Visualizing the Limiting Behavior of Iterated Conformal Mappings

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Tallahassee, FL

Big Picture

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The main mechanism for the latter will be Curt McMullen's software LIM.

Outline

Preliminaries

- Basic Complex Analysis
- Circles in $\mathbb{C}$ & $\widehat{\mathbb{C}}$
- Conformal Mappings, Linear Fractional Transformations, and the Matrix Groups $\mathrm{PGL}(2,\mathbb{C})$, $\mathrm{PSL}(2,\mathbb{C})$

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Limit Points and Limit Sets

- Definitions & Preliminaries
- Example—Apollonian Gasket

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Part I

Review of Basic Complex Analysis

- The complex number field $\mathbb{C}$ is the collection of all elements of the form $x + iy$ for $x, y \in \mathbb{R}$.

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- $\mathbb{C}$ is algebraically isomorphic to $\mathbb{R}^2$ under the map $(x, y) \longleftrightarrow x + iy$.
- One usually blurs the distinction between $\mathbb{C}$ and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the latter of which is useful for geometry.

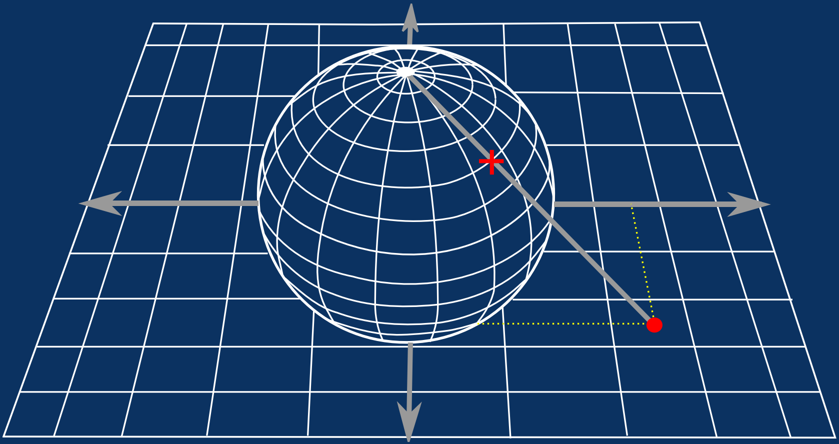

Figure 1

The identification of $\mathbb{C}$ (the plane) with $\widehat{\mathbb{C}}$ (the sphere) via *stereographic projection*

Review of Less-Basic Complex Analysis

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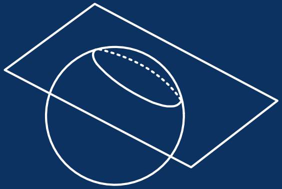

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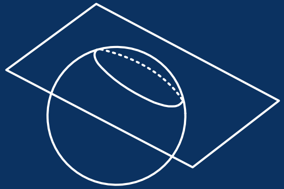

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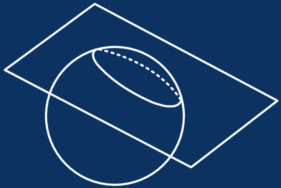

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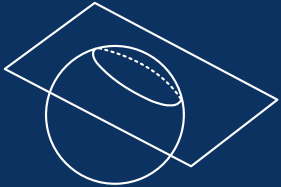

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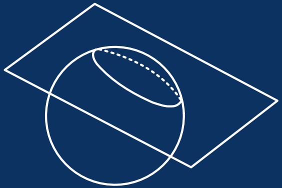

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  This fact will be important later when discussing the particulars of $\textsc{Lim}$.

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Definition—Linear Fractional Transformation

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- Every linear fractional transformation maps generalized circles to generalized circles.

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I really want to talk to you about iterating matrix multiplication for certain collections (subgroups) of matrices.

Part II

Definitions

Definitions

Definition—Limit Point

A point $\zeta \in \widehat{\mathbb{C}}$ is called a **limit point** of group $G = \{g_\alpha\}$ if there exists a point $z \in \widehat{\mathbb{C}}$ and a sequence of elements $\{g_i\}_{i=1}^{\infty}$ in G so that $g_i z \to \zeta$ as $i \to \infty$.

