An Introduction to Generalized (Complex) Geometry

Christopher Stover

Florida State University

Complex Analysis Seminar
April 10, 2014

Big Picture

- Generalized Geometry was invented by Nigel Hitchin in 2008.

Big Picture

- Generalized Geometry was invented by Nigel Hitchin in 2008.
- The goal of generalized geometry is to generalize usual notions from differential geometry to settings more easily-adaptable to modern physics.

Big Picture

- Generalized Geometry was invented by Nigel Hitchin in 2008.
- The goal of generalized geometry is to generalize usual notions from differential geometry to settings more easily-adaptable to modern physics.
- This is done by considering structures defined on $TM \oplus T^*M$ rather than TM, T^*M separately.

Big Picture

- Generalized Geometry was invented by Nigel Hitchin in 2008.
- The goal of generalized geometry is to generalize usual notions from differential geometry to settings more easily-adaptable to modern physics.
- This is done by considering structures defined on $TM \oplus T^*M$ rather than TM, T^*M separately.
- Via this method, one can define generalized analogues of things such as complex geometry, Symplectic geometry, Calabi-Yau geometry, etc.

Outline

Introduction

Outline

Outline

Outline

Outline

Outline

Outline

Introduction
- Tools and Techniques

Stuff about $V \oplus V^*$
- Algebraic Properties
- Transformations
- (Maximal) Isotropics

Stuff about $T \oplus T^*$

Outline

Outline

Outline

Outline

Outline

Part I

Getting Started

In order to make sense of generalized geometry, a new framework needs to be studied. We'll need to understand:

Getting Started

In order to make sense of generalized geometry, a new framework needs to be studied. We'll need to understand:

- $T \oplus T^*$, where $T = TM$.

Getting Started

In order to make sense of generalized geometry, a new framework needs to be studied. We'll need to understand:

- $T \oplus T^*$, where $T = TM$.
- Induced inner product $\langle \cdot\,, \cdot \rangle$ on $T \oplus T^*$.

Getting Started

In order to make sense of generalized geometry, a new framework needs to be studied. We'll need to understand:

- $T \oplus T^*$, where $T = TM$.
- Induced inner product $\langle \cdot\,, \cdot \rangle$ on $T \oplus T^*$.
- B-fields, aka B-transforms.

Getting Started

In order to make sense of generalized geometry, a new framework needs to be studied. We'll need to understand:

- $T \oplus T^*$, where $T = TM$.
- Induced inner product $\langle \cdot\, , \cdot \rangle$ on $T \oplus T^*$.
- B-fields, aka B-transforms.
- Lie Algebroids.

Getting Started

In order to make sense of generalized geometry, a new framework needs to be studied. We'll need to understand:

- $T \oplus T^*$, where $T = TM$.
- Induced inner product $\langle \cdot\,, \cdot \rangle$ on $T \oplus T^*$.
- B-fields, aka B-transforms.
- Lie Algebroids.
- Courant bracket $[\cdot\,, \cdot]$.

Getting Started

In order to make sense of generalized geometry, a new framework needs to be studied. We'll need to understand:

- $T \oplus T^*$, where $T = TM$.
- Induced inner product $\langle \cdot\,, \cdot \rangle$ on $T \oplus T^*$.
- B-fields, aka B-transforms.
- Lie Algebroids.
- Courant bracket $[\cdot\,, \cdot]$.
- Dirac structures.

Getting Started

In order to make sense of generalized geometry, a new framework needs to be studied. We'll need to understand:

- $T \oplus T^*$, where $T = TM$.
- Induced inner product $\langle \cdot\,, \cdot \rangle$ on $T \oplus T^*$.
- B-fields, aka B-transforms.
- Lie Algebroids.
- Courant bracket $[\cdot\,, \cdot]$.
- Dirac structures.

At that point, one can define generalized almost-structures and generalized structures using the developed machinery.

Notation

Unless otherwise noted:

Notation

Unless otherwise noted:

M = differentiable manifold of dimension m

Notation

Unless otherwise noted:

$$M = \text{differentiable manifold of dimension } m$$

$$V^* = \text{dual of vector space } V$$

Notation

Unless otherwise noted:

$$\begin{aligned} M &= \text{differentiable manifold of dimension } m \\ V^* &= \text{dual of vector space } V \\ T, T^* &= TM, \text{ resp. } T^*M \end{aligned}$$

Notation

Unless otherwise noted:

M	$=$	differentiable manifold of dimension m
V^*	$=$	dual of vector space V
T, T^*	$=$	TM, resp. T^*M
X, Y	$=$	C^∞-sections of T

