The Quest for Reebless Foliations in Sutured 3-Manifolds

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Outline

Section I

Recall [Foliation]

A dimension-k *foliation* of a manifold $M = M^n$ is a decomposition $\mathcal{F}$ of M into disjoint properly embedded submanifolds of dimension k which is locally homeomorphic to the decomposition $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.

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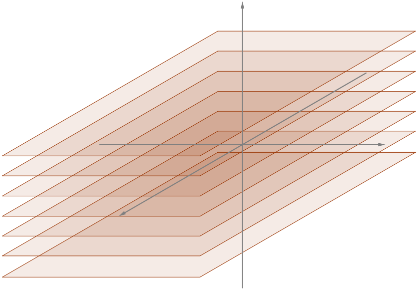

Figure 1
$\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ *for* $n = 3$ *and* $k = 2$

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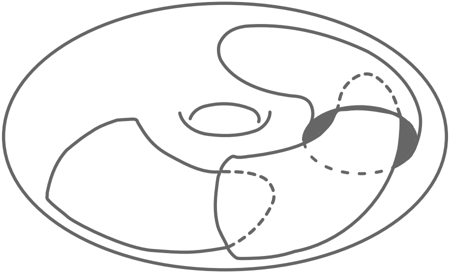

Figure 2
The Reeb foliation of V

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Definition 1.

Under the same assumptions as above, $\mathcal{F}$ is said to be *depth* k if

$$k = \max\{\text{depth}(L) : L \text{ is a leaf of } \mathcal{F}\}.$$

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Define $R_\pm = R_\pm(\gamma)$ to be the components of $R(\gamma)$ whose normal vectors point out of and into M, respectively.

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1. λ is a properly embedded nonseparating arc in γ.
2. λ is a simple closed curve in an annular component A of γ which is in the same homology class as $A \cap s(\gamma)$.
3. λ is a homotopically nontrivial curve in a toral component T of γ so that, if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $H_1(T)$.

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$$(M,\gamma) \xrightarrow{S} (M',\gamma')$$

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- $S_\pm = S'_\pm \cap R_\pm(\gamma')$.

Recall [Sutured Manifold Hierarchy]

A *sutured manifold hierarchy* is a sequence of sutured manifold decompositions

$$(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} (M_2, \gamma_2) \longrightarrow \cdots \xrightarrow{S_n} (M_n, \gamma_n)$$

where $(M_n, \gamma_n) = (R \times I, \partial R \times I)$ and $R_+(\gamma_n) = R \times \{1\}$ for some surface R. Here, $I = [0, 1]$ and R is some surface.

Section II

The Main Theorem

Theorem 1.

Suppose M is connected, and (M,γ) has a sutured manifold hierarchy

$$(M,\gamma) = (M_0,\gamma_0) \xrightarrow{S_1} (M_1,\gamma_1) \xrightarrow{S_2} (M_2,\gamma_2) \longrightarrow \cdots \xrightarrow{S_n} (M_n,\gamma_n)$$

so that no component of $R(\gamma_i)$ is a torus which is compressible. Then there exist transversely-oriented foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of M such that the following conditions hold:

The Main Theorem

Theorem 1 (Cont'd).

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3. If $H_2(M,\gamma) \neq 0$, then every leaf of $\mathcal{F}_0$ and $\mathcal{F}_1$ nontrivially intersects a transverse closed curve or a transverse arc with endpoints in $R(\gamma)$. However, if $\varnothing \neq \partial M \neq R_\pm(\gamma)$, then this holds only for interior leaves.

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4. There are no 2-dimensional Reeb components on $\mathcal{F}_i|\gamma$ for $i = 0, 1$.

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4. There are no 2-dimensional Reeb components on $\mathcal{F}_i|\gamma$ for $i = 0, 1$.
5. $\mathcal{F}_1$ is C^∞ except possibly along toral components of $R(\gamma)$ (if $\partial M \neq \varnothing$) or on S_1 (if $\partial M = \varnothing$).

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6. $\mathcal{F}_0$ is of finite depth.