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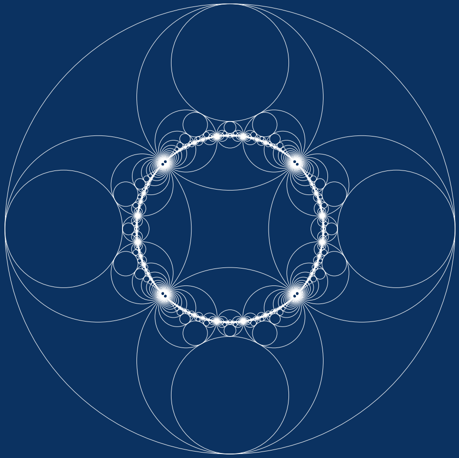

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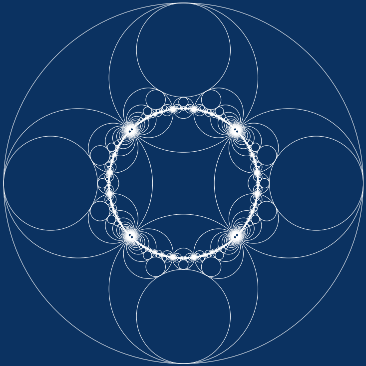

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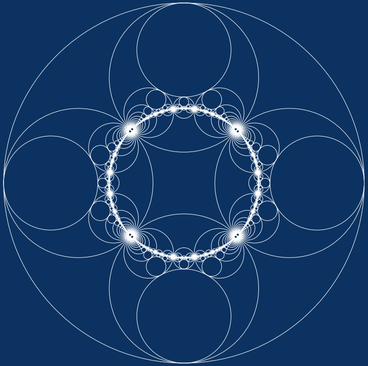

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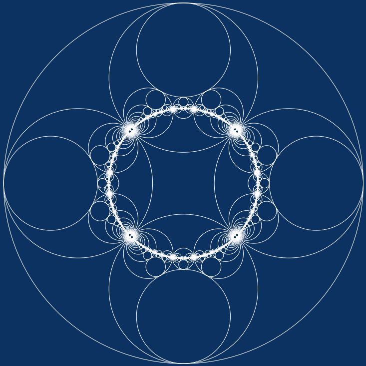

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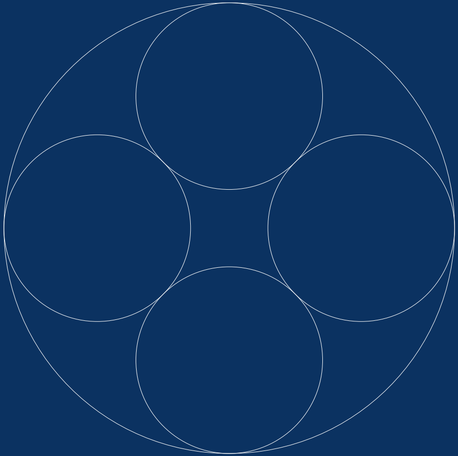

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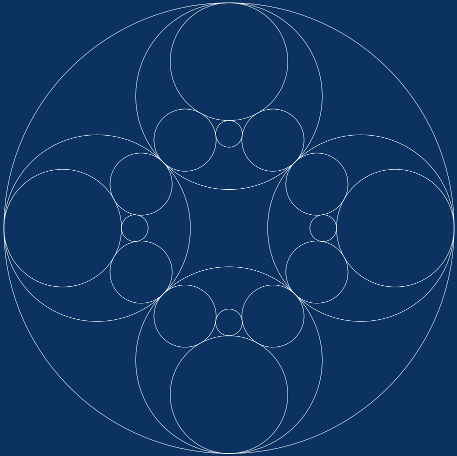

