an integral of this form with the power reduced by 1. Since this function involves $\frac{df}{dx}$ the new f may only be C^{n-1}.

After n repetitions we get an integral such as the following, for some continuous g. Since ln is integrable we can apply the dominated convergence theorem.

$$\lim_{\varepsilon\to 0}\int_{-\delta}^{\delta}\ln(x-i\varepsilon)gdx = \int_{-\delta}^{\delta}\ln(x)gdx.$$

Since the domain is bounded this integral is finite.

Now a finite sum is finite, so the original expression must be finite. $\square$

Note that ∂B_r intersects the cone $u = 0$ transversely, so we need only consider ∂M. Also, we only need $f \in C^5$. Our outline has now reduced to evaluating the right hand side of

$$\int_{\partial M} \Phi_\varepsilon(\star(dq)f) = \lim_{r\to 0}\int_{\partial B_r} \Phi_\varepsilon \frac{q}{r} dS f(0).$$

Evaluating $\int_{\partial B_r} \Phi_\varepsilon(\frac{q}{r} f(0))dS$.

$$\begin{aligned}
\int_{\partial B_r} \Phi_\varepsilon\left(\frac{q}{r}f(0)\right)dS &= \int_{\partial B_r} \frac{-6\alpha(\overline{q}^\dagger + i\varepsilon q^\dagger)}{N_\varepsilon(q)^4}\left(\frac{q}{r}f(0)\right)dS \\
&= -\frac{6\alpha}{r}\int_{\partial B_r} \frac{\overline{q}^\dagger(qf(0)) + i\varepsilon q^\dagger(qf(0))}{N_\varepsilon(q)^4}dS \\
&= -\frac{6\alpha}{r}\int_{\partial B_r} \frac{(\overline{q}^\dagger q)f(0) - [\overline{q}^\dagger, q, f(0)] + i\varepsilon N(q)f(0)}{N_\varepsilon(q)^4}dS.
\end{aligned}$$

Recall that $\overline{q}$ is an orientation preserving isometry on a sphere centered at the origin.

$$\int_{\partial B_r} \Phi_\varepsilon\left(\frac{q}{r}f(0)\right)dS = -\frac{6\alpha}{r}\int_{\partial B_r} \frac{(q^\dagger \overline{q})f(0) - [q^\dagger, \overline{q}, f(0)] + i\varepsilon N(\overline{q})f(0)}{N_\varepsilon(\overline{q})^4}dS.$$

$$[\overline{q}^\dagger, q, f(0)] = [\overline{q}, q^\dagger, f(0)] = -[q^\dagger, \overline{q}, f(0)].$$

These two expressions for this integral can be added together, with the result divided by 2. This yields a scalar valued function equal to both. This can be evaluated using polar coordinates:

$$\begin{aligned}
\int_{\partial B_r} \Phi_\varepsilon\left(\frac{q}{r}f(0)\right)dS &= -\frac{6\alpha}{r}\int_{\partial B_r} \frac{\|q\|^2 f(0) + i\varepsilon N(q)f(0)}{(N(q) + i\varepsilon\|q\|^2)^4}dS \\
&= -\frac{6\alpha}{r}\int_{\partial B_r} \frac{r^2 - i\varepsilon ur^2}{(-ur^2 + i\varepsilon r^2)^4}\frac{1-u^2}{16}r^7 dS_3^{\mathfrak{S}} dS_3^{\mathfrak{P}} du f(0) \\
&= -\frac{3\alpha}{8}\left(\int_{S_3} dS_3\right)^2 \int_{-1}^{1} \frac{(1 - i\varepsilon u)(1-u^2)}{(u - i\varepsilon)^4}du f(0) \\
&= -\frac{3\alpha\pi^4}{2}\int_{-1}^{1} \frac{(1 - i\varepsilon u - u^2 + i\varepsilon u^3)}{(u - i\varepsilon)^4}du f(0).
\end{aligned}$$

Partial fraction decomposition and odd symmetry can be used here:

$$
\begin{aligned}
&= -\frac{3\alpha\pi^4}{2}\int_{-1}^{1}\frac{(1+\varepsilon^2)^2}{(u-i\varepsilon)^4}-\frac{3i(\varepsilon+\varepsilon^3)}{(u-i\varepsilon)^3}-\frac{1+3\varepsilon^2}{(u-i\varepsilon)^2}+\frac{i\varepsilon}{u-i\varepsilon}du f(0)\\
&= -\frac{3\alpha\pi^4}{2}\left(\frac{-(1+\varepsilon^2)^2}{3(u-i\varepsilon)^3}+\frac{3i(\varepsilon+\varepsilon^3)}{2(u-i\varepsilon)^2}+\frac{1+3\varepsilon^2}{(u-i\varepsilon)}\Big|_{-1}^{1}+\int_{-1}^{1}\frac{i\varepsilon u-\varepsilon^2}{u^2+\varepsilon^2}du\right)f(0)\\
&= -\frac{3\alpha\pi^4}{2}\left(\frac{4+3\varepsilon^2}{3+3\varepsilon^2}+0-\varepsilon\tan^{-1}\left(\frac{u}{\varepsilon}\right)\Big|_{-1}^{1}\right)f(0)\\
&= -\frac{3\alpha\pi^4}{2}\left(\frac{4+3\varepsilon^2}{3+3\varepsilon^2}-2\varepsilon\tan^{-1}\left(\frac{1}{\varepsilon}\right)\right)f(0).
\end{aligned}
$$

