2

Mathematical Background

This chapter provides an overview of the main tools for analytic studies. We briefly review differential equations including linear, constant coefficient equations, separable equations, linearization and qualitative analysis.

2.1 Mathematical Preliminaries

Differential equations are relations between unknown function, $y(t)$, and its' derivatives, $F\left(\frac{d^n y}{dt^n}, \frac{d^{(n-1)}y}{dt^{(n-1)}}, \ldots, \frac{dy}{dt}, y(t), f(t)\right)$. As in algebra, one of the goals is to determine the unknown that satisfies some constraint this relationship – for example $F\left(\frac{d^n y}{dt^n}, \frac{d^{(n-1)}y}{dt^{(n-1)}}, \ldots, \frac{dy}{dt}, y(t), f(t)\right) = 0$. A few examples are,

$$\begin{aligned}
\frac{dy}{dt} - f(t) &= 0, \\
\left(\frac{d^2y}{dt^2}\right)^2 + \frac{d^2y}{dt^2}\frac{dy}{dt} + \sin(t) &= 0, \\
\left(\frac{d^3y}{dt^3}\right)\frac{dy}{dt} + y &= e^t.
\end{aligned}$$

One could naively think of the goal of determining $y(t)$ as some sort of integration. For the first example above,

$$\begin{aligned}
\frac{dy}{dt} - f(t) &= 0, \\
\int \frac{dy}{dt}dt &= \int f(t)dt, \\
y(t) &= \int f(t)dt.
\end{aligned}$$

Sometimes we can undo the derivatives. The solution to the equation $\frac{dy}{dt} - f(t) = 0$ is $y(t) = \int f(t)dt$. Recalling from calculus that until we have specified the domain, these integrals are indefinite and only determined up to a constant. For an equation that has a highest derivative n (referred to as an n^{th} order differential equation), we find n integration constants that need to be determined.

The theory of differential equations is very well developed, especially for the types of models that we will mainly be considering, namely *initial value problems*. For these types of differential equations, the integration constants are determined by providing information at a specific time, Information is typically the value of the function and enough derivatives to determine the constants (remember the initial position and velocity in the projectile motion section). The most widely used alternative to this is to provide information about the unknown function at multiple points. These equations are typically referred to as *boundary value problems* and are more technically challenging.

For initial value problems, broadly speaking as long as all the functions involved are well-behaved (are able to be differentiated a suitable number of times and have no singularities), one can show that there is a unique solution to the equation. We will focus exclusively on these and for the most part initial value problems have unique solutions as long as we prescribe initial values. We can then turn to finding these solutions. To do this, it is often useful to classify initial value problems since certain methods are only useful for certain classes of equations. The reasons to cover analytic methods (that is methods that you can do on paper and completely understand) are two-fold. First, our analysis will lead to insight reality-checks for our models. Second, since we will be doing a lot of of numerical simulations, having certain behaviors in hand will help us interpret our simulations.

Differential equations can be differentiated into two broad categories – *linear* or *nonlinear* depending on whether the relationship $F(\frac{d^{(n)}y}{dt^{(n)}}, \frac{d^{(n-1)}y}{dt^{(n-1)}}, \ldots, \frac{dy}{dt}, y(t), f(t))$ is linear, with no terms with products of terms containing $y(t)$, or not. Solution techniques and questions about uniqueness of solutions depend a lot on the classification.

2.1.1 Linear

Linear initial value problems can be written as,

$$\begin{aligned} a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{(n-1)}y}{dt^{(n-1)}} + \ldots + a_1(t)\frac{dy}{dt} + a_0(t)y(t) &= f(t) \\ y(0) &= y_0 \\ \frac{dy}{dt}(0) &= y_1 \\ &\vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(0) &= y_{n-1} \end{aligned}$$

If the right-hand-side is zero, this is a homogenous equation and $y(t) = 0$ is a candidate solution, depending on the initial conditions provided.