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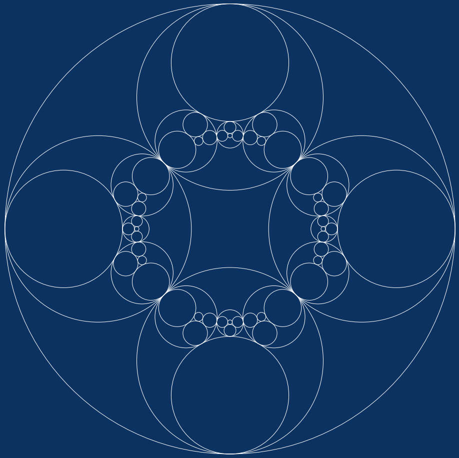

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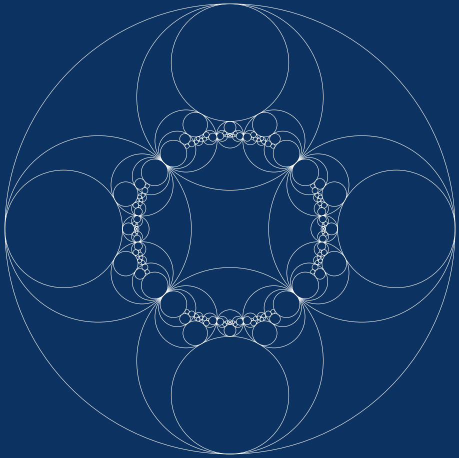

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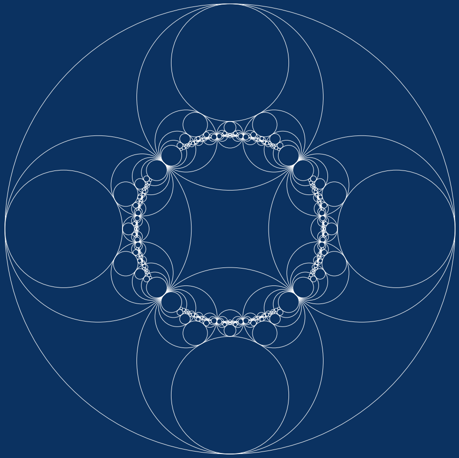

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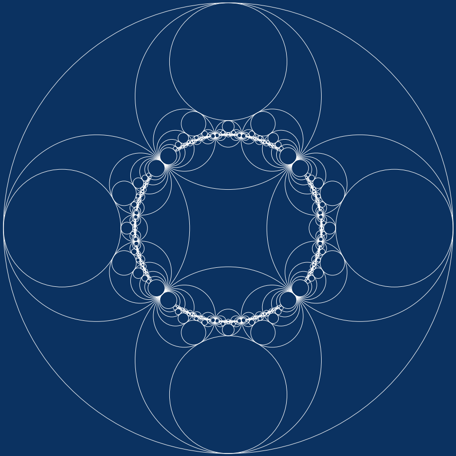

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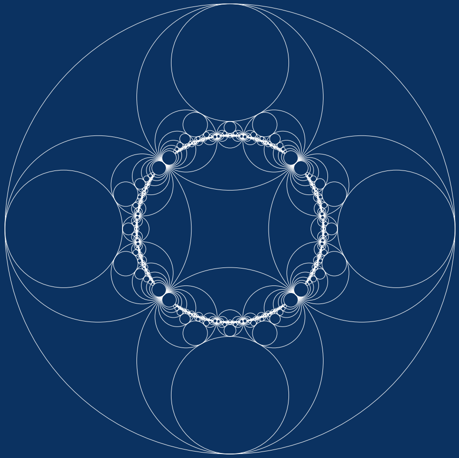

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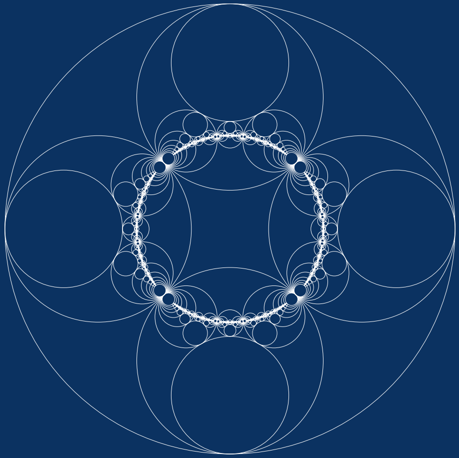