Now taking the limit as ϵ goes to 0 and setting $-6\alpha = \frac{3}{\pi^4}$ gives us the desired result.

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\partial B_r}\Phi_\varepsilon\left(\frac{q}{r}f(0)\right)dS &= -\frac{3\alpha\pi^4}{2}\lim_{\varepsilon\to 0}\left(\frac{4+3\varepsilon^2}{3+3\varepsilon^2}-2\varepsilon\tan^{-1}\left(\frac{1}{\varepsilon}\right)\right)f(0)\\
&= -6\alpha\frac{\pi^4}{3}f(0) = f(0).
\end{aligned}
$$

But this now completes the proof of a split-octonionic Cauchy integral formula.

Theorem 4.4.5 (Split-Octonionic Cauchy Integral Formula, Prather)**.** *Let Ω be an open connected region in $\mathbb{O}^-$. Let $M\subset\Omega$ be compact with smooth boundary ∂M that intersects $N(q)=0$ transversely. Let $f\in C^5(\Omega,\mathbb{O}^-)$ be left (right) $\mathbb{O}^-$-regular on Ω. Define $\Phi_\varepsilon(q) = \frac{3}{\pi^4}\frac{\bar{q}^\dagger + i\varepsilon q^\dagger}{(N(q)+i\varepsilon\|q\|^2)^4}$. Then:*

$$
\lim_{\varepsilon\to 0}\int_{\partial M}\Phi_\varepsilon(q-q_0)(\star(dq)f(q)) = 1_M(q_0)f(q_0),
$$
$$
\left(\lim_{\varepsilon\to 0}\int_{\partial M}(f(q)\star(dq))\Phi_\varepsilon(q-q_0) = 1_M(q_0)f(q_0)\right).
$$

Following this outline we can produce an integral formula for the split-quaternions. We can also simplify Φ_ε because we don't need to worry about associativity.

Theorem 4.4.6 (Split-Quaternionic Cauchy Integral Formula, Libine)**.** *Let Ω be an open connected region in $\mathbb{H}^-$. Let $M\subset\Omega$ be compact with smooth boundary ∂M that intersects $N(q)=0$ transversely. Let $f\in C^3(\Omega,\mathbb{O}^-)$ be left $\mathbb{H}^-$-regular on Ω. Define $\Phi_\varepsilon(q)=\frac{1}{2\pi^2}\frac{\bar{q}^\dagger}{(N(q)+i\varepsilon\|q\|^2)^2}$. Then:*

$$
\lim_{\varepsilon\to 0}\int_{\partial M}\Phi_\varepsilon(q-q_0)(\star(dq)f(q)) = 1_M(q_0)f(q_0).
$$

We note that Libine defines left regular to be $\overline{D}^{\dagger} f = 0$ [38]. This is motivated by the reversion involution being commonly viewed as more fundamental to Clifford algebras than conjugation. Thus if f is left regular in our sense, then $\overline{f}^{\dagger}$ will be left regular in Libine's sense, and vice versa. Since $\overline{q}^{\dagger}$ is an orientation reversing isometry this introduces a negative sign into his formula.

Over $\mathbb{C}^-$, $\overline{q}^{\dagger} = q^{\dagger}$ so the formula looks very much like the complex version. Again ∂M defines a degree 1 curve γ in $\mathbb{C}^-$. A more traditional looking form can be attained by noting $\star(dz)f(z) = jf(z)dz^{\dagger}$ where $j^2 = 1$.

Theorem 4.4.7 (Split-Complex Cauchy Integral Formula, Libine)**.** *Let Ω be an open connected region in $\mathbb{C}^-$. Let $M \subset \Omega$ be compact and simply connected with smooth boundary ∂M that intersects $N(q) = 0$ transversely. Let f be left $\mathbb{C}^-$-regular on Ω. Let γ be a degree 1 embedding of S_1 into ∂M. Define $\Phi_\varepsilon(z) = \frac{z^{\dagger}}{N(z)+i\varepsilon\|z\|^2}$. Then:*

$$\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\partial M} \Phi_\varepsilon(z - z_0)(\star(dz)f(z)) = 1_M(z_0)f(z_0),$$

$$\frac{1}{2\pi j} \lim_{\varepsilon \to 0} \int_{\gamma} \Phi_\varepsilon(z - z_0)f(z)dz^{\dagger} = 1_M(z_0)f(z_0).$$

Libine found a similar formula by factoring $N(q)$, perturbing the partial fraction decomposition and integrating along hyperboloids [37]. Emanuello has compiled an excellent review of our understanding of analysis on split signature composition algebras and their conformal geometries [18].