How Big Is “Big”?

This theorem is remarkable for a lot of reasons, not the least of which are the results it yields (almost) for free.

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A number of then-conjectures involving knots and links also follow as corollaries, as do a number of fundamental results such as the higher-genus Dehn's lemma.

How Does One Prove Such a Thing?

The proof is colossal and requires an enormous amount of work.

Outline of Proof

(O.I) First, "pre-process" the given hierarchy to get a "better-behaved" hierarchy $(M, \gamma) = (M_0, \gamma_0) \xrightarrow{T_1} (M_1, \gamma_1) \longrightarrow \cdots \xrightarrow{T_k} (M_k, \gamma_k)$.

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(O.III–V) Use other results to conclude that $R_\pm(\gamma)$ are norm-minimizing, to construct an "even better-behaved" hierarchy for (M,γ), and to inductively construct $\mathcal{F}^i_{0,1}$ for each level of this new hierarchy. These foliations satisfy all desired conditions.

The Proof

The constructions claimed in (O.II) are the main component of the proof.

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(H2) $\mathcal{F}_1^i$ is C^∞ except possibly along toral components of $\cup_{j=i+1}^{k} T_j \cup R(\gamma_i)$.

(H3) If δ is a curve on a nontoral component of $R(\gamma_i)$ and if $f : [0,\alpha) \to [0,\beta)$ is a representative of the germ of the holonomy map around δ for the foliation $\mathcal{F}_1^i$, then

$$\frac{d^n f}{dt^n}(0) = \begin{cases} 1, & i = 1 \\ 0, & i > 1 \end{cases}$$

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(H4) $\mathcal{F}^i_0$ is of finite depth if, for all $j \geq i$, $V \cap T_{j-1}$ is a union of parallel oriented simple curves for each component V of $R(\gamma_j)$ with $T_{j-1} \cap \partial V \neq \varnothing$.

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(H2) $\mathcal{F}^i_1$ is C^∞ except possibly along toral components of $\cup_{j=i+1}^k T_j \cup R(\gamma_i)$.

(H3) If δ is a curve on a nontoral component of $R(\gamma_i)$ and if $f : [0,\alpha) \to [0,\beta)$ is a representative of the germ of the holonomy map around δ for the foliation $\mathcal{F}^i_1$, then

$$\frac{d^n f}{dt^n}(0) = \begin{cases} 1, & i = 1 \\ 0, & i > 1 \end{cases}$$

(H4) $\mathcal{F}^i_0$ is of finite depth if, for all $j \geq i$, $V \cap T_{j-1}$ is a union of parallel oriented simple curves for each component V of $R(\gamma_j)$ with $T_{j-1} \cap \partial V \neq \varnothing$.

(H5) $\mathcal{F}^i_{0,1}$ has no Reeb components.

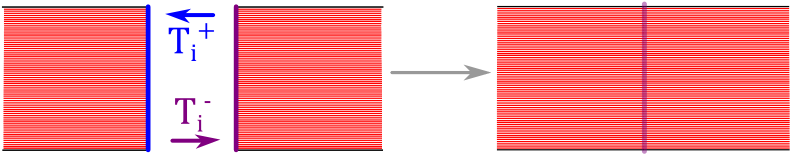
The Proof—The Gluings

Next, the goal is to glue T_i^+ to T_i^- to obtain a manifold Q and to see what needs to happen to the existing foliations $\mathcal{F}_{0,1}^i$ to get the desired foliations $\mathcal{F}_{0,1}^{i-1}$ on M_{i-1} (which contains Q).

The Proof—The Cases

The gluings can be classified based on properties of the manifolds (M_i, γ_i) and Q; there are three main cases to consider.

