Midterm II

MAP 4341 Fall 2007

Full Name:____________________________ Score: __________

Choose 4 of the following 5 questions. Show all of your work for full credit.

1. Grade: yes no

   Solve the wave equation with periodic boundary conditions using any method you wish (separation of variables comes to mind):

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} \\
\frac{\partial u}{\partial x}(0,t) &= 0 \\
u(\pi,t) &= 0 \\
u(x,0) &= f(x) \\
\frac{\partial u}{\partial t}(x,0) &= 0.
\end{aligned}
$$

$$
u = G(t)h(t)
$$

Plugging into the wave eqn

$$
\begin{aligned}
\frac{d^2 G}{dt^2} = -\lambda G \quad &; \quad \frac{d^2 h}{dt^2} = -\lambda h \\
G(0) = 0 \quad &; \quad \frac{dh}{dt} = 0 \\
\frac{dG}{dt}(\pi) \quad &; \quad h(\pi) = 0
\end{aligned}
$$

The Raleigh Quotient shows that the evals of the spatial problem are positive, so the general solution to the spatial problem is $h(x) = c_1 \cos\sqrt{\lambda x} + c_2 \sin\sqrt{\lambda x}$. The boundary conditions imply that $c_2 = 0$ and that the e-vals are $\lambda_n = 2n+1$.

Then the general solution to the time dependent problem is $G = b_1 \cos\sqrt{\lambda t} + b_2 \sin\sqrt{\lambda t}$ and to satisfy the general initial condition you must take a superposition solution:

$$
u(x,t) = \sum_{n=1}^{\infty} b_n \cos\sqrt{\lambda_n t}\cos\sqrt{\lambda_n x} + a_n \sin\sqrt{\lambda_n t}\cos\sqrt{\lambda_n x}
$$

The coefficients are found by evaluating at $t = 0$ and then integrating against $\cos\sqrt{\lambda_n x}$.

2. Grade: yes no

Consider the Sturm-Liouville equation:

$$
\begin{aligned}
\frac{d}{dx}(k(x)\frac{du}{dx}) + \lambda\sigma u &= 0 \qquad (1)\\
\frac{du}{dx}(0) &= 0\\
u(1) &= 0
\end{aligned}
$$

(a) Prove that the eigenfunctions are orthogonal with respect to the weight function σ.

In the book

(b) Prove that the eigenfunctions are unique. hint: Each of these starts by looking at the difference $\int_a^b \phi_n L(\phi_m) - \phi_m L(\phi_n) dx$, where $L = \frac{d}{dx}(k(x)\frac{du}{dx})$ then use Lagrange's or Green's identity.

In the book

3. Grade: yes no

Given the wave equation, with non-constant density $\rho_{min} \le \rho \le \rho_{max}$and tension $T_{min} \le T \le T_{max}$:

$$\begin{aligned}\rho(x)\frac{\partial^2 u}{\partial t^2} &= T(x)\frac{\partial^2 u}{\partial x^2}\\ u(0,t) &= 0\\ u(1,t) &= 0\end{aligned}$$

obtain an upper and (nonzero) lower bound on the lowest eigenvalue.

The Raleigh quotient for the spatial e-val problem:

$$\frac{d^2 h}{dx^2} = \rho/T\lambda h$$

implies that

$$\lambda_1 = min\frac{\int_0^1 \frac{d\phi}{dx}^2 dx}{\int_0^1 \rho(x)/T(x)\phi^2 dx}$$

which is bounded by

$$T_{min}/\rho_{max} min\frac{\int_0^1 \frac{d\phi}{dx}^2 dx}{\int_0^1 \phi^2 dx} \le min\frac{\int_0^1 \frac{d\phi}{dx}^2 dx}{\int_0^1 \rho(x)/T(x)\phi^2 dx} \le T_{max}/\rho_{min} min\frac{\int_0^1 \frac{d\phi}{dx}^2 dx}{\int_0^1 \phi^2 dx}$$

.

But

$$min\frac{\int_0^1 \frac{d\phi}{dx}^2 dx}{\int_0^1 \phi^2 dx}$$

is the minimum eval of the standard problem:

$$\frac{d^2 h}{dx^2} = \lambda h$$

which is π^2 (for the given boundary conditions).

4. Grade: yes no

Use the Rayleigh quotient to obtain an upper bound for the lowest eigenvalue of:

$$\begin{aligned}
\frac{d}{dx}((x+1)\frac{du}{dx}) + (\lambda - (1-x))\phi &= 0 \\
\frac{du}{dx}(0) + u(0) &= 0 \\
\frac{du}{dx}(1) &= 0
\end{aligned}$$

A test function is found by considering $u_t = ax^2 + bx + c$. Then boundary conditions imply that:

$$\begin{aligned}
b + c &= 0 \\
2a + b &= 0.
\end{aligned}$$

So $c = -b = -2a$ and $u_t = ax^2 - 2ax + 2a$. Plug into the raleigh quotient. Note that the boundary terms do NOT drop out.

5. Grade: yes   no

(a) Show that $\phi(x+ct)$ and $\psi(x-ct)$ both satisfy the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= \frac{d\phi}{dz}\frac{\partial z}{\partial t} \\
&= c\frac{d\phi}{dz} \\
\frac{\partial^2 \phi}{\partial t^2} &= c\frac{d}{dz}(\frac{d\phi}{dz})\frac{\partial z}{\partial t} \\
&= c^2\frac{d^2\phi}{dz^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{d\phi}{dz}\frac{\partial z}{\partial x} \\
&= \frac{d\phi}{dz} \\
\frac{\partial^2 \phi}{\partial x^2} &= \frac{d}{dz}(\frac{d\phi}{dz})\frac{\partial z}{\partial x} \\
&= \frac{d^2\phi}{dz^2}
\end{aligned}$$

So ϕ satisfies the wave equation.

$$\begin{aligned}
\frac{\partial \psi}{\partial t} &= \frac{d\psi}{dz}\frac{\partial z}{\partial t} \\
&= -c\frac{d\psi}{dz} \\
\frac{\partial^2 \psi}{\partial t^2} &= c\frac{d}{dz}(\frac{d\psi}{dz})\frac{\partial z}{\partial t} \\
&= c^2\frac{d^2\psi}{dz^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi}{\partial x} &= \frac{d\psi}{dz}\frac{\partial z}{\partial x} \\
&= \frac{d\psi}{dz} \\
\frac{\partial^2 \psi}{\partial x^2} &= \frac{d}{dz}(\frac{d\psi}{dz})\frac{\partial z}{\partial x} \\
&= \frac{d^2\psi}{dz^2}
\end{aligned}$$

(b) Show that $\psi(x,t) = (2t+3x)^2$ is also a solution of the wave equation for a particular value of c the wave-speed.

$$\begin{aligned} \frac{\partial^2 \psi}{\partial t^2} &= 8 \\ \frac{\partial^2 \psi}{\partial x^2} &= 18 \end{aligned}$$

So if $c^2 = 8/18$ the wave equation is satisfied (i.e. the wave speed is $\sqrt{8/18} = 2/3$.

(c) Sketch the graph of $f(x+ct)$, where $f(z) = x^2$. It is a parabola centered at the origin at time zero. As time increases, the parabola moves to the LEFT with speed c.

In three dimensions, with $z = f(x,t)$, it is a parabolic cylinder.