Over $\mathbb{R}$ the definition for regular would require f to be linear. While not a particularly interesting class of functions, the resulting theorem is trivial - the function must simply be the line defined by its two endpoints.

By Hurwitz' theorem there are just seven composition pre-algebras over $\mathbb{R}$. With the addition of the split-octonions, Cauchy integral formulas have now been found for all. This highlights just how special complex analysis is. In $\mathbb{C}$ differentiability, convergent Taylor series (analytic) and Cauchy-Riemann equations (regular) all yield the same class of functions. For the remaining pre-algebras these concepts are distinct.

For the real numbers the Cauchy-Riemann equations can be forced to work, resulting in only linear functions, while there are infinitely differentiable functions that are not analytic. For quaternionic and octonionic analysis this is reversed. Now only linear functions have a well defined difference quotient, and the Cauchy-Riemann equations generate a broad family of functions.

CHAPTER 5

DOUBLING PRE-ALGEBRAS

In Chapter 2 we found that for an pre-algebra to allow an inner product it must be a diagonal strong involution pre-algebra over an ordered field, in the sense of Albuquerque and Majid [1]. In this chapter we consider these pre-algebras further. Particular attention will be paid to various notions of sedenions.

5.1 Introduction

Dickson generalized Cayley's construction to unify the constructions of the complex numbers from the reals, the quaternions from the complex numbers and the octonions from the quaternions. This process can be repeated indefinitely, producing at the next step the sedenions [12][3]. These pre-algebras are all examples of the more general diagonal strong involution pre-algebras.

In 1989 Chesley identified a twisted version of the octonions [7]. In 2004 Cawagas demonstrated that this pseudo-octonion pre-algebra can be generated by basis elements of the sedenions [8]. Thus the pre-algebras generated by repeating the Cayley-Dickson construction are not closed under sub-pre-algebras, even when restricting to the basis elements.

In 2006 Chesley identified 52 distinct sedenion-like pre-algebras, and demonstrated that of the fifteen three generator sub-pre-algebras at most eight can be the octonions [11]. In 2009 Cawagas identified four non-isomorphic sedenion-like pre-algebras contained within the Cayley-Dickson 32-ions [9].

In Section 5.3 I identify classes of pre-algebras that are closed under both the Cayley-Dickson construction and sub-pre-algebras generated by closed subsets of basis elements.

With the octonions, every sub-pre-algebra with two generators is a copy of the quaternions. It is tempting to assume that every sub-pre-algebra of the sedenions with three generators must be the octonions. I show that no pre-algebra with four or more generators can be anti-associative for all triples, and thus must contain copies of the pseudo-octonions. We then examine the unique pre-algebra generated by any such triple.

5.2 Diagonal Strong Involution Pre-Algebras

We begin with a review of definitions from the theory of loops.

A **magma** is a set with a binary operation, called multiplication. A **quasi-group** is a magma where left and right multiplication by any element is an automorphism of the underlying set, allowing divisibility. A **loop** is a quasi-group with identity. An associative loop is a **group**.

$$
\begin{tikzcd}
 & \text{magma} \arrow[dl, "\text{div.}"'] \arrow[d, "\text{id}"] \arrow[dr, "\text{assoc.}"] & \\
\text{quasi-group} \arrow[d] \arrow[dr] & \text{unital magma} \arrow[dl] \arrow[dr] & \text{semi-group} \arrow[dl] \arrow[d] \\
\text{loop} \arrow[dr] & \text{inverse semi-group} \arrow[d] & \text{monoid} \arrow[dl] \\
 & \text{group} &
\end{tikzcd}
$$

Figure 5.1: Algebraic Structures. Parallel lines represent divisibility, identity element or associativity, as indicated in the first row.

A loop is **diassociative** if any subloop generated by any two elements is associative, i.e. a group. Diassociativity gives us two sided inverses. Also $a^{-1} \in \langle a \rangle$, so $(ab)a^{-1} = a(ba^{-1})$ and conjugation is well defined. This allows us to talk about cosets of subloops, normal subloops, the center of a loop and quotient loops in the usual way.

A **loop pre-algebra** kL over a loop L is the pre-k-algebra induced by bilinearity on finite sums of a field k with elements of L, where the elements of k and L commute.

