Representations of Once-Punctured Torus Bundles Using Flags

Alex Casella

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[Lie - Klein, 1982]: A geometric structure is a pair \((G, X)\) of some model topological space \(X\) and some group \(G\), acting on it.

For example:

- \(X = \mathbb{R}^n\) and \(G = O(n) \ltimes \mathbb{R}^n\): Euclidean Geometry
- \(X = \mathbb{H}^n\) and \(G = SO(1, n)\): Hyperbolic Geometry
- \(X = \mathbb{CP}^n, \mathbb{RP}^n\) and \(G = PGL(n, \mathbb{C}), PGL(n, \mathbb{R})\): Complex, Real Projective Geometry
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- \(X = \mathbb{C}P^n, \mathbb{R}P^n\) and \(G = PGL(n, \mathbb{C}), PGL(n, \mathbb{R})\): Complex, Real Projective Geometry

To endow a manifold \(M\) with a geometric structure \((G, X)\):
- developing map: \(\text{dev} : \tilde{M} \rightarrow X\)
- holonomy representation: \(\text{hol} : \pi_1(M) \rightarrow G\)

such that
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\text{dev}(\gamma \cdot x) = \text{hol}(\gamma) \cdot \text{dev}(x), \quad \gamma \in \pi_1(M), \; x \in \tilde{M}.
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Introduction

Geometric Structures and Representations

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Story of my PhD project:

Representation \(\rho: \pi_1(M) \to G\) \(\rightarrow\) A geometric structure \((G, X)\) on \(M\) with: \(\text{hol} = \rho\).
Once-Puncture Torus $\mathbb{T}$

$$G = \text{PGL}(3, \mathbb{C})$$
A Motivating Example

Once-Puncture Torus \( \mathbb{T} \)

[\text{Fock - Goncharov, 2007}]:\[ \pi_1(\mathbb{T}) \to G = \text{PGL}(3, \mathbb{C}) \]

Using \textbf{ideal triangulations} and \textbf{flags}.
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Once-Puncture Torus $\mathbb{T}$

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Using ideal triangulations and flags.

\[ = (P, \eta), \quad P \in \mathbb{C}P^2, \quad \eta \subset \mathbb{C}P^2, \quad P \in \eta. \]
We think of $\pi_1(\mathbb{T}) = \langle \alpha, \beta \rangle$ as deck transformations:

$$\rho : \alpha \mapsto A \in \text{PGL}(3, \mathbb{C}),$$

$$\beta \mapsto B \in \text{PGL}(3, \mathbb{C}).$$
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Some Generalisations

- This construction works for any $S_{g,n}$ with $n > 0$ and $\chi(S_{g,n}) < 0$.

$$\phi : \mathbb{C}^{16g+8n-16} \rightarrow [\rho : \pi_1(S_{g,n}) \rightarrow PGL(3, \mathbb{C})].$$
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- Generalise to $PGL(m, \mathbb{C})$ using \textit{m--dimensional flags}:

an array $(V_0, \ldots, V_{m-2})$ of subspaces of $\mathbb{CP}^{m-1}$ s.t.

\[ V_0 \subset V_1 \subset \cdots \subset V_{m-2} \quad \text{and} \quad \dim(V_i) = i. \]
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  $(m - 2)(m - 1)/2$ numbers per triangle and $m - 1$ numbers per edge:

  $$\phi : \mathbb{C}^{(m^2-1)(2g+n-2)} \to [\rho : \pi_1(S_{g,n}) \to PGL(m, \mathbb{C})].$$
A Particular Case

- **Positive Representations:** when we restrict to positive real parameters.

The restriction map

$$\phi : \mathbb{R}^{16g+8n-16} > 0 \rightarrow [\rho : \pi_1(S_{g,n}) \rightarrow PGL(3, \mathbb{R})]$$

is a branched covering, generically $6^n$–to–$1$.

All representations in its image are discrete and faithful, and they are all holonomies of **convex projective structures**.
Once-Punctured Torus Bundle

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\[ \pi_1(M_f) = \pi_1(\mathbb{T})_{f*} := \langle \alpha, \beta, t \mid t\alpha t^{-1} = f_*(\alpha), \ t\beta t^{-1} = f_*(\beta) \rangle, \]

where \( \pi_1(\mathbb{T}) = \langle \alpha, \beta \rangle \) and \( f_* : \pi_1(\mathbb{T}) \rightarrow \pi_1(\mathbb{T}) \).

**Question**

*Which representations of \( \pi_1(\mathbb{T}) \) into \( \text{PGL}(3, \mathbb{C}) \) do extend to representations of \( \pi_1(M_f) \)?*
[Floyd-Hatcher, 1982]: when $f$ is pseudo-Anosov, we can associate to $f$ a string of letters “R” and “L” which can be used to construct $M_f$ by layering tetrahedra.
Floyd-Hatcher Triangulation

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The Figure Eight Knot Complement $K_8$:

$K_8$ is a once-puncture torus bundle for $f : T \to T$ such that

$$f_*(\alpha) = \alpha \beta^{-1} \quad \text{and} \quad f_*(\beta) = \beta^2 \alpha^{-1}.$$  

The string associated to $f$ is “LR”.
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Floyd-Hatcher Triangulation: The Figure Eight Knot Complement

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Once Punctured Torus Bundles

Flip Functions

\[(a, \ldots, z) \xrightarrow{\text{Flip Function}} (a', \ldots, z')\]
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Flip Functions

Theorem

A representation of $\pi_1(\mathbb{T})$ extends to $\pi_1(M_f)$ if and only if

$$(a, \ldots, z) = (a'', \ldots, z'')$$
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There is a **global fixed point** for every automorphism $f$. This corresponds to a special representation: it is the holonomy of a CR-structure. CR-structures are geometric structures modelled on the pair $(PU(2, 1), S^3)$. For the figure eight knot complement, this structure was already known by Falbel, but not for all torus bundles.
Present and Future Directions

- Understand better these CR-structures (e.g. developing map, local behaviours, edge branching, etc.).
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- Extend this construction in two directions:
  1) 3–manifolds which are circle bundles over generic punctures surfaces;
  2) representations into $PGL(m, \mathbb{C})$. 
Thank you very much for your attention!