The Proof—Case I



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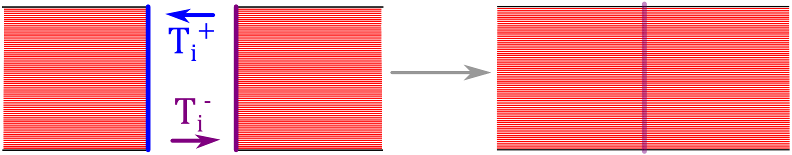

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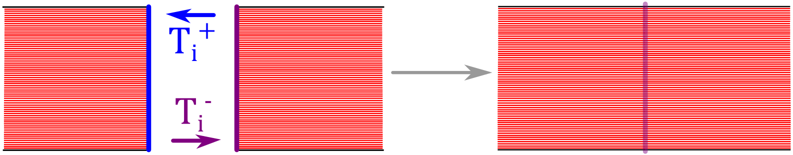

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The gluing happens in such a way that the existing (pre-glued) foliations are compatible.

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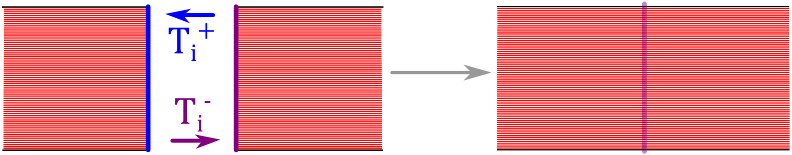
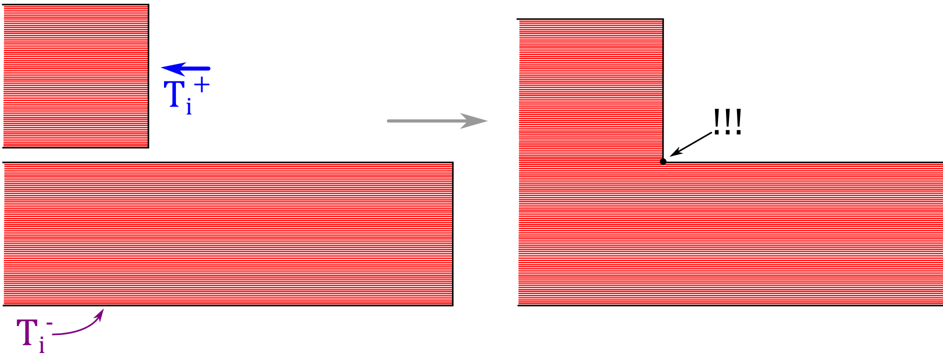


Case I is by far the easiest:

The gluing happens in such a way that the existing (pre-glued) foliations are compatible. Define $\mathcal{F}_{0,1}^{i-1}$ to be equal to the foliations induced by $\mathcal{F}_{0,1}^{i}$ and note that the desired properties are trivially satisfied.

The Proof—Case II



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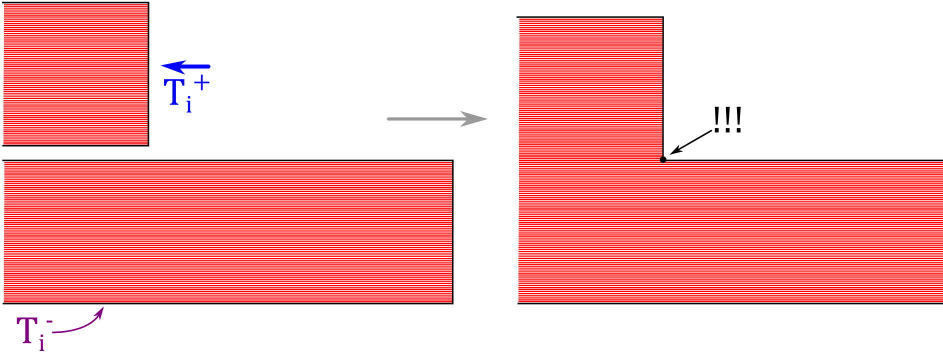