For this chapter we assume $1 \neq -1$ in k, i.e. k is not trivial and has characteristic other than 2.

5.2.1 Twist Pre-Algebras

The definitions in this subsection are a generalization of the twisted group pre-algebras of Albuquerque and Majid[1].

Let k be a field, kL be the loop pre-algebra over L and $F : L \otimes L \to k^*$. Then a **twist pre-algebra**, $k_F L$, consists of kL with the product extended linearly from $a \cdot_F b = abF(a, b)$, where

$a, b \in L$. In particular, we are interested in $F = \pm 1_k$, and $F(a, 1_L) = F(1_L, a) = 1_k$ for all a. Now $k_F L$ is **diagonal** if every element of L has order 2. For finite groups this restricts us to $\mathbb{Z}_2^n$. Further, $k_F L$ is anti-commutative if L is commutative and for any distinct non-unit $a, b \in L$, $F(a, b) = -F(b, a)$. Note that this is weaker than requiring $xy = -yx$ for all $x, y \in k_F L$.

Given a diagonal anti-commutative $k_F L$, we can extend $\overline{1_L} = 1_L$ and $\overline{a} = -a$ linearly to all of $k_F L$. It is straightforward to show that $x + \overline{x} \in k$ and $x\overline{x} \in k$ for any $x \in k_F L$. Thus any diagonal anti-commutative twist pre-algebra can be made into a diagonal strong involution pre-algebra. Further, if $F(a, a) = -1_k$ for all non-identity elements in L, and k is an ordered field, then the expression $x\overline{x}$ is positive definite.

A **sub-twist pre-algebra** is the twist pre-algebra induced on $k_F M$ by $k_F L$ for some subloop M of L. Note that not all sub-pre-algebras of a twist pre-algebra must be a sub-twist pre-algebra. In particular, we must also consider sub-pre-algebras spanned by the zero-divisors.

5.2.2 Non-Twist Sub-pre-algebras

Let the multiplication of the octonions be Table 5.1. Let $a = (e_1, e_2)$, $b = (e_5, e_6)$ and $c = (-e_7, e_4)$ be sedenions. Then $ba = cb = ac = 0$ but $ab = 2c$, $bc = 2a$ and $ca = 2b$. All three square to -2. Thus the pre-algebra generated by a, b and c is closed and four dimensional. Cawagas excludes sub-pre-algebras such as this by examining only sub-twist pre-algebras. A more thorough, and idiosyncratic, study of sub-pre-algebras generated in this manner can be found in the works of DeMarrais [13].

5.2.3 Steiner Pre-Algebras

The definitions of Steiner systems come from the study of block designs.

If we have a set of k element subsets of an n element set such that each t element subset is a subset of exactly one set in our collection, we have a **Steiner system** $S(t, k, n)$. We will mostly be concerned with **Steiner triple systems** (STS), of the form $S(2, 3, n)$ [54].

One combinatorial constraint on STSs is that the number of triples, $n(n-1)/6$, must be an integer. Further, the number of triples containing a fixed element is $(n-1)/2$. Since both of these must be integers, this means n must be 1 or 3 $\pmod 6$. In general there are t such constraints, and these are not sufficient. However, for STSs this is sufficient [54].

Given an STS we can generate a loop by adding a unit, making each element square to the unit and the product of any two the unique element other element in triple the two occupy.

This can be twisted to a diagonal strong involution pre-algebra by assigning a sign convention to each triple, such that each triple corresponds to a sub-twist pre-algebra isomorphic to the quaternions. Such pre-algebras exist for $n > 2$ equal 2 or 4 $\pmod 6$.

A **Steiner pre-algebra** is constructed in this manner, k, or one of $k[i]$ or $k \oplus k$, depending on whether -1 is a square in k.

5.2.4 XOR Pre-Algebras

This definition is motivated by the desire to emphasize pre-algebras with a $\mathbb{Z}_2$ grading.

An **XOR pre-algebra** is a twist pre-algebra over a group $\mathbb{Z}_2^n$. This means, up to sign, the product of basis elements is the bit-wise XOR, $\oplus$, of their binary representation of indices.

Clifford algebras over a vector space V form a prototypical example. The pre-algebras generated using the Cayley-Dickson construction below are another. A third example is the tensor product of two XOR pre-algebras A and B, $A \otimes_k B$. While neither the Clifford nor tensor product anti-commute, the Cayley-Dickson construction anti-commutes by construction.

If an XOR pre-algebra anti-commutes we can define a conjugation by negating the non-identity elements. XOR pre-algebras with conjugation are diagonal strong involution pre-algebras.

5.2.5 Cayley-Dickson Pre-Algebras

Even with the octonions one can find two versions of the Cayley-Dickson (CD) doubling rules, depending on whether one derived them with the imaginary units to the right or the left. The derivation of the doubling rules rely on the pre-algebra being a composition algebra, but once found can be repeated indefinitely. This results in an infinite family of pre-algebras.

