Curvature and the Shape of Space

Florida State University
Math Fun Day 2014

October 11, 2014

What Is Curvature?

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Flat

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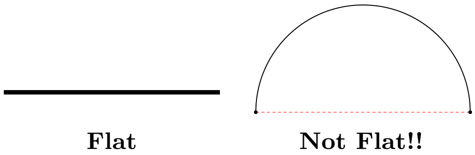

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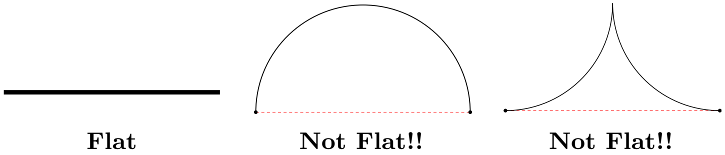

Not-Flat Things

Notice, too, that not all *not-flat* things are created equal!

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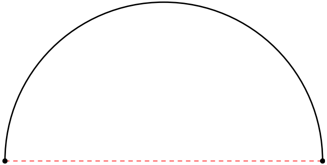

Not Flat + Convex
(...it pokes out...)

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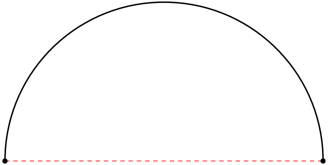

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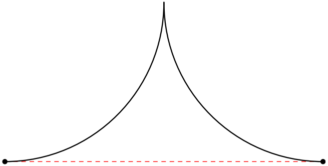

Not Flat + Concave
(...it caves in...)

Concave vs Convex

(Roughly speaking,) Things which are **not-flat and convex** are said to be *positively curved*...†

† This is extremely hand-wavy and imprecise....

Concave vs Convex

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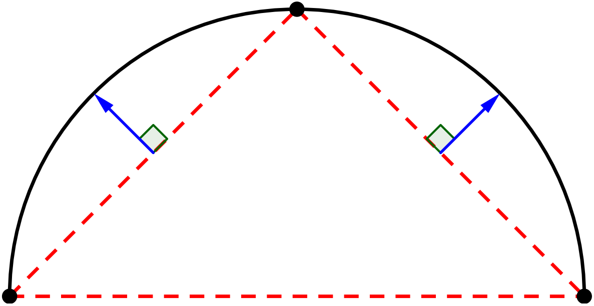

Positively Curved

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Concave vs Convex

...and (roughly speaking) things which are **not-flat and concave** are said to be *negatively curved*...†

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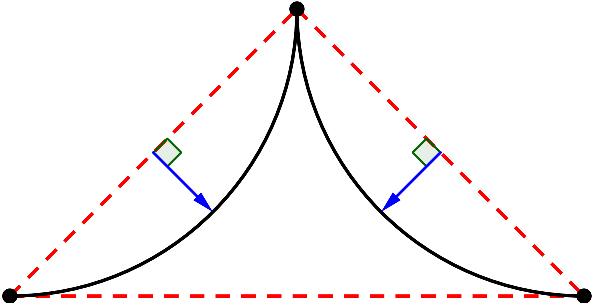

Negatively Curved

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Concave vs Convex

Flat things are said to have *zero curvature*.

All The Dimensions!

These ideas also apply to higher dimensions as well...

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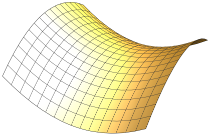

Negative

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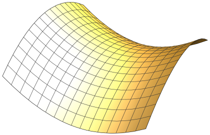

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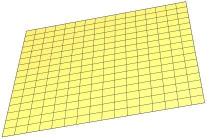

Zero

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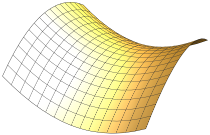

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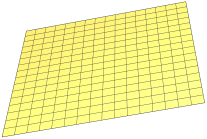

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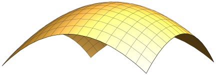

Positive

Dimensions Are Hard...

...but things are considerably more complciated....

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$$\delta\tau = \delta\int K\mathrm{d}\lambda =$$

$$= \delta\int\sqrt{-\frac{1}{c^2}g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda =$$

$$= \int\frac{-\frac{1}{c^2}\left((\delta g_{\mu\nu})\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + g_{\mu\nu}\left(\delta\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\left(\delta\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)\right)}{2K}\mathrm{d}\lambda =$$

$$= \frac{1}{c^2}\int\frac{\left(-\frac{1}{2}(\delta g_{\mu\nu})\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} - g_{\mu\nu}\left(\delta\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)}{K}\mathrm{d}\lambda =$$

$$= \frac{1}{c^2}\int\frac{\left(-\frac{1}{2}(\delta x^\alpha)\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} - g_{\alpha\nu}\frac{\mathrm{d}(\delta x^\alpha)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)}{K}\mathrm{d}\lambda =$$

$$= \frac{1}{c^2}\int\left(\frac{-\frac{1}{2}(\delta x^\alpha)\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}{K} + \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{g_{\alpha\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}{K}\right)(\delta x^\alpha)\right)\mathrm{d}\lambda =$$

$$= \frac{1}{c^2}\int\frac{1}{K}\left(K\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{g_{\alpha\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}{K}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)(\delta x^\alpha)\mathrm{d}\lambda =$$

$$= \frac{1}{c^2}\int\frac{1}{K}\left(K\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{K}\right) + \frac{\mathrm{d}g_{\alpha\nu}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + g_{\alpha\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)(\delta x^\alpha)\mathrm{d}\lambda =$$

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$$= \frac{1}{c^2}\int\frac{1}{K}\left(K\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{1}{K}\right)g^{\alpha\rho} + \delta_\nu{}^\rho\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\lambda^2} + \frac{1}{2}g^{\alpha\rho}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\right)(\delta x_\rho)\mathrm{d}\lambda =$$

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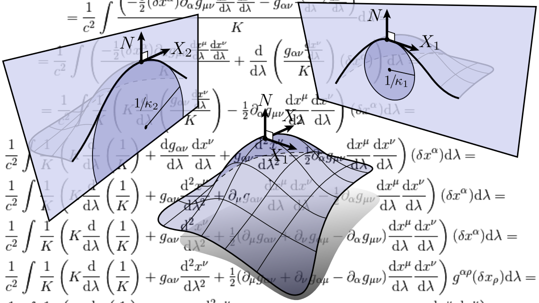

Dimensions Are Hard...