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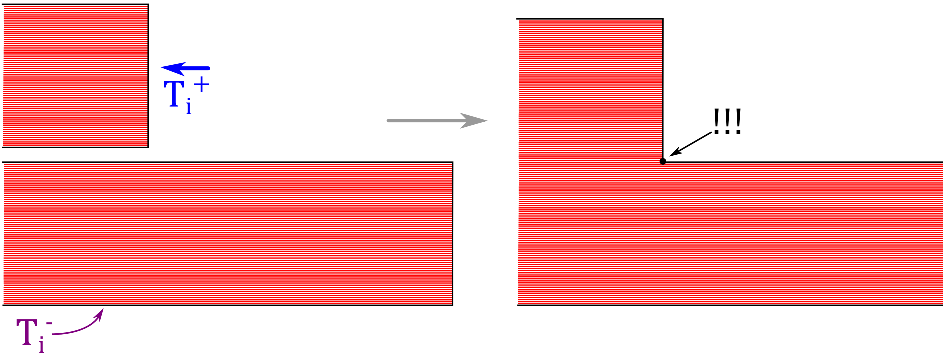

Case II is considerably harder:

The gluing here yields a point of non-convexity where the induced foliations are inconsistent. Substantially more work has to be done.

The Proof—Case II (Cont'd)

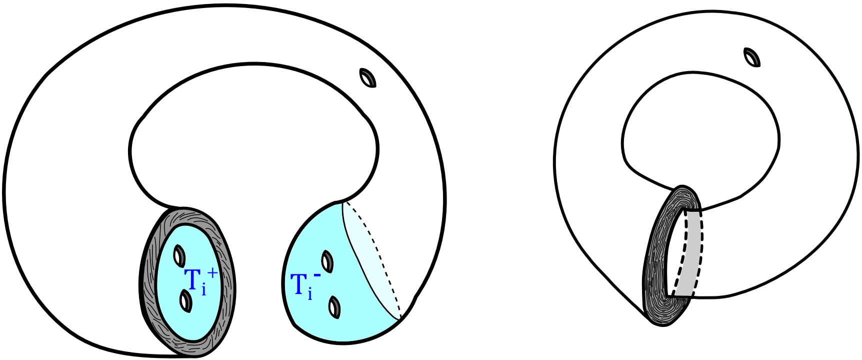

Figure 3
Gluing T_i^+ and T_i^- to get Q (from Gabai's perspective)

The Proof—Case II (Cont'd)

Two *different* processes must be undertaken in order to get the desired foliations $\mathcal{F}_{0,1}^{i-1}$ on M_{i-1}:

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- To get $\mathcal{F}_0^{i-1}$, the desired technique is to *spiral*.
- To get $\mathcal{F}_1^{i-1}$, there are a number of subcases to consider. The main issue at-hand, however, is the *holonomy*.

The Proof—Case II (Cont'd)

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1. Foliate a number of intermediate spaces.
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3. Identify a subspace Z of $V \times [-\infty, \infty]$ which is diffeomorphic to $M_{i-1} - \mathring{Q}$. Z has the foliation induced by V.
4. Glue Z to Q so that the foliations on each are compatible. This is done in a way so that depth $\mathcal{F}_0^{i-1} =$ depth $\mathcal{F}_0^{i} + 1$.

Define $\mathcal{F}_0^{i-1}$ to be the resulting foliation on M_{i-1}.

The Proof—Case II (Cont'd)

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- (i) If $\partial V \neq \varnothing$, the holonomy can be "pushed to the boundary" to reduce to case (C1).

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(C1) If $f = \text{id}$, then spiraling will yield a C^∞ foliation $\mathcal{F}_1^{i-1}$.

(C2) If $f \neq \text{id}$,:

- (i) If $\partial V \neq \varnothing$, the holonomy can be "pushed to the boundary" to reduce to case (C1).
- (ii) If $\partial V = \varnothing$ and $V = T^2$, things are screwed: $\mathcal{F}_1^{i-1}$ being C^0 is as good as it gets.

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To get $\mathcal{F}_1^{i-1}$: Write $\mathcal{F}^1$ for the foliation induced by $\mathcal{F}_1^i$ on Q, and let f be the holonomy of $\mathcal{F}^1$ along the transverse annulus A.