Let a, b, c and d be elements of an existing twist pre-algebra with conjugation. Then the relations $(a, b)^\dagger = (a^\dagger, -b)$ and $(a, b)(c, d) = (ac - d^\dagger b, da + bc^\dagger)$ define a double of our pre-algebra.

The Cayley-Dickson pre-algebras are those generated by repeated application of this rule, starting from k itself. After the first step we get either $k[i]$ or $k \otimes k$, depending on whether -1 is a square in k (as with $\mathbb{Z}_5$ or $\mathbb{C}$.)

5.2.6 Smith's Sedenions

Smith produced a generalization that allows the composition rule to be extended indefinitely [55]. If b is 0 we use the above rule, otherwise we use $(a,b)(c,d) = (ac - d^{\dagger}b, b(b^{-1}d \cdot a) + bc^{\dagger})$.

This breaks right distributivity, so this product is not bilinear. Further, if a, b and d are orthogonal pure generators of the octonions, this fully inverts the da component, no matter how near zero b becomes. However, if $b = 0$ and we perturb a we get the original da component. Thus this product fails to be continuous.

Actually, there are 16 forms similar to this, all of which suffer from these same issues, though we can choose whether to give up left or right distributivity and which triple produces the discontinuity. Once we get here, this process can be repeated indefinitely, to create a composition pre-algebra that is either left or right distributive for any 2^n [55].

5.2.7 Classification

It is useful to visually represent the nesting of the structures described here, including those from Chapter 1, as in Figure 5.2.

5.3 Properties

Cawagas demonstrated that the sedenions contain eight dimensional sub-twist pre-algebras distinct from the octonions. Further, Cawagas showed that the basis elements of the CD double of the sedenions contained four non-isomorphic sixteen dimensional sub-twist pre-algebras. Thus the CD pre-algebras are not closed under sub-twist pre-algebras. Is there a class of twist pre-algebras closed under both the Cayley-Dickson construction, and sub-twist pre-algebras? We will show that both Steiner and XOR pre-algebras do, and examine the intersection of these. A **doubling pre-algebra** is both a Steiner and XOR pre-algebra.

It is well known that the CD construction looses significant properties over its first four iterations. Which properties are preserved by the CD construction?

Closure under Sub-Twist Pre-Algebras.

Proposition 5.3.1 (Prather)**.** *A sub-twist pre-algebra of an XOR pre-algebra is an XOR pre-algebra.*

$$
\begin{tikzcd}
 & \text{pre-}k\text{-algebra} \arrow[dl] \arrow[d] & \\
\text{twist} \arrow[d] \arrow[dr] & * \arrow[d] \arrow[dr] & \\
\text{XOR} \arrow[d] \arrow[dr] & \text{Steiner} \arrow[d] & \text{diagonal strong involution} \arrow[l] \\
\text{Clifford} \arrow[dr] & \text{Doubling} \arrow[d] & \\
 & \text{Cayley-Dickson} \arrow[d] & \\
 & \text{composition} &
\end{tikzcd}
$$

Figure 5.2: Pre-algebra Dependencies. The pre-algebras from Chapter 1 and Chapter 5, arranged by inclusion.

Proof. Let L generate an XOR pre-algebra. Then $L/\mathbb{Z}_2 = \mathbb{Z}_2^n$. Since -1 in L must be in the subloop S used to generate a sub-twist pre-algebra, the cosets of $\{1, -1\}$ in S must be a subloop of $\mathbb{Z}_2^n$. But this is some loop $\mathbb{Z}_2^m$ with $m \leq n$.

But then $S/\mathbb{Z}_2 = \mathbb{Z}_2^m$, and S is an XOR pre-algebra.

Further, if the pre-algebra has conjugation, the restriction to the sub-pre-algebra yields a conjugation on the sub-pre-algebra. □

Proposition 5.3.2 (Prather)**.** *A sub-twist pre-algebra of a Steiner pre-algebra is a Steiner pre-algebra.*

Proof. Let L be any loop induced by some STS. Let S be any subloop. If $|S| < 3$, then either S is the trivial group or S is $\mathbb{Z}_2$, and the sub-twist algebra is k, $k[i]$ or $k \oplus k$. Otherwise S has two distinct non-identity elements.

Any two distinct non-identity elements of $S/\mathbb{Z}_2$ define a triple in this STS, yielding a third element. By the closure of S this must also be in S.

But then we have a set of triples of the non-identity cosets of S such that each pair is in exactly one triple. Thus there is an STS on the non-identity cosets of S. $\square$

Corollary 5.3.3 (Prather)**.** *A sub-twist pre-algebra of a doubling pre-algebra is a doubling pre-algebra.*

Closure under Cayley-Dickson Construction.

