Choose 4 of the following 5 questions. Show all of your work for full credit.

1. Grade: yes  no

Solve the wave equation with periodic boundary conditions using any method you wish (separation of variables comes to mind):

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}
\]

\[
\frac{\partial u}{\partial x}(0, t) = 0
\]

\[
u(\pi, t) = 0
\]

\[
u(x, 0) = f(x)
\]

\[
\frac{\partial u}{\partial t}(x, 0) = 0.
\]

\[
u = G(t)h(t)
\]

Plugging into the wave eqn

\[
\frac{d^2 G}{dt^2} = -\lambda G \quad ; \quad \frac{d^2 h}{dt^2} = -\lambda h
\]

\[
G(0) = 0 \quad ; \quad \frac{dh}{dt} = 0
\]

\[
\frac{dG}{dt}(\pi) \quad ; \quad h(\pi) = 0
\]

The Raleigh Quotient shows that the evals of the spatial problem are positive, so the general solution to the spatial problem is \( h(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x \). The boundary conditions imply that \( c_2 = 0 \) and that the e-vals are \( \lambda_n = (2n + 1)/2 \).

Then the general solution to the time dependent problem is \( G = b_1 \cos \sqrt{\lambda}t + b_2 \sin \sqrt{\lambda}t \) and to satisfy the general initial condition you must take a superposition solution:

\[
u(x, t) = \sum_{n=1}^{\infty} b_n \cos \sqrt{\lambda_n}t \cos \sqrt{\lambda_n}x + a_n \sin \sqrt{\lambda_n}t \cos \sqrt{\lambda_n}x
\]

The coefficients are found by evaluating at \( t = 0 \) and then integrating against \( \cos \sqrt{\lambda_n}x \).
Consider the Sturm-Liouville equation:

\[
\frac{d}{dx} \left( k(x) \frac{du}{dx} \right) + \lambda \sigma u = 0 \quad (1)
\]

\[
\frac{du}{dx}(0) = 0
\]

\[
u(1) = 0
\]

(a) Prove that the eigenfunctions are orthogonal with respect to the weight function \( \sigma \).

(b) Prove that the eigenfunctions are unique. hint: Each of these starts by looking at the difference \( \int_a^b \phi_n L(\phi_m) - \phi_m L(\phi_n) dx \), where \( L = \frac{d}{dx} \left( k(x) \frac{du}{dx} \right) \) then use Lagrange’s or Green’s identity.

In the book
3. Grade: yes  no

Given the wave equation, with non-constant density $\rho_{\text{min}} \leq \rho \leq \rho_{\text{max}}$ and tension $T_{\text{min}} \leq T \leq T_{\text{max}}$:

$$\rho(x) \frac{\partial^2 u}{\partial t^2} = T(x) \frac{\partial^2 u}{\partial x^2}$$

$$u(0, t) = 0$$

$$u(1, t) = 0$$

obtain an upper and (nonzero) lower bound on the lowest eigenvalue.

The Raleigh quotient for the spatial e-val problem:

$$\frac{d^2 h}{dx^2} = \rho/T\lambda h$$

implies that

$$\lambda_1 = \min \frac{\int_0^1 \frac{d\phi^2}{dx} \, dx}{\int_0^1 \frac{\rho(x)}{T(x)} \phi^2 \, dx}$$

which is bounded by

$$T_{\text{min}}/\rho_{\text{max}} \min \frac{\int_0^1 \frac{d\phi^2}{dx} \, dx}{\int_0^1 \phi^2 \, dx} \leq \min \frac{\int_0^1 \frac{d\phi^2}{dx} \, dx}{\int_0^1 \rho(x)/T(x) \phi^2 \, dx} \leq T_{\text{max}}/\rho_{\text{min}} \min \frac{\int_0^1 \frac{d\phi^2}{dx} \, dx}{\int_0^1 \phi^2 \, dx}$$

But

$$\min \frac{\int_0^1 \frac{d\phi^2}{dx} \, dx}{\int_0^1 \phi^2 \, dx}$$

is the minimum eval of the standard problem:

$$\frac{d^2 h}{dx^2} = \lambda h$$

which is $\pi^2$ (for the given boundary conditions).
4. Grade: yes  no
Use the Rayleigh quotient to obtain an upper bound for the lowest eigenvalue of:
\[
\frac{d}{dx}((x + 1) \frac{du}{dx}) + (\lambda - (1 - x))\phi = 0
\]
\[
\frac{du}{dx}(0) + u(0) = 0
\]
\[
\frac{du}{dx}(1) = 0
\]

A test function is found by considering \( u_t = ax^2 + bx + c \). Then boundary conditions imply that:
\[
b + c = 0
\]
\[
2a + b = 0.
\]
So \( c = -b = -2a \) and \( u_t = ax^2 - 2ax + 2a \). Plug into the raleigh quotient. Note that the boundary terms do NOT drop out.
5. Grade: yes no

(a) Show that \( \phi(x + ct) \) and \( \psi(x - ct) \) both satisfy the wave equation:

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

\[
\frac{\partial \phi}{\partial t} = \frac{d\phi}{dz} \frac{\partial z}{\partial t}
= \frac{dc}{dz} \frac{d\phi}{dz}
\]

\[
\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{d}{dz} \frac{d}{dz} \frac{d\phi}{dz} \frac{\partial z}{\partial t}
= c^2 \frac{d^2 \phi}{dz^2}
\]

\[
\frac{\partial \phi}{\partial x} = \frac{d\phi}{dz} \frac{\partial z}{\partial x}
= \frac{d\phi}{dz}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{d}{dz} \frac{d}{dz} \frac{d\phi}{dz} \frac{\partial z}{\partial x}
= \frac{d^2 \phi}{dz^2}
\]

So \( \phi \) satisfies the wave equation.

\[
\frac{\partial \psi}{\partial t} = \frac{d\psi}{dz} \frac{\partial z}{\partial t}
= -c \frac{d\psi}{dz}
\]

\[
\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{d}{dz} \frac{d}{dz} \frac{d}{dz} \frac{d\psi}{dz} \frac{\partial z}{\partial t}
= c^2 \frac{d^2 \psi}{dz^2}
\]

\[
\frac{\partial \psi}{\partial x} = \frac{d\psi}{dz} \frac{\partial z}{\partial x}
= \frac{d\psi}{dz}
\]

\[
\frac{\partial^2 \psi}{\partial x^2} = \frac{d}{dz} \frac{d}{dz} \frac{d}{dz} \frac{d\psi}{dz} \frac{\partial z}{\partial x}
= \frac{d^2 \psi}{dz^2}
\]
(b) Show that \( \psi(x, t) = (2t + 3x)^2 \) is also a solution of the wave equation for a particular value of \( c \) the wave-speed.

\[
\frac{\partial^2 \psi}{\partial t^2} = 8 \\
\frac{\partial^2 \psi}{\partial x^2} = 18
\]

So if \( c^2 = 8/18 \) the wave equation is satisfied (i.e. the wave speed is \( \sqrt{8/18} = 2/3 \).

(c) Sketch the graph of \( f(x + ct) \), where \( f(z) = x^2 \). It is a parabola centered at the origin at time zero. As time increases, the parabola moves to the LEFT with speed \( c \).

In three dimensions, with \( z = f(x, t) \), it is a parabolic cylinder.