There are many techniques for analyzing these, but we will focus on one more restrictions. If the coefficients are constant, there is a completely algorithmic way to understand the solution. For a concrete example, we can consider a second order example,

$$\begin{aligned} a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy &= 0 \\ y(0) &= y_0 \\ \frac{dy}{dt}(0) &= y_1. \end{aligned} \tag{2.1}$$

We will come up with a solution by noticing that we already know a function whose derivatives are related to that function. This is the definition of the exponential, $e^{\lambda t}$. So we can guess this as a solution and substitute this into Equation 2.2. This gives us, $a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0$. Since the exponential is never zero, we can simplify this to find the *characteristic equation*,

$$a\lambda^2 + b\lambda + c = 0.$$

This is an algebraic equation for λ, so we know that there are two values of λ that satisfy the equation,

$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These are referred to as *eigenvalues*, which is a term that may be familiar if you have had linear algebra. The most important thing to notice at this point is that $y(t) = Ae^{\lambda t}$ is a solution for each of these values of λ and for any value of A. Linear combinations of both of the solutions leads to a solution since the differential equation is linear. In general any solution to the differential Equation 2.2 can be written $y(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}$. The initial conditions determine A and B.

There are certain things to be aware of. First, what if the values of $\lambda_\pm$ are complex? How do we understand $e^{\lambda_+ t}$ in this case? What happens if $b^2 - 4ac = 0$? For a short version that is applicable to this material, we note that if the eigenvalues are real and distinct, the solutions are exponential and the solutions either exponential increase or decrease to zero. If the eigenvalues are complex conjugate pairs, $\lambda_\pm = R \pm iI$ the solutions can be written as combinations of $e^{Rt}\sin(It)$ and $e^{Rt}\cos(It)$. These solutions oscillate with amplitudes that grow or decay according to whether R is positive, negative or zero.

There is also the case where the eigenvalues are real and repeated. In a differential equations course you learn methods to deal with this. In this book it is less useful to address this. This is what is often referred to as a non-generic case. That is, it requires a precise relationship in the parameters. The point of view that comes from sensitivity and as a practical applied mathematician is that parameters are not absolute, fixed values. They may vary between individual experiments, between individual species, between times of the year or other small differences. As such, it tends to be less important to understand things that require exact values of parameters and we focus on things that occur in wider parameter spaces.

One more useful piece of information is that any higher order differential equation can be written as a system of lower-order differential equations. If we define a new variable, $u = \frac{dy}{dt}$, we can see that $\frac{du}{dt} = \frac{d^2y}{dt^2} = -\frac{b}{a}\frac{dy}{dt} - \frac{c}{a}y$. Equation 2.2 can be written differently,

$$\begin{aligned} \frac{dy}{dt} &= u \\ \frac{du}{dt} &= -\frac{b}{a}\frac{dy}{dt} - \frac{c}{a}y = -\frac{b}{a}w - \frac{c}{a}y. \end{aligned}$$

This can be written very succinctly as

$$\frac{d\mathbf{y}}{dt} = M\mathbf{y} \tag{2.2}$$

, where,

$$M = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}$$

We define $\mathbf{y} = < y(t), u(t) >$.

It turns out that the eigenvalues of this matrix are equivalent to the roots of the characteristic equation 2.2. We will see that our numerical methods are typically written in this form and, in fact, many of the models come in this form.

That works well for a very restricted class of equations – linear, constant coefficient, homogenous initial value problems. However, most processes in nature are nonlinear. This means that we have to have some way to analyze nonlinear equations. To do this, we will introduce two analytic methods that will be supplemented with direct numerical simulations. The first method illustrates one of the most useful insights in mathematics, namely if we look close enough, most nonlinear processes can be approximated by linear processes. This is used in all areas of mathematics including topology, algebra, differential equations, etc. Any student has seen this in calculus where nonlinear functions are approximated by linear functions using Taylors' theorem. We can do this for differential equations. The second tool is to look at the qualitative behavior of solutions by considering how the derivatives of y control the rate of change of aspects of the graph of y. We will do this in the context of first order equations in this chapter, but in subsequent chapters we will explore the same processes for higher order equations/systems.

2.1.2 Nonlinear Equations

Nonlinear equations are much more difficult to solve analytically. There are several classes that are generally solvable and are focused on in a course in differential equations.