Proposition 5.3.4 (Prather)**.** *The CD double of an XOR pre-algebra with conjugation is an XOR pre-algebra with conjugation.*

Proof. Note that the basis of (a, b) can be labeled with the same labeling for a, and 2^k plus the label of b, where 2^k is the dimension of the original pre-algebra.

Now the four terms in $(a, b)(c, d) = (ac - d^\dagger b, da + bc^\dagger)$ preserve the XOR relation since the underlying pre-algebra is an XOR algebra and the 2^k cancels for the indices in the $d^\dagger b$ term. $\square$

Proposition 5.3.5 (Prather)**.** *The CD double of a Steiner pre-algebra is is a Steiner pre-algebra.*

Proof. By inspection, the CD doubles of k, $k[i]$ and $k \otimes k$ are all Steiner pre-algebras. Otherwise we have two distinct basis elements e_i and e_j.

Note that $(1, 0)$ is the identity for the new algebra. The non identity basis elements of the source Steiner algebra is generated by some sign convention for some STS. We add new basis elements to represent the $(0, e_i)$ elements.

The CD construction gives us:

- $(e_i, 0)(0, 1) = (0, e_i)$,
- $(0, 1)(0, e_i) = (e_i, 0)$,
- $(0, e_i)(e_i, 0) = (0, N(e_i))$,
- $(0, 1)(e_i, 0) = (0, -e_i)$,
- $(0, e_i)(0, 1) = (-e_i, 0)$,
- $(e_i, 0)(0, e_i) = (0, -N(e_i))$.

Thus $(0, 1), (0, e_i), (e_i, 0)$ form a quaternionic triple.

Let i, j be distinct indices. The following quaternionic triples are similarly shown:

- $(e_i, 0), (e_j, 0), (e_ie_j, 0)$,
- $(e_i, 0), (0, e_ie_j), (0, e_j)$,
- $(0, e_i), (0, e_j), (e_je_i, 0)$.

But now every pair of non-identity basis elements in the new pre-algebra are in precisely one triple, so we have an STS. In fact, the sign convention is fixed by the CD construction. □

Corollary 5.3.6 (Prather)**.** *The CD double of a doubling pre-algebra is a doubling pre-algebra.*

Theorem 5.3.7 (Prather)**.** *Doubling pre-algebras are closed as sub-twist pre-algebras and under CD doubling.*

5.4 Classification of Small Doubling Pre-Algebras

From the XOR structure, a doubling pre-algebra must have a dimension that is a power of 2. For $d = 1$, it is simply the field itself, and for $d = 2$ it is isomorphic to $k[i]$ or $k \otimes k$, depending on whether -1 is a square in k. For $d = 4$, the Steiner structure has a unique triple, and the algebra is determined by the sign convention of this triple. The two cases are isomorphic, with conjugation providing the isomorphism.

5.4.1 The Pseudo-Octonions

Chesley demonstrated that there are exactly two positive definite doubling pre-algebras with three generators [7]. Cawagas demonstrated that the CD sedenions contain this pre-algebra as a sub-pre-algebra [8]. We now reproduce Chelsey's result.

Observe that once e_1 and e_2 are set we can define $e_3 = e_1e_2$. This fixes the upper right quadrant. Further, once e_4 is added we can define $e_5 = e_1e_4$ and likewise for e_6 and e_7 (choosing the eyes-right convention here). This fixes the diagonals and axis of each remaining quadrant.

This leaves three independent choices of sign for $e_6e_5 = \alpha e_3$, $e_5e_7 = \beta e_2$ and $e_7e_6 = \gamma e_1$. If $\alpha = \beta = \gamma = 1$ we get the octonions. This is shown in Table 5.1.

Let $\langle + - - \rangle$ represent the table generated by letting α be positive while β and γ are negative. Now generate a table mapping $e_2 \to e_1$, $e_4 \to e_2$ and $e_1 \to e_4$. To restore the sign convention we need to then map $e_1 \to -e_1$ and $e_7 \to -\gamma e_7$. This gives us a map from one table to another with $\alpha \to \alpha$, $\gamma \to \beta$ and $\beta\gamma \to \gamma$.

Table 5.1: Three generator xor pre-algebra multiplication table. The octonions have $\alpha = \beta = \gamma = 1$. All others are isomorphic to the table with $\alpha = \beta = \gamma = -1$.

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-\gamma e_7$	γe_6
e_2	$-e_3$	$-e_0$	e_1	e_6	βe_7	$-e_4$	$-\beta e_5$
e_3	e_2	$-e_1$	$-e_0$	e_7	$-\alpha e_6$	αe_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_4	$-\beta e_7$	αe_6	$-e_1$	$-e_0$	$-\alpha e_3$	βe_2
e_6	γe_7	e_4	$-\alpha e_5$	$-e_2$	αe_3	$-e_0$	$-\gamma e_1$
e_7	$-\gamma e_6$	βe_5	e_4	$-e_3$	$-\beta e_2$	γe_1	$-e_0$

This gives us four isomorphism classes.

