Midterm 1 - Solutions.

Math 224 - Section 05 — Spring 2003

Full Name:______________________ Score: __________

Show all of your work for full credit.

1. Find the general solution to the differential equation

$$y' = \frac{y}{x-1} + \frac{y}{x+1}$$

Solution: The equation is separable,

$$\begin{aligned}
y' &= \frac{y}{x-1} + \frac{y}{x+1} \\
\frac{dy}{y} &= \frac{1}{x-1} + \frac{1}{x+1} \\
\ln y &= \ln(x-1) + \ln(x+1) + C_0 \\
&= \ln\left((x-1)(x+1)\right) + C_0 \\
y &= C(x-1)(x+1)
\end{aligned}$$

2. Find the general solution to the differential equation

$$x' = x \sin t + 2te^{-\cos t}$$

Solution: The equation is linear so we can use integrating factor or variation of parameters. Using the integrating factor $\mu = e^{-\int \sin t \; dt}$ gives

$$\begin{aligned} (xe^{\cos t})' &= 2t \\ xe^{\cos t} &= t^2 + C \\ x &= t^2 e^{-\cos t} + Ce^{-\cos t} \end{aligned}$$

3. Find the particular solution to the differential equation

$$\begin{aligned} x' - \frac{n}{t}x &= e^t t^n \text{ , n a positive integer} \\ x(1) &= 1 \end{aligned}$$

Solution: Solution: The equation is linear so we can use integrating factor or variation of parameters. Using the integrating factor $\mu = e^{-\int \frac{n}{t}\,dt} = e^{-n\ln t} = t^{-n}$, gives

$$\begin{aligned} x' - \frac{n}{t}x &= e^t t^n \\ (t^{-n}x)' &= e^t \\ t^{-n}x &= e^t + C \\ x &= t^n e^t + Ct^n \end{aligned}$$

The initial condition implies that

$$\begin{aligned} 1 &= e + C \\ C &= 1 - e \end{aligned}$$

Hence, $x(t) = t^n e^t + (1-e)t^n$

4. Given the differential equation

$$y\ dx + (2x - ye^y)\ dy \quad = \quad 0$$

(a) Show that $\mu = y$ is an integrating factor. Solution:

$$\begin{aligned} M &= y^2 \\ M_y &= 2y \\ N &= 2xy - y^2e^y \\ N_x &= 2y \end{aligned}$$

So the after multiplying by the integrating factor the equation is exact.

(b) Solve the resulting exact differential equation. You will probably need to know that $\int y^2e^y dy = y^2e^y - 2ye^y + 2e^y + C$. Solution: Because the equation is exact we know that there exists a function Ψ so that

$$\begin{aligned} \Psi_x &= M = y^2 \\ \Psi &= xy^2 + \phi(y) \\ \Psi_y &= 2xy + \phi' = N = 2xy - y^2e^y \\ \phi' &= -y^2e^y \\ \phi &= -(y^2e^y - 2ye^y + 2e^y + C) \\ \Psi &= xy^2 - (y^2e^y - 2ye^y + 2e^y + C) \end{aligned}$$

Choose one of the next two problems. Indicate your choice by marking yes or no in the space provided.

5. grade(yes/no): ____________

A young person opens an account with 1000\$. The account earns $r\%$ interest compounded continuously.

(a) Find the time required for the initial deposit to double as a function of r assuming that there are no deposits. Solution: The balance in the account at time t is given by

$$\begin{aligned} P' &= rP \\ P &= Ce^{rt} \\ P(0) = 1000 &= C \\ P &= 1000e^{rt} \\ \text{to find the doubling time we solve} \\ 2000 &= 1000e^{rt} \\ t &= \frac{\ln 2}{r} \end{aligned}$$

(b) Suppose that the same person also deposits k dollars per year into the account. Determine k so that one million dollars will be available for retirement in 40 years given that $r = 7.5\%$.

