Pricing Digital Options With High Resolution Schemes

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Abstract

High order finite difference methods, which are standard techniques in the computational finance literature, fail to handle the discontinuities in the payoff functions of digital options when solving the Black-Scholes-Merton partial differential equation numerically. Finite difference solutions produce spurious oscillations in the neighborhood of the discontinuities, which make the numerical option prices impractical for hedging. We extend the linear finite difference methods to overcome these difficulties by developing high resolution non-linear schemes that resolve discontinuities and facilitate pricing and hedging digital options with higher accuracy.

1. Digital Options

Digital options are exotic options traded in the over-the-counter (OTC) market between financial institutions. They are called digital or binary because they either pay the full obligation (1) or they pay nothing (0). Digital options have two main styles: The European style where the option can be exercised at the expiry date only, and the American style where the option may be exercised at any time before the expiry date. The main two types of digital options are the cash-or-nothing which pays one unit of money if the option expires in-the-money, and the asset-or-nothing digital options, which pays the value of the underlying asset if the option expires in-the-money.

2. Black-Scholes-Merton Model

If an investor invests in a stock and in a money market account, then the portfolio value \( V(t) \) at time \( t \) of this investment should agree with the value of the option \( c(t, S(t)) \) at every time \( t \), or

\[
\frac{dV}{dt} = rV - \frac{\partial c}{\partial t} - \frac{\partial c}{\partial S} \frac{dS}{dt} = 0
\]

Starting from this argument, and using the Itô-Doeblin formula for Brownian motion to differentiate both sides of (1) we get both the delta-hedging rule and the Black-Scholes-Merton partial differential equation (BSM PDE)

\[
&c_t(t, S(t)) + r c_S(t, S(t)) = \frac{1}{2} \sigma^2 c_{SS}(t, S(t))
\]

This PDE is a backward parabolic equation which has in the case of European Digital call option

\[
&\text{The terminal condition}
\]

\[
&\lim_{x \to \infty} c(t, x) = 1, \quad \lim_{x \to -\infty} c(t, x) = 0
\]

3. Conservation Laws

Let's consider the initial value problem

\[
\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0
\]

where \( u(x, t) \) is the unknown conserved quantity, \( f(u) \) is the flux, and \( -\infty < x < \infty, \quad t \geq 0 \). We say that (4) is a conservation law because if we integrate it over the domain \( -\infty < x < \infty \), we get

\[
\frac{d}{dt} \int_{-\infty}^{\infty} u(x, t) dx = 0
\]

assuming that \( f(u) \) vanishes as \( x \to \infty \). If we integrate the flux function over a fixed domain \( \Omega = [a, b] \), the time of the amount of \( u \) in that interval is

\[
\int_{a}^{b} \frac{df}{dx} \Delta x = \int_{u(a)}^{u(b)} \Delta f
\]

where \( u_b = u(x, \tilde{y}) \) and \( u_c = u(x, \tilde{z}) \). For the linear advection diffusion equation, the flux is

\[
\text{flux} = au - \alpha \frac{du}{dx}
\]

If the numerical solution for (4) converges everywhere to a function \( w(x, t) \), then it is guaranteed that \( w(x, t) \) is a weak solution to (4) as Lax and Wendroff Theorem says, and it satisfies Rankine-Hugoniot conditions when discontinuities exist.

4. Finite Volume Formulation of Schemes and Limiters

All linear monotone schemes for the convection equation are necessarily first order accurate. The only way to get non-oscillatory higher order schemes is to move away from linear numerical schemes (monotonicity conditions) and introduce non-linear correction terms called the limiters (use total variation diminishing concept).

We consider a uniform grid where \( x_j = j \Delta x \) and \( c_n = n \Delta t \).

In finite difference schemes, the discretized value of the function at the spatial point \( i \) and at time \( t \) is

\[
u^n_i = u(x, f(t, x))
\]

\[
u^n_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t) dx
\]

In (9), \( x_{i-1/2} \) and \( x_{i+1/2} \) are called the cell faces. One defines the numerical flux, \( F_i^n \), at the cell face \( i+1/2 \) to be

\[
F_{i+1/2} = F_{i+1} - \frac{1}{\Delta x} \int_{x_{i+1/2}}^{x_{i+1}} u(x, t) dx
\]

High order accuracy schemes can be found by piecewise interpolation of the fluxes. An example is the \( \sigma \)-scheme, which has a four grid point support \( i-2, i-1, i, i+1 \):

\[
F_{i+1/2} = \frac{1}{4} (1 + \sigma)(F_i - F_{i+1}) + \frac{1}{4} (1 + \sigma)(F_{i+1} - F_i)
\]

where different values of \( \sigma \) give first, second and third order accurate schemes; Table 1 shows \( \sigma \) and \( \alpha \) values for some of these schemes.

5. Deriving the Black Scholes Merton PDE in Flux Form

\[
e(x, t) = \frac{\partial}{\partial x} \left[ e(x, t) \right] = -\frac{\sigma^2}{2} e(x, t) + r e(x, t)
\]

By doing some algebraic manipulations, we get

\[
e(x, t) = \left( e^2 - 1 \right) F_i + \left( \frac{1}{2} \sigma^2 \right) F_{i+1} - q c_i c_{i+1}
\]

We write the approximation of the flux at the cell \( i \) in Equation (10) as

\[
\frac{\partial}{\partial x} \left[ c_i \right] = -\frac{\partial}{\partial x} \left[ e(x, t) \right] = \frac{F_{i+1} - F_i}{\Delta x}
\]

6. First Order Scheme

\[
F_{i+1/2} = F_i + \frac{1}{2} \left[ e_i + e_{i+1} \right] (F_{i+1/2} - F_i)
\]

We can get \( F_{i+1/2} \) similarly by replacing \( i \) by \( i-1 \) in \( F_{i+1/2} \) where

\[
\cdot r = \frac{c_{i+1} - c_i}{\Delta t}
\]

\[
\cdot q = \frac{c_{i+1} - c_i}{\Delta x}
\]

\[
\cdot e_i \text{ is the limiter function like van Leer limiter: } e_i = \frac{e_{i+1} + e_i}{2}
\]

Let’s consider the change in the numerical flux through the cell \( i \) at time \( n \)

\[
F_i^n - \frac{1}{\Delta t} \left[ e_i^n - e_{i+1}^n \right] < 0
\]

7. Towards a Second Order Scheme

\[
e_{i+1/2}^{temp} = e_i - \frac{M}{2} \left( F_i^n + F_{i+1}^n \right)
\]

\[
e_{i+1}^{temp} = e_i - \Delta t \left( \text{Flux}_{i+1/2}^{temp} + q c_i^{temp} \right)
\]

\[
\text{Flux}_{i+1/2}^{temp} = \frac{F_{i+1/2} - F_{i-1/2}}{\Delta x}
\]

8. Preliminary Numerical Results

Figure 1: Rate of Accuracy for the FVM solution with van-Leer limiter (log-log)

Figure 2: Finite Difference Scheme - Central Scheme

Figure 3: Finite volume scheme with van Leer limiter

References