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- (ii) If $\partial V = \varnothing$ and $V = T^2$, things are screwed: $\mathcal{F}_1^{i-1}$ being C^0 is as good as it gets.
- (iii) If $\partial V = \varnothing$ and $V = S_g$, $g > 1$, then holonomy can be reduced to case (C1) by attaching thick bands to A and appealing a result of Mather, Sergeraert, and Thurston.

The Proof—Case II (Cont'd)

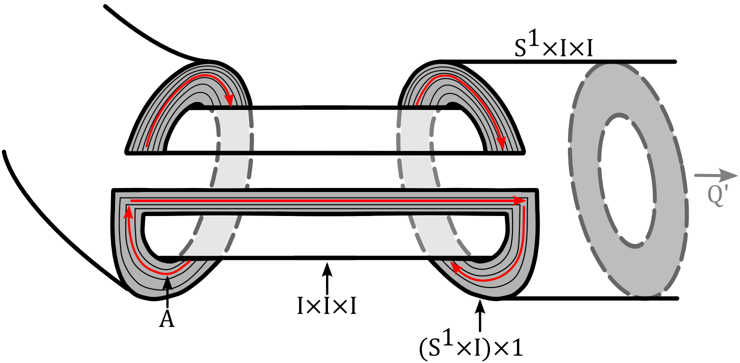


Figure 4
Pushing holonomy to the boundary in case (C2.i)

The Proof—Case II (Cont'd)

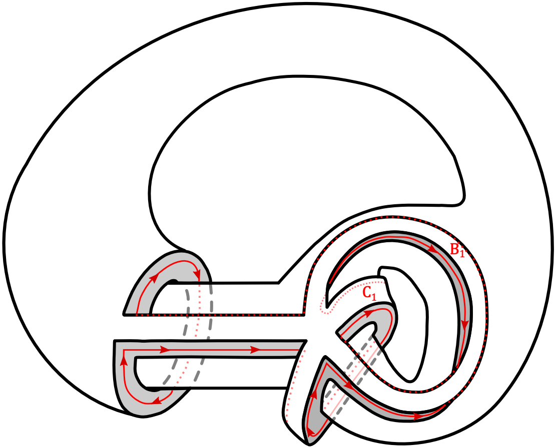
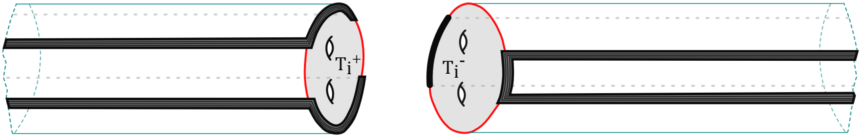
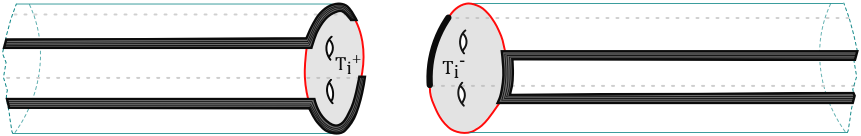


Figure 5
Attaching thick bands to A in case (C2.iii)

The Proof—Case III



The Proof—Case III



Case III is similar to Case II but is more involved still:

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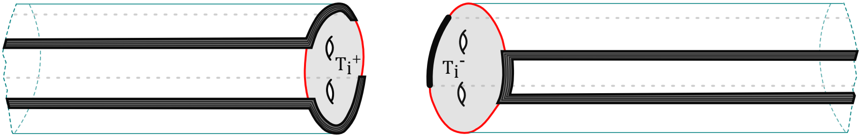

Case III is similar to Case II but is more involved still:

The gluing again yields inconsistent induced foliations. Because holonomy lies along an arc (and hence is trivial), the goal is to *smooth* (similar to *spiraling* in Case II).