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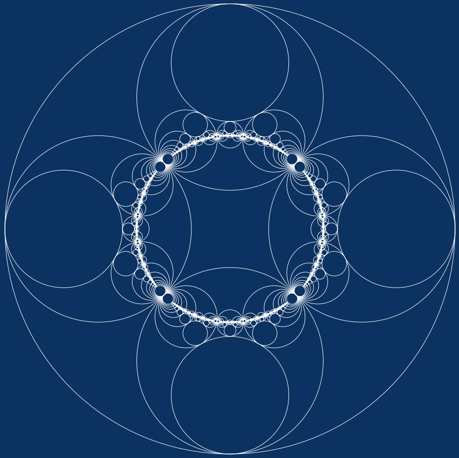

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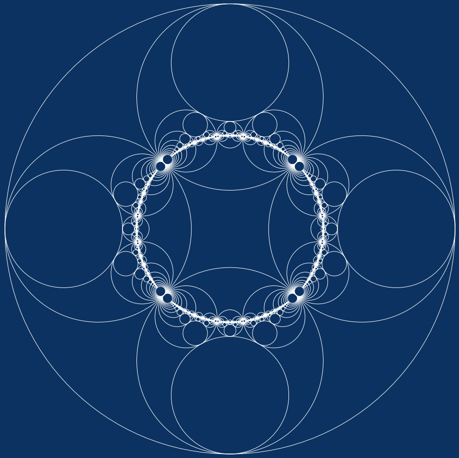

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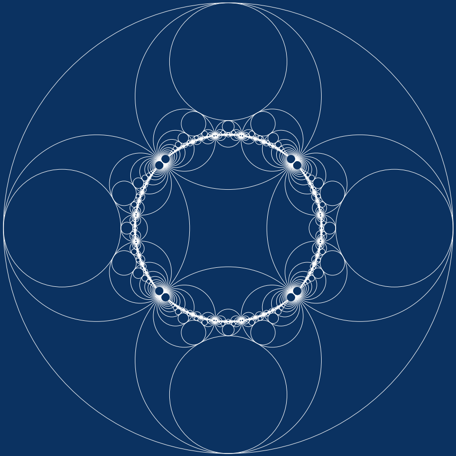

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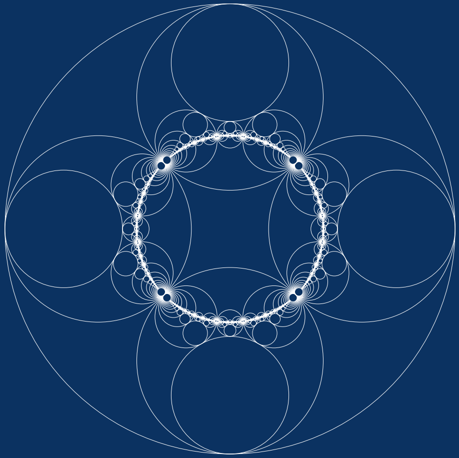

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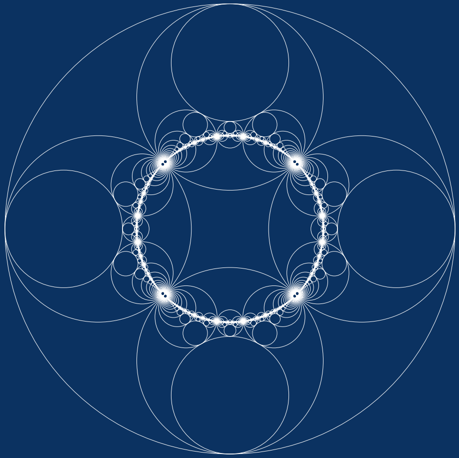

The Take-Away

What Does the Apollonian Gasket Tell Us?

