Continued Fractions:
What They Are, Why They're Important, and Who Really Cares

Christopher A. Stover   cstover@math.fsu.edu   Florida State University, Tallahassee, Florida

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Introduction

From a mathematical standpoint, there are several seemingly-different ways to define the notion of a continued fraction. The situation in which continued fractions are most often seen at an elementary level occurs with regards to rational numbers.

Example

First, consider the rational 37/14. Clearly, 37/14 can be written as $2+\frac{9}{14}$. Next, 9/14 can be rewritten as $\frac{1}{14/9}=\frac{1}{1+5/9}$. Similarly, $\frac{5}{9}=\frac{1}{9/5}=\frac{1}{1+4/5}$, and $\frac{4}{5}=\frac{1}{5/4}=\frac{1}{1+1/4}$. Note that this ad hoc process seems to "terminate" naturally because $\frac{1}{4}=\frac{1}{4+0}$ has a denominator which can no longer be simplified in a "meaningful way".

Without knowing it, we've just computed the continued fraction representation of the number $\frac{37}{14}$:

$$\frac{37}{14}=2+\frac{1}{14/9}=2+\frac{1}{1+\frac{5}{9}}=2+\frac{1}{1+\frac{1}{9/5}}$$

$$=2+\frac{1}{1+\frac{1}{1+\frac{4}{5}}}=2+\frac{1}{1+\frac{1}{1+\frac{1}{5/4}}}$$

$$=2+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{4}}}}. \tag{1}$$

Motivation

The previous example sheds some light on how one may go about constructing continued fractions for rational numbers $p/q\in\mathbb{Q}$ but it also leaves many an unanswered question.

- How does one go about considering continued fractions of irrational numbers? Can continued fractions be constructed for numbers not in $\mathbb{R}$? For multi-dimensional elements like vectors and matrices?
- Are the continued fraction representations of all rational numbers finite? All real numbers?
- What does a continued fraction representation say about the object it represents?
- In what areas do specialists care about and/or apply the theory of continued fractions?

In order to answer questions like these, a more general mathematical theory is needed.

General Theory: Real Continued Fractions

An expression ξ of the form

$$\xi=b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\ddots}}} \tag{2}$$

is a **(real) continued fraction** provided that $a_m,b_m\in\mathbb{Z}$, $a_m\neq 0$ for all m. The terms a_m, resp. b_m are called the **partial numerators**, resp. **partial denominators** of ξ, and collectively a_m and b_m are called its **elements**. ξ is said to be **finite** if it has finitely many elements and **infinite** otherwise. Continued fractions are called **regular** if $a_m=1$ for all m [as in (1)] and are called **generalized** otherwise.

As noted above, continued fractions may be infinite. To interpret an infinite fraction, it's often helpful to view (2) by way of the sequence of finite continued fractions ξ_n which, in Gauss notation [see (5)], have the form

$$\xi_n=b_0+\mathop{\mathrm{K}}_{m=1}^{n}\frac{a_m}{b_m}. \tag{3}$$

In this way, discussing the convergence of infinite continued fractions becomes a matter of discussing convergence of sequences: A fraction ξ represents a real number $r\in\mathbb{R}$ if either

1. ξ is finite [in which case $r\in\mathbb{Q}$], or
2. ξ is infinite and $\lim_{n\to\infty}\xi_n=r$ as a sequence of rational approximations.

For each n, the fraction ξ_n is called the n^{th} **approximant** (or **convergent**) of ξ.

Notation

It's not hard to imagine that the notation in (2) can quickly get out of hand. Fortunately, there are a number of more concise notations. Generalized fractions ξ as in (2) can be expressed as:

$$\xi=(a_1,a_2,\ldots;b_0,b_1,b_2,\ldots) \tag{4}$$

$$=b_0+\mathop{\mathrm{K}}_{m=1}^{\infty}\frac{a_m}{b_m} \tag{5}$$

$$=b_0+\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\frac{a_3|}{|b_3}+\cdots. \tag{6}$$

The notation (5) [and in (3) above] is called "Gauss notation" after Karl Friedrich Gauss and the **K** therein stands for the German "Kettenbruch" (literally, "continued fraction"). The notation in (6) is called "Pringsheim notation" for German mathematician Alfred Pringsheim. Applied to (1), one can write:

$$\frac{37}{14}=[2;1,1,1,4] \tag{7}$$

$$=2+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|4}.$$

The notation (7) is a specialized version of that in (4) which is reserved especially for regular continued fractions; the numbers therein represent the denominators $b_0,b_1,b_2,\ldots$.

Properties

Continued fractions are the source of many interesting number theoretic properties:

- Define $B_{-1}=0$, $A_{-1}=B_0=1$, and $A_0=b_0$, and write $\xi_n=A_n/B_n$ for the n^{th} convergent of a continued fraction ξ [see (3)]. Iterated substitution shows that ξ_n satisfies the so-called **three-term recurrence relations**
$$\begin{aligned}A_n&=b_nA_{n-1}+a_nA_{n-2}\\B_n&=b_nB_{n-1}+b_nB_{n-2}\end{aligned} \tag{8}$$
for $n=1,2,3,\ldots$.
- The **best rational approximations** (BRA) of an irrational number $r\in\mathbb{R}\setminus\mathbb{Q}$ come from its finite approximants. For example, one can easily verify that $\pi=[3;7,15,1,292,\ldots]$, and the famous Archimedean approximation $\pi\approx 22/7$ is precisely $22/7=[3;7]$. Other BRA of π thus include
$$[3;7,15]=\frac{333}{106}\approx 3.1415094$$
$$[3;7,15,1]=\frac{355}{113}\approx 3.14159292$$
$$[3,7,15,1,292]=\frac{103993}{33102}\approx 3.14159265.$$
These rational approximations play an important role in many unexpected areas such as computing optimal leap year schemes.

Extensions

Various generalizations of the above theory exist.

- A **complex continued fraction** is with elements $a_k,b_k\in\mathbb{C}$.
- Several notions of **multidimensional fractions** exist including those for which the recurrences (8) are k-term, $k\geq 3$, and those whose elements which are vectors in $\mathbb{R}^d$, $d>1$.
- **Branched continued fractions** have elements which themselves are continued fractions.

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