Choose 4 of the following 5 questions. Show all of your work for full credit.

1. Grade: yes no

Solve the wave equation with periodic boundary conditions using any method you wish (separation of variables comes to mind):

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}
\]

\[
\frac{\partial u}{\partial x}(0, t) = 0
\]

\[
u(\pi, t) = 0
\]

\[
u(x, 0) = f(x)
\]

\[
\frac{\partial u}{\partial t}(x, 0) = 0.
\]

\[
u = G(t)h(t)
\]

Plugging into the wave eqn

\[
\frac{d^2 G}{dt^2} = -\lambda G \quad ; \quad \frac{d^2 h}{dt^2} = -\lambda h
\]

\[
G(0) = 0 \quad ; \quad \frac{dh}{dt} = 0
\]

\[
\frac{dG}{dt}(\pi) \quad ; \quad h(\pi) = 0
\]

The Raleigh Quotient shows that the evals of the spatial problem are positive, so the general solution to the spatial problem is \( h(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x \). The boundary conditions imply that \( c_2 = 0 \) and that the e-vals are \( \lambda_n = 2n + 1 \).

Then the general solution to the time dependent problem is \( G = b_1 \cos \sqrt{\lambda} t + b_2 \sin \sqrt{\lambda} t \) and to satisfy the general initial condition you must take a superposition solution:

\[
u(x, t) = \sum_{n=1}^{\infty} b_n \cos \sqrt{\lambda_n} t \cos \sqrt{\lambda_n} x + a_n \sin \sqrt{\lambda_n} t \cos \sqrt{\lambda_n} x
\]

The coefficients are found by evaluating at \( t = 0 \) and then integrating against \( \cos \sqrt{\lambda_n} x \).
Consider the Sturm-Liouville equation:

\[ \frac{d}{dx}(k(x) \frac{du}{dx}) + \lambda \sigma u = 0 \]  

(a) Prove that the eigenfunctions are orthogonal with respect to the weight function \( \sigma \).

(b) Prove that the eigenfunctions are unique. hint: Each of these starts by looking at the difference \( \int_a^b \phi_n L(\phi_m) - \phi_m L(\phi_n) \) \( dx \), where \( L = \frac{d}{dx}(k(x) \frac{du}{dx}) \) then use Lagrange’s or Green’s identity.

In the book
3. Grade: yes  no  

Given the wave equation, with non-constant density \( \rho_{\text{min}} \leq \rho \leq \rho_{\text{max}} \) and tension \( T_{\text{min}} \leq T \leq T_{\text{max}} \):

\[
\rho(x) \frac{\partial^2 u}{\partial t^2} = T(x) \frac{\partial^2 u}{\partial x^2}
\]

\[
u(0, t) = 0
\]

\[
u(1, t) = 0
\]

obtain an upper and (nonzero) lower bound on the lowest eigenvalue.

The Raleigh quotient for the spatial e-val problem:

\[
\frac{d^2 h}{dx^2} = \rho/T \lambda h
\]

implies that

\[
\lambda_1 = \min \frac{\int_0^1 \frac{d\phi}{dx} \frac{d\phi}{dx} dx}{\int_0^1 \rho(x)/T(x) \phi^2 dx}
\]

which is bounded by

\[
T_{\text{min}}/\rho_{\text{max}} \min \frac{\int_0^1 \frac{d\phi}{dx} \frac{d\phi}{dx} dx}{\int_0^1 \rho(x)/T(x) \phi^2 dx} \leq \min \frac{\int_0^1 \frac{d\phi}{dx} \frac{d\phi}{dx} dx}{\int_0^1 \rho(x)/T(x) \phi^2 dx} \leq T_{\text{max}}/\rho_{\text{min}} \min \frac{\int_0^1 \frac{d\phi}{dx} \frac{d\phi}{dx} dx}{\int_0^1 \phi^2 dx}
\]

But

\[
\min \frac{\int_0^1 \frac{d\phi}{dx} \frac{d\phi}{dx} dx}{\int_0^1 \phi^2 dx}
\]

is the minimum eval of the standard problem:

\[
\frac{d^2 h}{dx^2} = \lambda h
\]

which is \( \pi^2 \) (for the given boundary conditions).
4. Grade: yes  no  
Use the Rayleigh quotient to obtain an upper bound for the lowest eigenvalue of:

\[
\frac{d}{dx}((x + 1)\frac{du}{dx}) + (\lambda - (1 - x))\phi = 0
\]

\[
\frac{du}{dx}(0) + u(0) = 0
\]

\[
\frac{du}{dx}(1) = 0
\]

A test function is found by considering \( u_t = ax^2 + bx + c \). Then boundary conditions imply that:

\[
b + c = 0
\]

\[
2a + b = 0.
\]

So \( c = -b = -2a \) and \( u_t = ax^2 - 2ax + 2a \). Plug into the raleigh quotient. Note that the boundary terms do NOT drop out.
5. Grade: yes no

(a) Show that $\phi(x + ct)$ and $\psi(x - ct)$ both satisfy the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{dz} \frac{\partial z}{\partial t}$$
$$= c \frac{d\phi}{dz}$$
$$\frac{\partial^2 \phi}{\partial t^2} = c \frac{d \left( \frac{d\phi}{dz} \right)}{dz} \frac{\partial z}{\partial t}$$
$$= c^2 \frac{d^2 \phi}{dz^2}$$

$$\frac{\partial \phi}{\partial x} = \frac{d\phi}{dz} \frac{\partial z}{\partial x}$$
$$= \frac{d\phi}{dz}$$
$$\frac{\partial^2 \phi}{\partial x^2} = \frac{d \left( \frac{d\phi}{dz} \right)}{dz} \frac{\partial z}{\partial x}$$
$$= \frac{d^2 \phi}{dz^2}$$

So $\phi$ satisfies the wave equation.

$$\frac{\partial \psi}{\partial t} = \frac{d\psi}{dz} \frac{\partial z}{\partial t}$$
$$= -c \frac{d\psi}{dz}$$
$$\frac{\partial^2 \psi}{\partial t^2} = c \frac{d \left( \frac{d\psi}{dz} \right)}{dz} \frac{\partial z}{\partial t}$$
$$= c^2 \frac{d^2 \psi}{dz^2}$$

$$\frac{\partial \psi}{\partial x} = \frac{d\psi}{dz} \frac{\partial z}{\partial x}$$
$$= \frac{d\psi}{dz}$$
$$\frac{\partial^2 \psi}{\partial x^2} = \frac{d \left( \frac{d\psi}{dz} \right)}{dz} \frac{\partial z}{\partial x}$$
$$= \frac{d^2 \psi}{dz^2}$$
(b) Show that $\psi(x, t) = (2t + 3x)^2$ is also a solution of the wave equation for a particular value of $c$ the wave-speed.

$$\frac{\partial^2 \psi}{\partial t^2} = 8$$
$$\frac{\partial^2 \psi}{\partial x^2} = 18$$

So if $c^2 = 8/18$ the wave equation is satisfied (i.e. the wave speed is $\sqrt{8/18} = 2/3$).

(c) Sketch the graph of $f(x + ct)$, where $f(z) = x^2$. It is a parabola centered at the origin at time zero. As time increases, the parabola moves to the LEFT with speed $c$.

In three dimensions, with $z = f(x, t)$, it is a parabolic cylinder.