It is typical to write the implicit relationship between $y(t)$ and its' deriviatives, $F\left(\frac{d^n y}{dt^n}, \frac{d^{(n-1)}y}{dt^{(n-1)}}, \ldots, \frac{dy}{dt}, y(t), f(t)\right) = 0$, in terms of the highest derivative,

$$\frac{d^n y}{dt^n} = G\left(\frac{d^{(n-1)}y}{dt^{(n-1)}}, \ldots, \frac{dy}{dt}, y(t), t\right).$$

Where G encodes all of the steps needed to isolate $\frac{d^n y}{dt^n}$.

To be very clear, we can start with first order equations which can be written as,

$$\frac{dy}{dt} = G(y,t). \tag{2.3}$$

These cannot be solved by integrating both sides since the unknown y occurs on both sides. There are some cases where we can *almost* do this though. If $G(y,t) = G_1(y)G_2(t)$ we can rewrite the equation,

Separation of Variables

Consider the differential equation,

$$\begin{aligned} \frac{dy}{dt} &= \frac{t}{y}, \\ y(0) &= 1. \end{aligned}$$

We can solve this using separation of variables:

$$\begin{aligned} \frac{dy}{dt} &= \frac{t}{y}, \\ y\,dy &= t\,dt, \\ \int y\,dy &= \int t\,dt, \\ y^2 &= t^2 + c, \end{aligned}$$

which gives an implicit solution for y.
The initial condition, $y(0) = 1$, implies that $c = 1$ and we can write an explicit solution $y(t) = \pm\sqrt{t^2+1}$.

$$\frac{dy}{dt} = G(y,t) = G_1(y)G_2(t),$$
$$G_1(y)dy = G_2(t)dt.$$

This is a slight abuse of notation but can be made formally correct. Then integrating both sides with respect to the arguments gives an implicit solution,

$$\int G_1(y)dy = \int G_2(t)dt.$$

It should be noted, however, that this does not work in general. Even when it does work, the solution is often only written implicitly and may be quite complicated and unwieldy.

2.2 Linearization

Linearization is one of the major concepts in mathematics. Almost all branches use linearization to approximate the behavior of nonlinear processes. Applied mathematics relies on linearization to a wide extent. We will begin with a brief review from calculus since this is often the first place we see the formal idea of linearization.

Start with a function of an independent variable t, say $f(t)$. Taylors' theorem states that under some conditions on f, we can write $f(t)$ as a polynomial, $P(t) = \sum_i \alpha_i t^i$. Moreover, there is an interval about any point in the domain of f (where f obeys certain restrictions), where $f(t) = P(t)$. This is a remarkable result that provides insight into how to integrate and differentiate a range of functions, since we know that integration and differentiation of polynomials has a predictable patterns. But how do we determine $P(t)$? Taylors' theorem states that, near any point, t, we can write $f(t)$ as,

$$f(t) = f(a) + \frac{df}{dt}(a)(t-a) + \frac{\frac{d^2f}{dt^2}(a)}{2!}(t-a)^2 +$$
$$\ldots + \frac{\frac{d^nf}{dt^n}(a)}{n!}(t-a)^n) + \text{error}.$$

Therefore the coefficients of the polynomials are related to the derivatives of f. It also means that as long as t is close to t, $(t-a)^i$ is very small. So we could approximate f,

$$f(t) \approx f(a) + \frac{df}{dt}(a)(t-a),$$

where the error term can usually be estimated using the mean value theorem. Graphically, this means that near any point, a nice enough function can be thought of as a line.

This idea can be used to approximate nonlinear differential equations by linear equations. To introduce the topic, we will start with the simplest case of a scalar equation of the form,

$$\begin{aligned} \frac{dy}{dt} &= f(y), \\ y(0) &= y_0, \end{aligned} \tag{2.4}$$

although this idea can be generalized to other forms of differential equations. One of the most important restrictions that we are requiring is that the right-hand-side cannot depend explicitly on time. These equations are referred to as autonomous equations and are the main focus of this text. There are other methods that are used for non-autonomous equations and can be found in textbooks on differential equations (for example [7]).