The Proof—Case III (Cont'd)

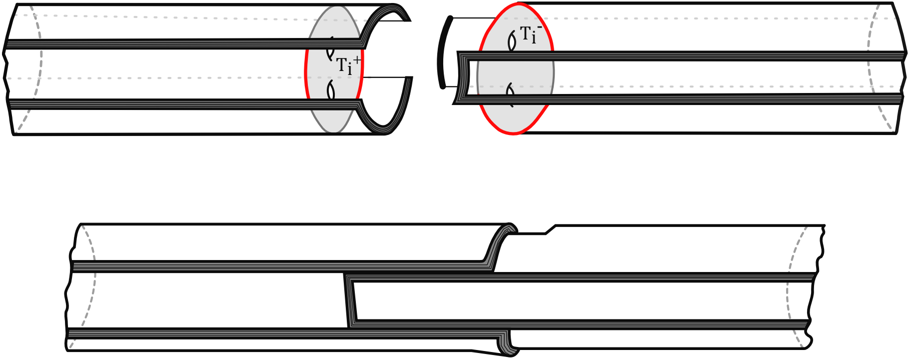

Figure 6

Gluing (bottom) happens after first "stretching" the pieces of γ_i which contain $\partial T_i^+ \cup \partial T_i^-$ (top).

The Proof—Case III (Cont'd)

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This means that whatever smoothing procedure is devised to handle Case III must be done for *every* component V of $R(\gamma_{i-1})$ (satisfying $\partial T_i \cap V \neq \varnothing$).

The Proof—Case III (Cont'd)

Because of this difference, the construction of the foliations $\mathcal{F}_{0,1}^{i-1}$ requires one to examine manifolds of the form $P(V) = N(V) \cap Q$:

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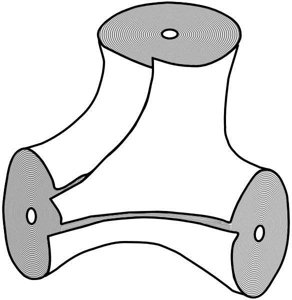

Figure 7
Prototypical $P(V)$

The Proof—Case III (Cont'd)

After constructing an intricate gluing procedure on $P(V)$ for general V, the foliations $\mathcal{F}_{0,1}^{i-1}$ are constructed on M_{i-1}.

The Proof—Case III (Cont'd)

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Figure 8
A diagrammatic representation M_{i-1}, foliated.

The Proof—Case III (Cont'd)

The gist of the gluing procedure on $P(V)$:

1. Define a number of intermediate spaces.

The Proof—Case III (Cont'd)

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1. Define a number of intermediate spaces. One will be Q_1, which looks like (M_{i-1}, γ_{i-1}) with "ditches" drilled out.

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The Proof—Case III (Cont'd)

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The Proof—Case III (Cont'd)

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   - A very particular gluing is required to (sometimes) yield finite depth. Call the resulting foliation $\mathcal{F}_0^{i-1}$.

Note that finite depth isn't always possible for $\mathcal{F}_0^{i-1}$ depending on how $P(V)$ looks; when it *is* possible, the gluing always yields depth $\mathcal{F}_0^{i-1} = \text{depth}\ \mathcal{F}_0^i + 1$.1

Conclusion

As a result of the procedure outlined above, there are foliations $\mathcal{F}_{0,1}$ on M which in general satisfy only a subset of the desired properties.

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Conclusion

As a result of the procedure outlined above, there are foliations $\mathcal{F}_{0,1}$ on M which in general satisfy only a subset of the desired properties.

To get the results as claimed, a number of outside results are used to get a "better" initial hierarchy for (M,γ). By completing the above procedure for this new hierarchy, there exist foliations (again called $\mathcal{F}_{0,1}$) on (M,γ) which satisfy all conditions of the theorem. $\square$

Section I

Why Study...

- ...3-Manifolds?

Why Study...

- ...3-Manifolds?
- ...Foliations?

Why Study...

- ...3-Manifolds?
- ...Foliations?
- ...Reeblessness?

Why Study...

- ...3-Manifolds?
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Why Study...

- ...3-Manifolds?
- ...Foliations?
- ...Reeblessness?
- ...Sutured Manifolds?
- ...the Work of Gabai?

Thank You!