Understanding 3–Manifolds by Their Character Variety

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Joint work with C. Katerba and S. Tillmann

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Dictionary

$n$-manifold: a topological space that locally looks like $\mathbb{R}^n$ (assume connected, compact and orientable);
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![Diagram of a 3-manifold and a surface embedded in it]
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  a) $S^3$ is irreducible;
  
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**Theorem (Waldhausen, 1976)**

Suppose $M$ is closed (empty boundary) and Haken. If $M_1$ is an irreducible 3-manifold such that $\pi_1(M) \cong \pi_1(M_1)$, then $M \cong M_1$. 

**Theorem (Thurston, 1986)**

Hyperbolization Theorem: If $M$ is a Haken manifold with torus boundary, then the interior of $M$ admits a complete hyperbolic structure of finite volume.
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Stalling’s Construction of Essential Surfaces

How can we construct essential surfaces?

Stalling’s Construction:

An action of $\pi_1(M)$ on a tree $T$ without inversions (no edge is flipped)

$(ii)$ non-trivial (no vertex globally fixed) $\rightarrow$ A (non-canonical) essential surface

By $(i)$ the quotient $G := T/\pi_1(M)$ is a graph. There is a map $f: M \rightarrow G$, where, for any $x \in G \setminus G(0)$, $f^{-1}(x)$ is a collection of disjoint surfaces properly embedded in $M$.

By $(ii)$, $f^{-1}(x) \neq \emptyset$.

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The character variety $\mathcal{X}$ is the categorical quotient of $\mathcal{R}$ by $SL_2(\mathbb{C})$ under conjugation:

$$\mathcal{X} := \mathcal{R} // SL_2(\mathbb{C}).$$

Categorical quotient: a point of $\mathcal{X}$ is the closure of an $SL_2(\mathbb{C})$-orbits of a point in $\mathcal{R}$.

For future reference, let $t : \mathcal{R} \to \mathcal{X}$ be the quotient map.
Ideal Points

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Let $X$ be one such irreducible curve.

By first resolving singularities, and then projectivizing, there is a smooth projective curve $\tilde{X}$ (unique up to isomorphism) and a bi-rational map

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The set of **ideal points** of $X$ is $\mathcal{I}(X) := \tilde{X} \setminus \iota^{-1}(X)$. 
For each ideal point $P \in \mathcal{I}(X)$, there is natural valuation $v_P$ associated to it.

Every rational map $f \in \mathbb{C}(X)$ uniquely extends to an element of $\mathbb{C}(\tilde{X})$, hence $f$ is locally a meromorphic function.

**Lemma**

Let $R$ be an irreducible component of $\mathcal{R}$ such that $t(R) = X$. Then $v_P$ extends uniquely (up to a scalar factor) to a valuation on the function field $\mathbb{C}(R)$. 
Valuation

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We define $v_P : \mathbb{C}(X) \to \mathbb{Z}$,

$$v_P(f) := \begin{cases} -(\text{order of the pole of } f \text{ at } P) & \text{if } f(P) = \infty, \\ 0 & \text{if } f(P) \in \mathbb{C} \setminus \{0\}, \\ \text{order of the zero of } f \text{ at } P & \text{if } f(P) = 0. \end{cases}$$
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Lemma

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A construction due to Tits, Bass and Serre associates:

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\text{a valuation } v \text{ on a field } K \quad \rightarrow \quad \text{An action of } \text{SL}(2, K) \\
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From Valuation to Action on a Tree

A construction due to Tits, Bass and Serre associates:

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By applying this to the valuation \( \nu_P \) and the function field \( \mathbb{C}(R) \), we get an action of \( \text{SL}(2, \mathbb{C}(R)) \) on a tree \( T \).

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Finally, we compose the action with the \textit{tautological representation}

\[
\pi_1(M) \rightarrow \text{SL}(2, \mathbb{C}(R))
\]

to get an action of \( \pi_1(M) \) on \( T \).
Culler-Shalen Detection

Stalling’s theory applies to construct a non-empty essential surface $S$ in $M$. All surfaces arising in this way are said to be associated to the ideal point $P$. Essential surfaces associated to some ideal point of $X$ are said to be detected by $X$. 

**Theorem (Culler - Shalen)**

For $\gamma \in \pi_1(M)$, let $I_\gamma \in C(R)$ be the trace function:

$$I_\gamma : R \rightarrow C \rho \mapsto \text{tr}(\rho(\gamma))$$

Let $S$ be an essential surface in $M$ associated to an ideal point $P$:

1. If $v_P(I_\alpha) \geq 0$ for all $\alpha \in \pi_1(\partial M)$, then $S$ may be chosen to have empty boundary;
2. Otherwise, there is a unique simple curve $\alpha_0 \in \pi_1(\partial M)$ with $v_P(I_{\alpha_0}) \geq 0$ and every component of $\partial S$ is parallel to $\alpha_0$.

This theorem divides ideal points in two classes, type 1 and type 2.
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The Questions

Two questions naturally arise from this result.

Question (1):
Is every closed essential surface in $M$ detected by an ideal point of a curve in $X$?

Question (2):
Suppose $\alpha$ is a boundary component of an essential surface $S$. Is there an essential surface $S'$ with boundary $\alpha$ detected by a curve in $X$?

Question (2): negative answer (Chesebro and Tillmann).

Question (1): negative answer (C., Katerba and Tillmann).
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- **Ideal points:** show that none of them are of type 1. We appealed to:

**Theorem (Chesebro - Katerba, 2016)**

Let \( \text{Tr}_Q \) be \( \mathbb{Q} \)-algebra generated by all the trace functions \( I_\gamma \) on \( C[\mathfrak{x}] \). Then \( \text{Tr}_Q \) is naturally a \( \mathbb{Q}[I_\gamma] \)-module. Furthermore, if for some \( \alpha \in \pi_1(\partial M) \), \( \text{Tr}_Q \) is finitely generated and free as a \( \mathbb{Q}[I_\alpha] \)-module, then \( \mathfrak{x} \) does not detect a closed essential surface.

These computations were carried on using Macauly2.
Thank you very much for your attention!