In this case, the goal is to determine the behavior of the solution. In calculus, we need a specific place to linearize near. Linearization is inherently a local argument and cannot in general be used everywhere. In calculus we linearize at a point. In differential equations, we linearize around a known solution. This seems counter-intuitive at first, since the goal was to find a solution in the first place. However, it is often simple to find some special solutions to differential equations. For example, we can look for for solutions that do not depend on time. These are steady-state solutions and are constants that satisfy the differential equation. To find steady-state solutions, $\bar{y}$, we have to solve,

$$f(\bar{y}) = 0.$$

There may be no steady-states, in which case we cannot proceed with the linearization and we have to do something else such as direct numerical simulations. Otherwise, we have at least one solution, $\bar{y}$. We then want to find out how the solution behaves near this steady-state. We define the solution we are studying as,

$$y(t) = \bar{y} + \varepsilon Y(t). \tag{2.5}$$

The function $Y(t)$ is close to $\bar{y}$ as long as $\varepsilon Y(t)$ is 'small'. We will not be perfectly precise here about how small is small enough but the idea is that understanding the dynamics of $Y(t)$ can provide information about $y(t)$.

To derive an equation for $Y(t)$, we put the solution in Equation 2.5 into Equation 2.4,

$$\begin{aligned} \frac{dy}{dt} &= f(y), \\ \frac{d\bar{y} + \varepsilon Y(t)}{dt} &= f(\bar{y} + \varepsilon Y(t)), \\ \varepsilon \frac{dY(t)}{dt} &= f(\bar{y} + \varepsilon Y(t)), \\ &= f(\bar{y}) + \varepsilon f'(\bar{y})Y(t) + \text{error}. \end{aligned}$$

We have used the fact that $\bar{y}$ does not depend on time to simplify the left-hand-side and Taylors' theorem to approximate $f(y)$ with a linear approximation near $\bar{y}$. Notice that this provides an equation for $Y(t)$,

$$\frac{dY(t)}{dt} = f'(\bar{y})Y(t). \tag{2.6}$$

Since $\bar{y}$ is constant and this equation is linear, we know exactly how $Y(t)$ behaves since the solution is $Y(t) = ke^{f'(\bar{y})t}$. If $f'(\bar{y}) > 0$, $Y(t)$ increases and $y(t)$ moves away from $\bar{y}$. On the other hand, if $f'(\bar{y}) <=$, $Y(t)$ decreases and the solution moves towards $\bar{y}$. This says something specific about the steady-state solution, $\bar{y}$. If $f'(\bar{y}) > 0$, we refer to the steady-state as *unstable*. If $f'(\bar{y}) < 0$, $\bar{y}$ is stable. If $f'(\bar{y}) = 0$, the linearization fails and we have to use different arguments to understand the behavior.

We can generalize this idea. If we have a system of n nonlinear differential equations,

$$\begin{aligned}\frac{dy_1}{dt} &= f_1(y_1, y_2, ..., y_n),\\ \frac{dy_2}{dt} &= f_2(y_1, y_2, ..., y_n),\\ \dots & \quad \dots\\ \frac{dy_n}{dt} &= f_n(y_1, y_2, ..., y_n)\end{aligned}$$

we define the steady-state as a vector $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, ..., \bar{y}_n)$. We look near $\bar{\mathbf{y}}$,

$$\mathbf{y} = \bar{\mathbf{y}} + \varepsilon \mathbf{Y}(t),$$

by inserting this into the differential equations to find,

$$\frac{d\mathbf{Y}}{dt} = J\mathbf{Y}, \tag{2.7}$$

We use the notation J since the matrix we have obtained is referred to as the Jacobian. The Jacobian is a useful concept, and for us one of the uses is to shorten some of the calculations. The Jacobian,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{x_n} \end{bmatrix} \tag{2.8}$$

We show the details for planar systems in the appendix.

2.3 Qualitative Analysis

The last thing that we will review here is a method for understanding the broad behavior of the solutions of differential equations. This does not provide useful information for *quantifiable* predictions. However, it is quite useful for modeling as it can give insight into the general behavior of solutions to models. This often helps diagnose issues with the model that prevent reasonable physical interpretation. Just as linearization does not provide all information (but restricted to starting near a known solution), neither does qualitative analysis.

