More on Continued Fractions

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1 Introduction

Despite the brevity of the poster, the study of continued fractions (CFs) is extremely robust. Here's some (very brief) additional information on a (still very small) selection of associated topics.

2 Arithmetic

One of the first questions generally asked is, "Why on earth would anyone use continued fractions to represent a number when decimals work just fine?" The answer is that you may *not* want to use CFs but that lots of people *do* want to because CFs have several advantages. The natural question, then, is: "Well, how do they work?" More precisely: Given two decimal numbers $x = a_0.a_1a_2a_3\ldots$, $y = b_0.b_1b_2b_3\ldots$, computing $a \pm b$, $a \times b$, or a/b is a process that's well-known. Surely it's gotta be impossible to do all that with continued fractions, right?!

Surprisingly (or not), it's *not* impossible. It is, however, very tedious and requires substantially more than fits in the scope of this discussion. Fortunately, the presentation given at the following address does a good job breaking down the details:

http://perl.plover.com/classes/cftalk/TALK/slide037.html

The information therein is presented in a way that utilizes computer science ideas and matrix-like notation but is relatively easy to follow otherwise. The techniques themselves are due in large part to Gosper and, given how daunting they first appear, it's easy to see that CFs have obvious cons which, to some degree, counterbalance the pros.

3 Functions

Given functions a_m, b_m, one can define a new function ξ in terms of continued fractions:

$$\xi(x) = b_0(x) + \operatorname*{K}_{m=1}^{\infty} \frac{a_m(x)}{b_m(x)}.$$

Such functions ξ have been studied extensively with various conditions imposed and exist as the focus of a very rich area of study. For more information, see here:

http://www.wolframalpha.com/input/?i=continued+fraction+family+types

These are examples of generalized CFs and, as such, can be studied and understood by breaking them down into their finite convergents. In order to gain any sort of meaningful information from such functions, it's necessary to understand the convergence of the these convergents.

4 Convergence, Briefly

The literature on convergence of continued fractions is vast. As discussed on the poster, talking about convergence "makes sense" for any CF, numerical or otherwise. On the other hand, when the fraction in question defines a function, the conversation becomes more robust and can include various notions of convergence such as generalized, classical, uniform, absolute, etc. A large number of results can be found in [1] and its sources.

5 Applications

One "real-life" example of the usefulness of continued fractions is the interesting topic of leap year schemes, which is the foundation of [2] and is summarized here.

First, note that a day isn't *precisely* 24 hours long. Indeed, the ratio between seconds and years is defined in the literature (see note 2 in [2]) so that "one second is fixed as 1/31,556,925.9747 of the tropical year 1900...," and because there are 86,400 seconds in a day, there are 365.24219878125... days in a year. Rounding to 365.2422, consider the decimal part 0.2422 whose CF representation is $0.2422 = [0;4,7,1,3,4,1,1,1,2]$. One can consider the sequence of convergents: $\xi_0 = 0$, $\xi_1 = [0;4] = 1/4$, $\xi_2 = [0;4,7] = (4+1/7)^{-1} = 7/29$, etc., whose values continue as follows:

$$\xi_3 = \frac{8}{33}, \xi_4 = \frac{31}{128}, \xi_5 = \frac{132}{545}, \xi_6 = \frac{163}{673}, \xi_7 = \frac{458}{1891}, \xi_8 = \frac{1211}{5000}. \tag{1}$$

Interpretting each of the above rational numbers p/q as *add p days to every q years*, one can devise leap year schemes; in particular, the $\xi_1 = 1/4$ approximant corresponds to the standard *add 1 day to every 4 years* scheme utilized presently, and there are different ways to interpret each ξ_k. For example, a way of realizing ξ_8 within a 5,000 year period is to add a day to every four years (+1250 days) except at the end of a century (-50 days), with one leap year counted for every fifth century (+10 days) and a "double leap year" at the end of the 5,000 (+1 days).

What would be the benefit of such an alteration? A well-known fact about continued fractions is that the (increasing) sequence of odd convergents (i.e., ξ_k for k odd) all underestimate the actual value 0.2422 and that the (decreasing) sequence of even convergents (ξ_k for k even) all overestimate it. Therefore, the current standard ξ_1 is an overestimate and the fractions in (1) yield a variety of new schemes which would reduce the error in the current system. Returning to the example of ξ_8, [2] points out that this scheme is effectively perfect (essentially no error per 5,000 years), whereas the current system is equivalent to adding 39 "unnecessary" days per 5,000 years. These schemes can be perfected further by carrying more digits when rounding 0.24219878125....

6 References

[1] Cuyt et al., *Handbook of Continued Fractions for Special Functions*, Springer, 2008.

[2] Bernard Rasof, *Continued Fractions and "Leap" Years*, "The Mathematics Teacher", Vol 63, No 1, Jan 1970, pp 23-27. Accessed online: `http://www.jstor.org/stable/27958280`.