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It tells us that all those math words from before let us create pretty pictures!

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...sometimes...

...and for that, we appeal to Curt McMullen!

Part III

Preliminaries

- Basic Complex Analysis
- Circles in $\mathbb{C}$ & $\widehat{\mathbb{C}}$
- Conformal Mappings, Linear Fractional Transformations, and the Matrix Groups $\mathrm{PGL}(2,\mathbb{C})$, $\mathrm{PSL}(2,\mathbb{C})$

Limit Points and Limit Sets

- Definitions & Preliminaries
- Example—Apollonian Gasket

Curt McMullen's LIM

- Introduction to LIM
- Some Technical Stuff
- Examples and Output
  - Example 1: Hex † Example 2: Maskit's Teichmüller Embedding
  - Example 3: Schottky Group † Example 4: Hyperbolic Tiling
  - Example 5: Lattice
- Conclusions

The LIM Program

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McMullen's description in the "Read Me" file:

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> *Limit Sets of Kleinian Groups*
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> *The program lim draws the orbits of circles under the action of a group of Möbius transformations.*

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Definitions—Kleinian Group; Möbius Transformation

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Limit Sets of Kleinian Groups

The program lim draws the orbits of circles under the action of a group of Möbius transformations.

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- A **Kleinian Group** is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

The LIM Program

McMullen's description in the "Read Me" file:

> *Limit Sets of Kleinian Groups*
>
> *The program lim draws the orbits of circles under the action of a group of Möbius transformations.*

Definitions—Kleinian Group; Möbius Transformation

- A **Kleinian Group** is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.
- A **Möbius Transformation** is just a linear fractional transformation.

How It Works—Short Version

Required Input

- Circles $c_1, \ldots, c_i$ known to be in the limit set

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- Output style options

How It Works—Short Version

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Optional Input

- Circles $r_1, \ldots, r_k$ in which to define reflections for c_α

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- Matrices $m_1, m_2, \ldots, m_j$, $t_1, \ldots, t_\ell \in \mathrm{PSL}(2, \mathbb{C})$ to be applied to the c_α and to the coordinate system, respectively

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- Circles $u_1, \dots, u_n$ in which to define reflections of the coordinate system

How It Works—Short Version

Behind the Scenes

- LIM applies the group

  $$G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle$$

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- The loop ends when the stacks are full or when optional user-input thresholds are reached.

Output

- The raw output is data in .ps format.
- This can be converted to .pdf images.

Example 1

hex.run[1]

```
./lim -d 8 -s -h 3 <<eof > hex.ps
c 0.866025403784438 0.0 -0.5
c 0.25 0.433012701892219 -0.166666666666
c -0.25 0.433012701892219 -0.833333333333
r 0.866025403784438 0.0 -0.5
r 0.25 0.433012701892219 -0.166666666666
r -0.25 0.433012701892219 -0.833333333333
eof
```

[1] Graph on sphere; omit to graph in plane
Output file name
Two different threshold variables

Example 1

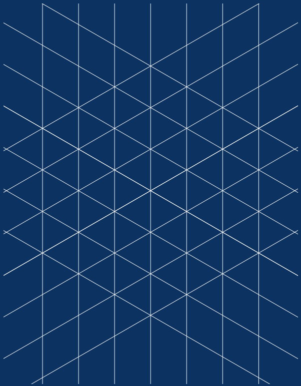
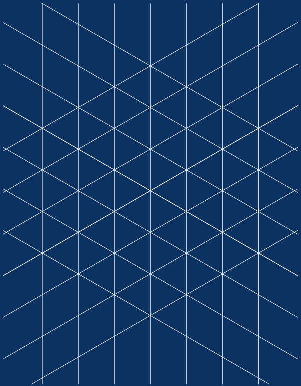