We will see that qualitative analysis is almost always restricted to scalar or planar systems and it is useful to look at each one separately. We start with scalar equations,

$$\frac{dy}{dt} = f(y).$$

We will again restrict our discussion to autonomous equations and note that we are not imposing initial conditions. Qualitative analysis provides information for the behavior for all initial conditions.

We can consider the graph of the function $f(y)$ (see Figure 2.1). Where f crosses the y-axis, $f(y) = 0$ which means $\frac{dy}{dt}$ is zero there. That is all roots of $f(y)$ are steady-state solutions. As long as f is nice enough (continuous and differentiable), if the sign of f changes between y_1 and y_2, there is a root. That means that $\frac{dy}{dt}$ has to be of one sign between the roots of f. That means that the solution y is either increasing or decreasing on intervals that do not contain a root. We can sketch the direction that the solution moves on the 'phase-line' (see Figure 2.1). This can be used to sketch the solution of the equation as a function of time.

The argument for planar curves is similar but a bit more involved. Consider the system of equations,

$$\begin{aligned} \frac{dx}{dt} &= f(x,y), \\ \frac{dy}{dt} &= g(x,y). \end{aligned}$$

The curves $f(x,y)$ and $g(x,y)$ can be drawn in the (x,y)-plane. Above f, x must be increasing. While below f, x is decreasing. Similarly, the evolution of y depends on which side of g we are. We can also see that the solution curve must be tangent to the vector field (f,g). So

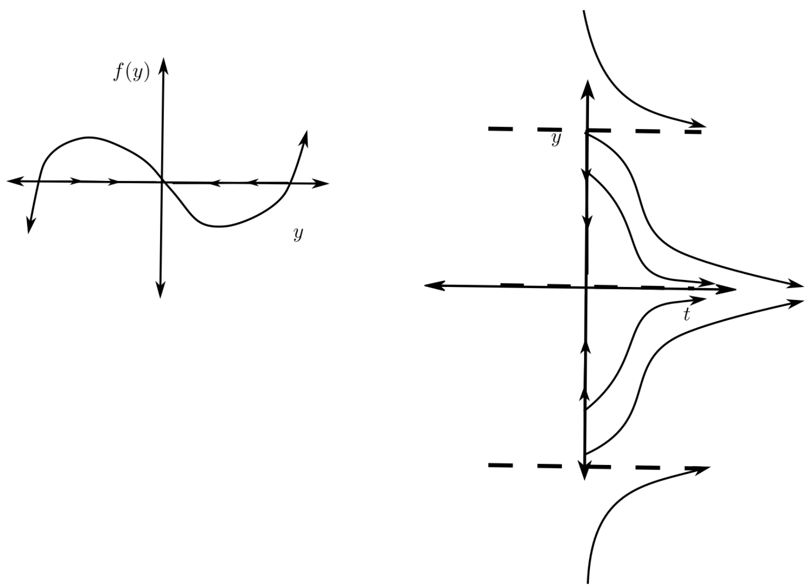

Figure 2.1: Phase line showing how $f(y)$ determines whether y increases or decreases and the qualitative sketch of the solutions, $y(t)$ for different initial conditions.

it is possible to see how trajectories move in the phase-plane – where we think of the solution as a parameterized curve $(x(t), y(t))$ (see Figure 2.2).

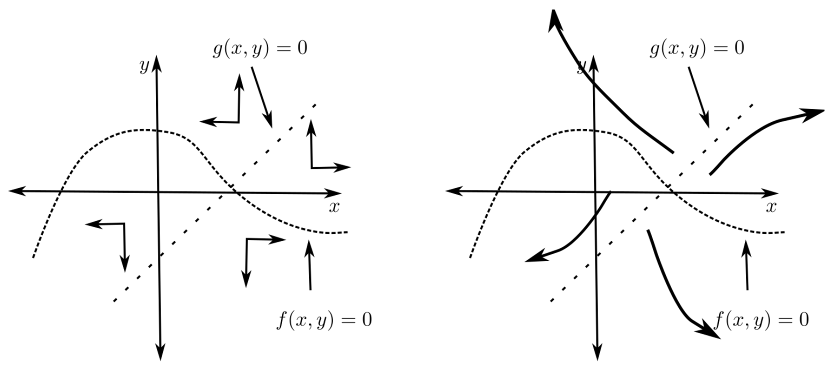

Figure 2.2: Phase plane showing the x and y-nullclines. The direction of increase/decrease in the solution x and y are shown. The same phase plane is shown with a sketch of the trajectories.