- $\langle + + + \rangle$,
- $\langle - + + \rangle$,
- $\langle + + - \rangle \to \langle + - - \rangle \to \langle + - + \rangle \to \langle + + - \rangle$ and
- $\langle - + - \rangle \to \langle - - - \rangle \to \langle - - + \rangle \to \langle - + - \rangle$.

Similarly, $e_1 \to e_1$, $e_2 \to e_2$ and $e_5 \to e_4$ requires $e_4 \to -e_4$, $e_7 \to \beta e_7$ and $e_6 \to -\alpha e_6$. This gives us $\beta \to \alpha$, $\alpha \to \beta$ and $\alpha\beta\gamma \to \gamma$. This introduces the involutions $\langle + - + \rangle \leftrightarrow \langle - + - \rangle$ and $\langle + - - \rangle \leftrightarrow \langle - + + \rangle$. The first shows that the latter two classes are isomorphic, and the second the middle two. Thus all seven non-octonionic choices are isomorphic.

Theorem 5.4.1 (Chelsey)**.** *There is a unique eight dimensional doubling pre-algebra that is not isomorphic to the octonions, the* **pseudo-octonions***.*

Let us choose to use the $\langle - - - \rangle$ representation. Let e_i, e_j and e_k be associating generators, $(e_i e_j)e_k = e_i(e_j e_k)$, since order does matter. Then $(e_i - e_k)(e_i e_j + e_j e_k) = 0$. Now the norm of any such element is 2, so $-(e_i - e_k)/2$ is a two sided inverse of $e_i - e_k$. Scale any such element by $1/\sqrt{2}$ and you have invertible unit norm zero divisors. Unlike the octonions, we need to keep track not only of which elements are units, but which units preserve the norm on multiplication.

Note that the pseudo-octonions have a unique quaternionic triple that would restore an octonionic table if its sign convention is changed. For $\langle - - - \rangle$ it is the triple $\{e_1, e_2, e_3\}$.

Four Generator Doubling Algebras. With $d = 16$ we now have 35 signs to choose in our Steiner pre-algebra, 11 of which can be specified by selecting four generators and fixing the remaining non-unit basis. This leaves 2^{24} cases to consider. One might hope that one of these pre-algebras has the property that any $d = 8$ sub-twist pre-algebra is isomorphic to the octonions.

Theorem 5.4.2 (Prather)**.** *Any XOR pre-algebra over a field with $-1 \neq 1$ and four independent generators must have three independent generators that produce the pseudo-octonions.*

Proof. Let i, j, k and l be independent generators. Since we are dealing with an XOR loop, $(ij)k = i(jk)$ up to sign. Thus the only choices are associative or anti-associative.

Observe that if four generators are independent, then any two and the product of the remaining must also be independent. Suppose that all independent triples are anti-associative. Then we have the following 5-cycle of parenthesis changes:

$(ij)(kl) = -((ij)k)l = (i(jk))l = -i((jk)l) = i(j(kl)) = -(ij)(kl)$

But then all would be identically 0. Thus at least one step in the cycle must be associative. But each step only involves three independent basis elements.

From the classification of doubling pre-algebras with $d = 8$, this triple does not generate the octonions, so must generate the pseudo-octonions. $\square$

Now we can consider how many isomorphism classes of the 2^{24} pre-algebras there are. Cawagas found four non-isomorphic sedenionic sub-twist pre-algebras within the double of the CD sedenions [9].

Chelsey demonstrated that there are at least 52 positive definite four generator doubling pre-algebras [11]. First he found they could be separated into 9 classes by the number of octonionic sub-pre-algebras they contain, with an exhaustive search showing there are none with more than 8. By examining how these octonionic sub-pre-algebras intersect he was able to increase this to 52 classes, establishing a lower bound.

The upper bound can be reduced by building equivalence classes as we did for the pseudo-octonions. There are $15 * 14 * 12 * 8 = 20160$ ways to select four independent generators, each yielding an isomorphism. Several of these may be automorphisms, reducing this. There are also 16 sign choices as well, producing more possibilities.

This suggests that most of Chelsey's 52 cases are isomorphism classes, but some of the larger ones may be composed of a few separate classes. It is not clear what invariants will distinguish any such pre-algebras.

5.4.2 Loss of Symmetry

The automorphism group of the octonions is the 14 dimensional Lie group G_2. This acts on $\mathbb{R}^8$ by fixing the real axis and rotating the unit sphere in the orthogonal space. Thus e_1 can go to any point in this 6 dimensional manifold. This leaves a 5 dimensional manifold for e_2. Now e_3 is fixed by the pre-algebra, leaving a 3 dimensional manifold for e_4. Thus there is no distinction between unit vectors and basis vectors.