Solution: The balnce in the account is given by the ODE

$$\begin{aligned} P' &= .075P + k \\ \text{Solving this with initial condition } P(0) = 1000 \\ (Pe^{-.075t})' &= e^{-.075t}k \\ P &= -\frac{k}{.075} + Ce^{.075t} \\ P(0) = 1000 &= -\frac{k}{.075} + C \\ C &= 1000 + \frac{k}{.075} \\ P &= -\frac{k}{.075} + (1000 + \frac{k}{.075})e^{.075t} \end{aligned}$$

We need to find k so that $P(40) = 1 million\$$.

$$\begin{aligned} 1000000 &= k(-\frac{1}{.075} + \frac{e^{.075(40)}}{.075}) + 1000e^{.075(40)} \\ k &= 3850.70 \end{aligned}$$

Choose either this problem or the previous problem. Indicate your choice by marking yes or no in the space provided.

6. grade(yes/no): ____________

In the investigation of a homicide it is often important to estimate the time of death. One method used is to approximate the rate of change of the body temperature by Newton's law of cooling. That is

$$\frac{d\theta}{dt} = -k(\theta - T),$$

where $\theta(t)$ is the body temperature at time t, T is the ambient temperature and $k > 0$ is a constant.

(a) Solve the differential equation (Note:The differential equation is linear, so there are at least two methods one can use). Solution: The equation is separable,

$$\begin{aligned} \frac{d\theta}{dt} &= -k(\theta - T) \\ \frac{d\theta}{\theta - T} &= -kdt \\ \ln(\theta - T) &= -kt + C_0 \\ \theta &= Ce^{-kt} + T \end{aligned}$$

(b) Using the solution found in the previous part, find the particular solution given that the temperature of the corpse is $85°F$ when discovered. Assume that $k = .52$ when the ambient temperature,T, is $68°$. Solution:

$$\begin{aligned} \theta &= Ce^{-kt} + T \\ \theta(0) &= C + 68 = 85 \\ C &= 17 \\ \theta &= 17e^{-kt} + 68 \end{aligned}$$

(c) Using the previous parts, approximately how long ago was the corpse murdered, assuming that the temperature of the living person was $98.6°F$?

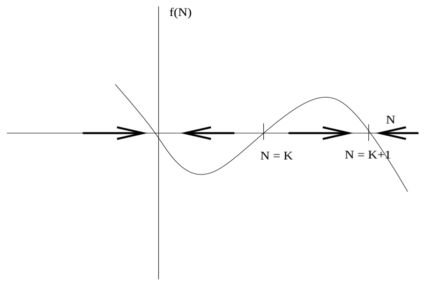
$$\begin{aligned} \theta &= 17e^{-kt} + 68 \\ 98.6 &= 17e^{-.52t} + 68 \\ t &= \ln\left((98.6 - 68)/17\right)/(-.52) \\ &\approx 1.13 \; hours\ ago \end{aligned}$$

7. Given the differential equation

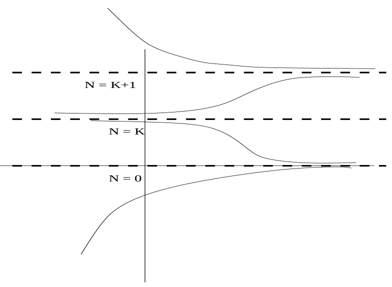
$$\frac{dN}{dt} = -r(1 - \frac{N}{K})(1 - \frac{N}{K+1})N$$

where r and K are positive constants.

(a) Use the graph of the right-hand-side to draw the phase line.



(b) Use the previous part to sketch the qualitative behavior of solutions, for different initial conditions.



(c) Classify the equilibrium solutions.

$$\begin{aligned} N = 0 &\quad \textit{stable} \\ N = K &\quad \textit{unstable} \\ N = K+1 &\quad \textit{stable} \end{aligned}$$

Bonus: Solve

$$\frac{dN}{dt} = -r(1-\frac{N}{K})(1-\frac{N}{K+1})N$$

Solution:

$$\begin{aligned}
\frac{dN}{dt} &= -r(1-\frac{N}{K})(1-\frac{N}{K+1})N \\
\frac{dN}{(1-\frac{N}{K})(1-\frac{N}{K+1})N} &= -rdt \\
\left(\frac{1/(1-K/(K+1))}{1-\frac{N}{K}} + \frac{-\frac{1}{1+1/k}}{1-N/(K+1)} + \frac{1}{N}\right) dN &= -rdt \\
(1-\frac{N}{K})^{1/(1-K/(K+1))} \left((1-N/(K+1))^{-\frac{1}{1+1/k}}\right) N &= Ce^{-rt}
\end{aligned}$$