2.4 Problems

Problems 2.1 *Classify the following differential equations as linear or nonlinear. Also indicate the order of the differential equation*

(a) $\frac{dy}{dt}+2y=\sin(t)$

(b) $t\frac{d^2y}{dt^2}+\frac{dy}{dt}=\left(\frac{1}{(1+t^3)}\right)-\left(\frac{3t^2}{(1+t^2)}\right)y$

(c) $-\left(\frac{d^2y}{dt^2}\right)^4\frac{dy}{dt}=4$

(d) $4\frac{d^5y}{dt^5}+\cos(t)=0$

(e) $y\frac{d^3y}{dt^3}-t^2\frac{dy}{dt}+y=0$

(f) $t^5\frac{d^2y}{dt^2}+t^2\frac{dy}{dt}+y=\sin(t)$

Problems 2.2 *A relation or operator, $F(x)$ is linear if two properties hold: $F(x+y)=F(x)+F(y)$ and $F(cx)=xF(x)$.*

(a) *Show that the derivative operator is linear*

(b) *Suppose that $y_1(t)$ and $y_2(t)$ both solve the equation,*

$$\begin{aligned} \frac{d^2y}{dt^2} + \sin(t)\frac{dy}{dt} + \cos(t)y &= 10, \\ \frac{dy}{dt}(0) &= 1, \\ y(0) &= 1. \end{aligned}$$

Show that $y_1(t) + y_2(t)$ is also a solution.

(c) *Show that a linear combination of any two solutions to the differential equation,*

$$\begin{aligned} \frac{d^2y}{dt^2}\frac{dy}{dt} + y^2 &= 0, \\ \frac{dy}{dt}(0) &= 10, \\ y(0) &= 10. \end{aligned}$$

is not *a solution to the equation.* (Hint: There is no reason to try and solve this equation but see if $y = ay_1 + by_2$ is a solution if y_1 and y_2 are.)

Problems 2.3 *Euler's formula states that* $e^{ix} = \cos(x) + i\sin(x)$.

(a) *Use Euler's identity to show that* $e^{R \pm iI} = e^R \cos(I) \pm ie^R \sin(t)$

(b) *Find two linear combinations of* $y_1(t) = e^R \cos(I) + ie^R \sin(t)$ *and* $y_2(t) = e^R \cos(I) - ie^R \sin(t)$ *that involve either* $\cos(t)$ *or* $\sin(t)$.

Problems 2.4 *The following steps show how to relate the solution to a second order, linear, constant coefficient differential equation with complex eigenvalues to real-valued solutions.*

Consider the differential equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0$.

(a) *Find the eigenvalues*

(b) *Use Euler's formula to write the two solutions in terms of* $\cos(\alpha t)$ *and* $\sin(\alpha t)$.

(c) *Show that the real and complex part of the solutions are themselves solutions.*

Problems 2.5 *The eigenvalues of a 2×2 matrix,*

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Can be found by finding the roots of the determinant of $A - \lambda I$, where I is the 2×2 identity matrix,

(a) *Find the eigenvalues of*

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$$

(b) *Show that the eigenvalues of the matrix associated with Equation 2.2 are the same as the roots of the characteristic polynomial of $\frac{d^2y}{dt^2} = -\frac{b}{a}\frac{dy}{dt} - \frac{c}{a}y$.*

Problems 2.6 *Use separation of variables to solve,*

(a)

$$\begin{aligned} \frac{dy}{dt} &= y(y+2), \\ y(0) &= 1, \end{aligned}$$

(b)

$$\begin{aligned} \frac{dy}{dt} &= y(t+3), \\ y(0) &= 1, \end{aligned}$$

Problems 2.7 (a) *Sketch the graph of a nonlinear function $f(x)$, along with the linearization at a point, x_0.*

(b) *Use the sketch to show how the linearization provides an approximation of the function value at $x_0 + \delta x$.*

Problems 2.8 *Find and classify the steady-states of the following differential equations:*

(a) $\frac{dy}{dt} = y(1-y)$

(b) $\frac{dy}{dt} = -1+x^2$

(c) $\frac{dy}{dt} = \sin(x)$.