One difficulty is that some objects tend to be visualized in a "distorted" manner....

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(think of a hollowed-out donut)

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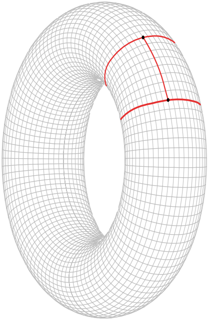

Is it convex?

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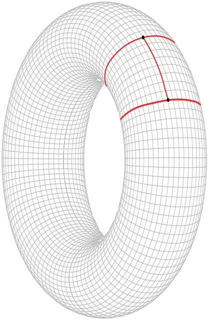

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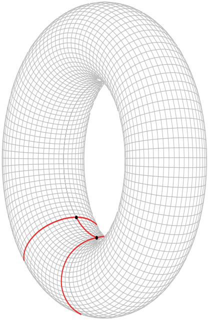

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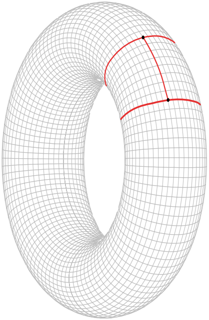

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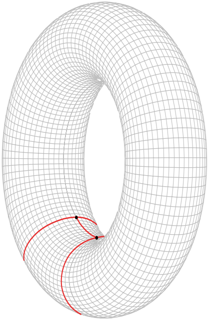

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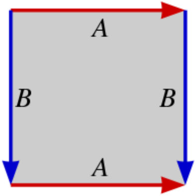


No. It's flat.

Example: The Torus
(think of a hollowed-out donut)

We LOVE Examples...

Even so, curvature is everywhere!

Negative Curvature

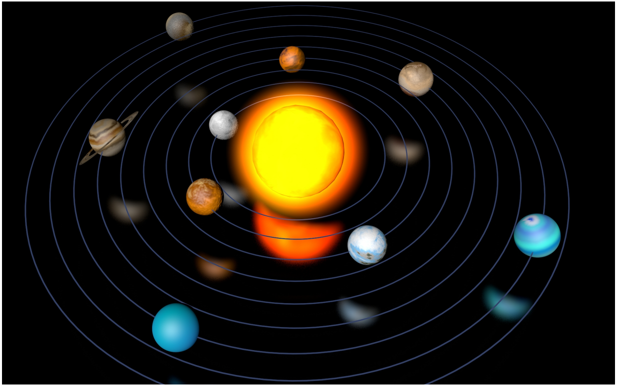
...Examples Make People Care!

...curvature, curvature, curvature,...

Positive Curvature

Positive Curvature Is *Everywhere*...

Seriously...just *look up*!



Space Stuff

Space Things

QUESTION:

Space Things

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What about *space itself*? What shape does it have?

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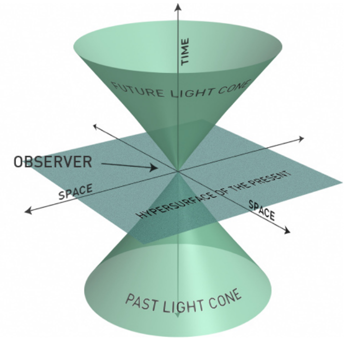

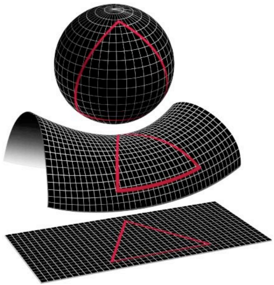

Space Things

ANSWER:

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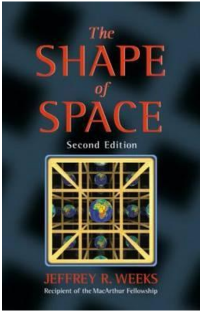
Nobody *really* knows...
...but Dr. Jeff Weeks has some ideas.



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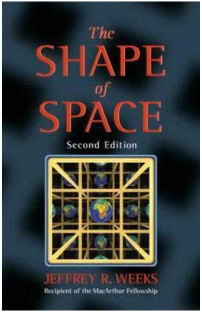

You can hear some of them today (4p–5p, Moore Auditorium)!

More Curvature/Shape-Related Things

You can also see Dr. Hironaka's construction of a hyperbolic blanket for examples of building not-flat things out of flat things!

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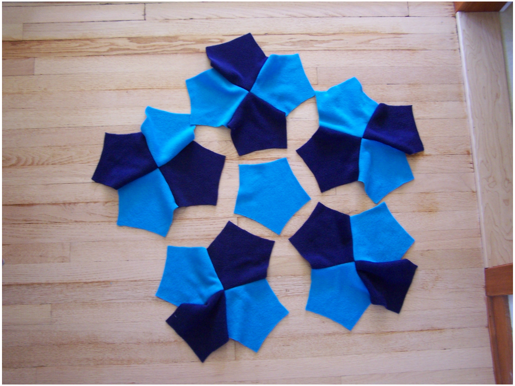

(image belongs to Dr. Jeff Weeks, who owns all rights thereto)

Thank you!