Figure 2

`hex.ps` without -s

Example 1



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`hex.ps` without -s

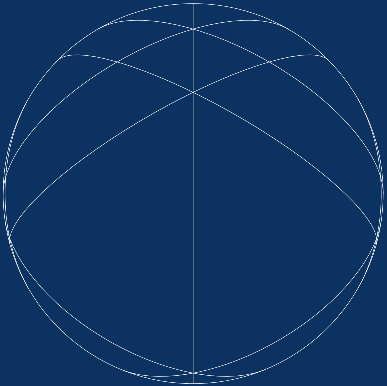

Figure 3
`hex.ps` with -s

Example 2

Example.run[2]

```
./lim -s -d 60 -e 0.0001 <<eof > Example.ps
c 0.  0.  1
m 1 1 0 1 0 -1 1 -1
m 1 -1 0 -1 0 1 1 1
m 0.955 -0.025 0.045 0.025 -1.955 0.025 0.955 -0.025
m 0.955 -0.025 -0.045 -0.025 1.955 -0.025 0.955 -0.025
eof
```

[2] A different threshold variable

Example 2

Example.run[2]

```
./lim -s -d 60 -e 0.0001 <<eof > Example.ps
c 0.  0.  1
m 1 1 0 1 0 -1 1 -1
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m 0.955 -0.025 0.045 0.025 -1.955 0.025 0.955 -0.025
m 0.955 -0.025 -0.045 -0.025 1.955 -0.025 0.955 -0.025
eof
```

Remark

[2] A different threshold variable

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m 0.955 -0.025 0.045 0.025 -1.955 0.025 0.955 -0.025
m 0.955 -0.025 -0.045 -0.025 1.955 -0.025 0.955 -0.025
eof
```

Remark

According to McMullen: “This [corresponds to] a picture of the limit set of a Kleinian group on the boundary of Maskit’s embedding of the Teichmuller space of a once-punctured torus.”

[2] A different threshold variable

Example 2

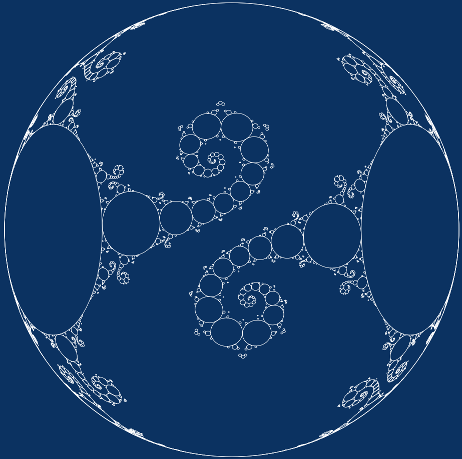

Figure 4

`Example.ps`

Example 3

Schottky.run

```
./lim -d 10 -e .001 <<eof > schottky2.ps
r 0 1 .7
r 0.866025 -.5 .8
r -0.866025 -.5 .8
c 0 1 .7
c 0.866025 -.5 .8
c -0.866025 -.5 .8
eof
```

Example 3

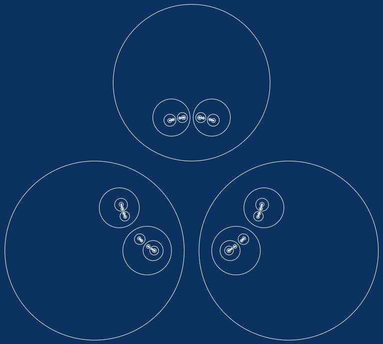

Figure 5
In the plane

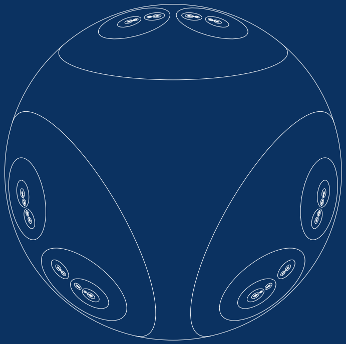

Figure 6
On the sphere

Example 4

ngon4.run[3]