Problems 2.9 *Suppose we have the planar system of differential equations,*

$$\begin{aligned}\frac{dx}{dt} &= x - xy,\\ \frac{dy}{dt} &= -y + xy.\end{aligned}$$

(a) Find all equilibiria

(b) Find the Jacobian at the equilibria

(c) Classify the equilibria.

2.5 Appendix: Planar Example

Many of the examples in this book are planar systems where linearization is very well described. We will briefly cover this and show some of the linear algebra details in this context.

Consider a planar system of equations,

$$\begin{aligned}\frac{dx}{dt} &= f(x,y),\\ \frac{dy}{dt} &= g(x,y).\end{aligned} \tag{2.9}$$

We assume that we have found a steady-state, $(\bar{x},\bar{y})$ so that $f(\bar{x},\bar{y}) = g(\bar{x},\bar{y}) = 0$. We consider the behavior near this solution,

$$\begin{aligned}x(t) &= \bar{x} + \varepsilon X(t),\\ y(t) &= \bar{y} + \varepsilon Y(t).\end{aligned}$$

Plugging this into the differential equations, using Taylors' theorem for multivariable functions and dropping the nonlinear terms, we find,

$$\begin{aligned}
\frac{dx}{dt} &= f(x,y), \\
\frac{d\bar{x}+\varepsilon X(t)}{dt} &= f(\bar{x}+\varepsilon X(t),\bar{y}+\varepsilon Y(t)), \\
\varepsilon\frac{dX(t)}{dt} &= f(\bar{x},\bar{y})+\varepsilon\frac{\partial f(\bar{x},\bar{y})}{\partial x}X+\varepsilon\frac{\partial f(\bar{x},\bar{y})}{\partial y}Y, \\
\frac{dX(t)}{dt} &= \frac{\partial f(\bar{x},\bar{y})}{\partial x}X+\frac{\partial f(\bar{x},\bar{y})}{\partial y}Y,
\end{aligned}$$

for the x-component. Similarly,

$$\begin{aligned}
\frac{d\bar{y}+\varepsilon Y(t)}{dt} &= g(\bar{x}+\varepsilon X(t),\bar{y}+\varepsilon Y(t)), \\
\varepsilon\frac{dY(t)}{dt} &= g(\bar{x},\bar{y})+\varepsilon\frac{\partial g(\bar{x},\bar{y})}{\partial x}X+\varepsilon\frac{\partial g(\bar{x},\bar{y})}{\partial y}Y, \\
\frac{dY(t)}{dt} &= \frac{\partial g(\bar{x},\bar{y})}{\partial x}X+\frac{\partial g(\bar{x},\bar{y})}{\partial y}Y,
\end{aligned}$$

for the y-component.

The Jacobian matrix is,

$$J=\begin{bmatrix} \frac{\partial f(\bar{x},\bar{y})}{\partial x} & \frac{\partial g(\bar{x},\bar{y})}{\partial x} \\ \frac{\partial g(\bar{x},\bar{y})}{\partial x} & \frac{\partial g(\bar{x},\bar{y})}{\partial y} \end{bmatrix}.$$

For short-hand, we can write this

$$J=\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

To find the eigenvalues of J we find the roots of the determinant of $J-\lambda I$,

$$\begin{aligned}
\left|\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right| &= 0, \\
(a-\lambda)(d-\lambda)-bc &= 0, \\
\lambda^2-(a+d)\lambda+ad-bc &= 0, \\
\lambda^2-\mathbf{Tr}J\lambda+\mathbf{Det}J &= 0.
\end{aligned}$$

Where $\mathbf{Tr}J$ is the trace of J (the sum of the diagonals) and $\mathbf{Det}J$ is the determinant of J. Therefore, the eigenvalues of the Jacobian are,

$$\lambda + \pm \quad = \quad \frac{\mathbf{Tr}J \pm \sqrt{\mathbf{Tr}J^2 - 4\mathbf{Det}J}}{2}.$$

The components of the Jacobian completely determine whether the real part of the eigenvalues is positive, negative or zero.