For the sedenions there are restrictions to which unit elements can be used as basis elements. Further, even when restricting to the basis elements, only 8 of the 15 eight dimensional doubling algebras yield the octonions. The automorphism group is $G_2 \otimes S_3$, that of the octonions times a choice of permutation related to triality [28].

5.5 Conclusion

So why should we care about these pre-algebras?

If we are going to embrace Graves' octonions, we need to answer his question, "If with our alchemy we can make seven pounds of gold, why should we stop there?"

At the fourth iteration of the Cayley-Dickson process we encounter pre-algebras containing XOR pre-algebras with associative generators. This introduces invertible zero divisors with unit norm. This distinguishes the basis elements, breaking the beautiful symmetry between unit norm elements found in the octonions.

While alternate products keep the composition rule, this is done at a cost of left or right distributivity and continuity.

These are a few of the most significant reasons why the 15 basis elements of the sedenions seem more like lead than gold. But if one does decide to wade into these waters, the pseudo-octonions and the non-twist pre-algebras of DeMarrais will hold clues to their most salient idiosyncrasies.

EPILOGUE

Despite the lack of associativity, the octonions have useful properties that allow for work arounds to several common mathematical constructions. The first is alternativity, a weaker notion of associativity. This allows us to manipulate any expression having only two octonions as if they were associative.

The second is the composition rule. This gives the octonions a very geometric nature. As outlined in the introduction, physicists have begun making attempts to use this nature to find more concise ways to describe the highly precise dynamics we now routinely observe. Indeed, $\mathbb{O}^-$ can be viewed as a composition pre-algebra analogous to paravector representations of Minkowski $3+1$ space in $\mathrm{Cl}(0,3)$.

Modules, Hilbert spaces, projective planes, and a useful Dirac operator are fundamental tools in a physics toolbox. Here we have provided an outline for how to properly use these tools in a non-associative setting, particularly pertaining to the octonionic pre-algebras.

In contemporary literature, geometric algebras are synonymous with Clifford algebras. With these tools the octonionic pre-algebras certainly seem to be closely related. If geometries of similar dimensions are compared by their symmetries, then the automorphism group of $\mathbb{O}$, G_2, is eight dimensions higher than that of $\mathrm{Cl}(3)$, $\mathrm{SO}(3)$. Indeed, since the symmetries extend to the entire pre-algebra, rather than a generating set, an argument could be made that the term geometric algebras better describes the composition pre-algebras. But that is a battle for another day.

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BIOGRAPHICAL SKETCH

Ben Prather was born on January 5th 1975 in Silver Spring, Maryland; just outside Washington D.C.. In 1989 his parents moved to Panama City Beach, Florida, where he graduated from Bay High School in 1993. There he was a standout on the math team, becoming the first student from Bay High School to qualify for the AIME.

Ben then earned an A.A. in Pre-Engineering from Gulf Coast Community College (now Gulf Coast State College) in 1995. At GCCC he participated on the Brain Bowl team as a math/science specialist both years, and as captain in 1995.

Ben earned a B.S. in Mechanical Engineering, with an emphasis in Automation and Robotics, from The University of Utah in 1998. His senior project involved the design of a prototype cancer treatment positioning device. He then pursued a second B.S. in Computer Science from The U which was preempted by a bout of depression in 2001. During this time he represented The U on their second team at the 2000 International Collegiate Programming Contest - finishing second in their subregional to Utah's first team. He also worked with Ganesh Gopalakrishnan doing research in formal verification of computer protocol. This research was funded in part by Intel, as part of the development of multi-core processors.

In 2005 Ben graduated Top Waffle from Waffle House University (a corporate training seminar), before depression undermined his restaurant career. In 2006 he returned to Gulf Coast State College as a Mathematics Learning Manager, overseeing a mathematics tutoring lab. This period reignited his academic interests. An accreditation review uncovered that a few faculty in the Department of Mathematics were not fully qualified. This resulted in a partnership with Florida State University, Panama City to offer graduate level mathematics courses for these individuals. Ben was able to participate in these courses, meeting Dr. Wolfgang Heil. In 2012 Ben's position at GCSC was dissolved due to a restructuring of the tutoring services.

He eventually applied to the Florida State Universities' Ph.D in Mathematics program - with a recommendation from Dr. Heil as a cornerstone of his application. Ben accepted an unfunded M.S. program offer as a part-time student in 2015. After his first term he reapplied for the Ph.D. program and was accepted. He received an M.S. in Mathematics in 2017.

Ruth Moufang was on Dr. Heil's M.S. thesis committee at Goethe University in 1967.