```
./lim -a 1000 -b -d 100 -e 0.001
    -c 0 0 1 -w -1.1 -1.1 1.1 1.1
    <<eof > ngon4.ps
r 1.553773974030037 0 1.189207115002721
c 1.553773974030037 0 1.189207115002721
r 0 1.553773974030037 1.189207115002721
c 0 1.553773974030037 1.189207115002721
r -1.553773974030037 0 1.189207115002721
c -1.553773974030037 0 1.189207115002721
r 0 -1.553773974030037 1.189207115002721
c 0 -1.553773974030037 1.189207115002721
eof
```

[3] Optional style parameter; A *different* threshold variable; Clipping circle; Window parameters

Example 4

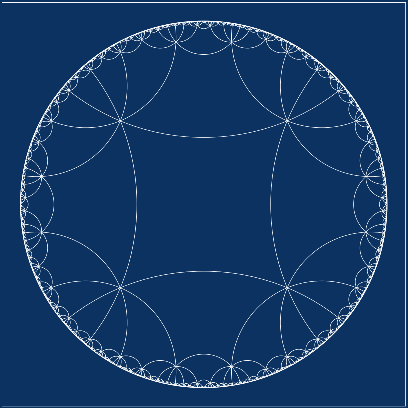

Figure 7
`ngon4.ps` + a box because of -b

Remark:

According to McMullen: "Tiling of $\mathbb{H}$ for torus with orbifold point of order 2."

Example 5

lattice.run

```
./lim -s -d 10 <<eof > lattice.ps
c 0 0 0.0
c 0 0 -.5
m 1 0 1 0 0 0 1 0
m 1 0 0 1 0 0 1 0
m 1 1 0 0 0 0 1 0
u .3 .4 2
eof
```

Example 5

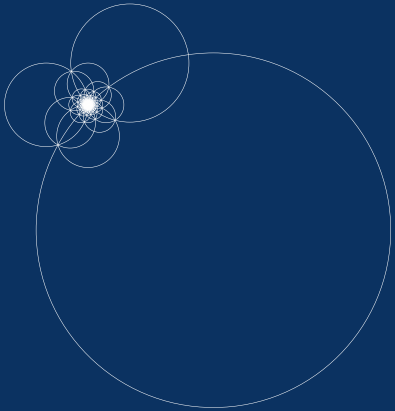

Figure 8
In the plane

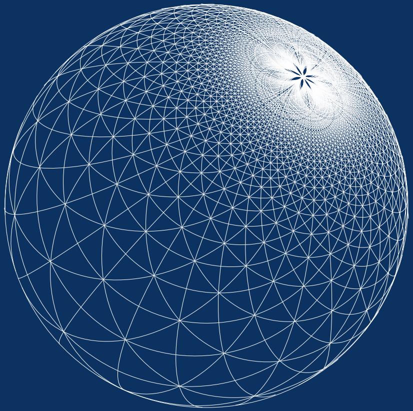

Figure 9
On the sphere

Cue the Applause!

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...so there *are* lots of pretty pictures!

...But There's Always a Caveat...

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...sometimes...

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Worth noticing is that all the examples shown have been carefully constructed from real-world (mathematical) situations.

...But There's Always a Caveat...

...sometimes...

Worth noticing is that all the examples shown have been carefully constructed from real-world (mathematical) situations. In almost every conceivable scenario, analyzing random collections of Möbius transformations yields nothing useful whatsoever!

The Synopsis:

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- McMullen's program is good for what it does...

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- McMullen's program is good for what it does...
- ...but getting useful information requires a considerable amount of pre-existing mathematical knowledge.
- It's also very hard to generalize because of this requisite knowledgeand because of this, attempting to visualize "more advanced" mathematical scenarios will almost certainly require devising something new rather than modifying LIM.

But Even So...

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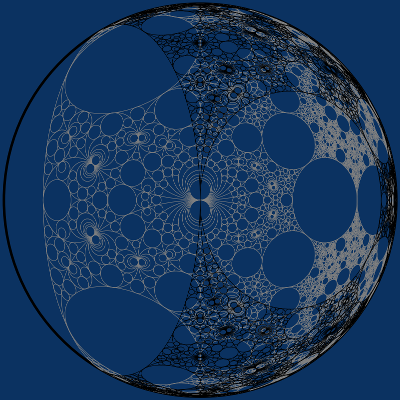

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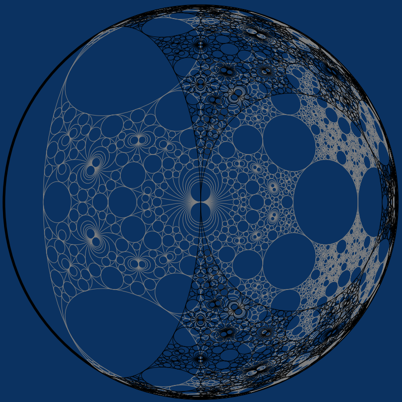

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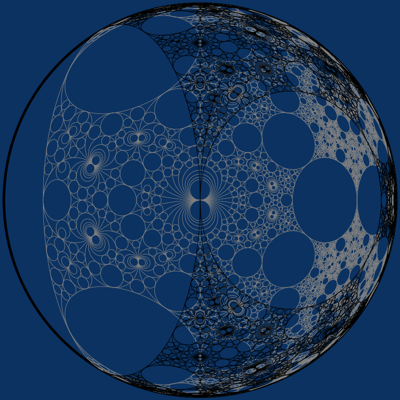

← Pretty pictures

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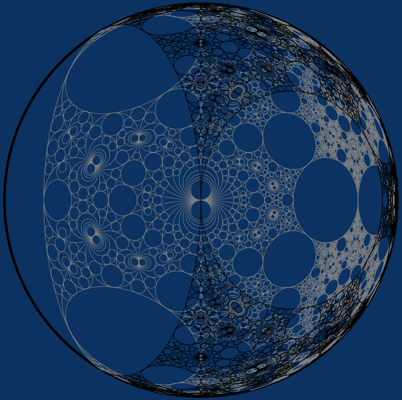
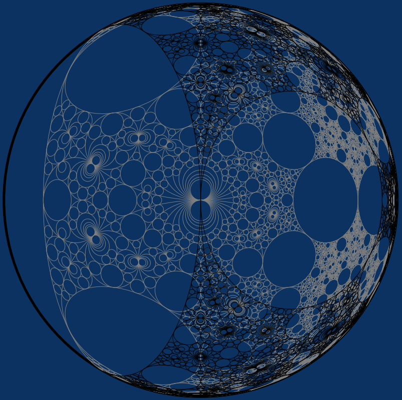


← Pretty pictures!!!!!

But Even So...



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Remark:

Transparency is obtained by first graphing on the sphere with `-s`

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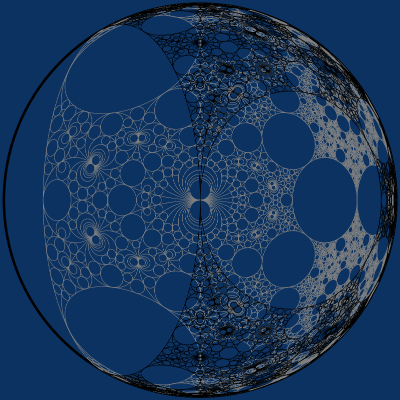

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Remark:

Transparency is obtained by first graphing on the sphere with `-s` and then by adding `-t num_`

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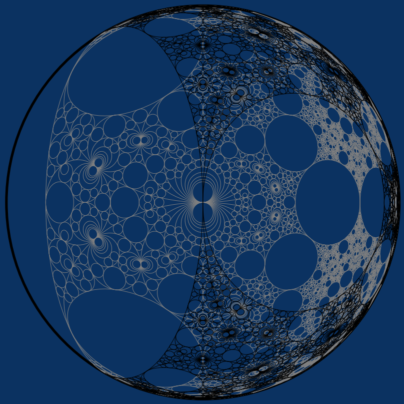

← Pretty pictures!!!!!

Remark:
Transparency is obtained by first graphing on the sphere with -s and then by adding -t num_ where num_ is a decimal value between 0.0 and 1.0, inclusive.

Thank you!