Notation

Unless otherwise noted:

$$
\begin{aligned}
M &= \text{differentiable manifold of dimension } m \\
V^* &= \text{dual of vector space } V \\
T, T^* &= TM, \text{ resp. } T^*M \\
X, Y &= C^\infty\text{-sections of } T \\
&= \text{vector fields on } M
\end{aligned}
$$

Notation

Unless otherwise noted:

$$
\begin{aligned}
M &= \text{differentiable manifold of dimension } m \\
V^* &= \text{dual of vector space } V \\
T, T^* &= TM, \text{ resp. } T^*M \\
X, Y &= C^\infty\text{-sections of } T \\
&= \text{vector fields on } M \\
\xi, \eta &= C^\infty\text{-sections of } T^*
\end{aligned}
$$

Notation

Unless otherwise noted:

$$
\begin{aligned}
M &= \text{differentiable manifold of dimension } m \\
V^* &= \text{dual of vector space } V \\
T, T^* &= TM\text{, resp. } T^*M \\
X, Y &= C^\infty\text{-sections of } T \\
&= \text{vector fields on } M \\
\xi, \eta &= C^\infty\text{-sections of } T^* \\
&= \text{1-forms on } M
\end{aligned}
$$

Notation

Unless otherwise noted:

$$
\begin{aligned}
M &= \text{differentiable manifold of dimension } m \\
V^* &= \text{dual of vector space } V \\
T, T^* &= TM\text{, resp. } T^*M \\
X, Y &= C^\infty\text{-sections of } T \\
&= \text{vector fields on } M \\
\xi, \eta &= C^\infty\text{-sections of } T^* \\
&= \text{1-forms on } M \\
\bigwedge^p V &= p\text{-fold wedge/exterior product of elements in } V
\end{aligned}
$$

Notation

Unless otherwise noted:

$$
\begin{aligned}
M &= \text{differentiable manifold of dimension } m \\
V^* &= \text{dual of vector space } V \\
T, T^* &= TM, \text{ resp. } T^*M \\
X, Y &= C^\infty\text{-sections of } T \\
&= \text{vector fields on } M \\
\xi, \eta &= C^\infty\text{-sections of } T^* \\
&= \text{1-forms on } M \\
\textstyle\bigwedge^p V &= p\text{-fold wedge/exterior product of elements in } V \\
&= \underbrace{V \wedge V \wedge \cdots \wedge V}_{p \text{ times}}
\end{aligned}
$$

Notation (Cont'd)

$$\iota_X \quad = \quad \text{the interior product } \iota_X : \bigwedge^k V \to \bigwedge^{k-1} V,\ \xi \mapsto (\iota_X \xi),$$
$$\text{such that } (\iota_X \xi)(X_1, \ldots, X_{k-1}) = \xi(X, X_1, \ldots, X_{k-1})$$

Notation (Cont'd)

$$\iota_X = \text{the interior product } \iota_X : \bigwedge^k V \to \bigwedge^{k-1} V,\ \xi \mapsto (\iota_X \xi),$$
$$\text{such that } (\iota_X \xi)(X_1, \ldots, X_{k-1}) = \xi(X, X_1, \ldots, X_{k-1})$$
$$d = \text{exterior derivative}$$

Notation (Cont'd)

$$\begin{aligned}
\iota_X &= \text{the interior product } \iota_X : \textstyle\bigwedge^k V \to \bigwedge^{k-1} V,\ \xi \mapsto (\iota_X \xi), \\
&\quad \text{such that } (\iota_X \xi)(X_1, \ldots, X_{k-1}) = \xi(X, X_1, \ldots, X_{k-1}) \\
d &= \text{exterior derivative} \\
\mathcal{L}_X &= \text{Lie derivative associated to } X
\end{aligned}$$

Notation (Cont'd)

$$\begin{aligned}
\iota_X &= \text{the interior product } \iota_X : \textstyle\bigwedge^k V \to \bigwedge^{k-1} V,\ \xi \mapsto (\iota_X \xi), \\
&\quad \text{such that } (\iota_X \xi)(X_1, \ldots, X_{k-1}) = \xi(X, X_1, \ldots, X_{k-1}) \\
d &= \text{exterior derivative} \\
\mathcal{L}_X &= \text{Lie derivative associated to } X \\
&= \iota_X d + d\iota_X
\end{aligned}$$

Notation (Cont'd)

$$\begin{aligned}
\iota_X &= \text{the interior product } \iota_X : \textstyle\bigwedge^k V \to \bigwedge^{k-1} V,\ \xi \mapsto (\iota_X \xi), \\
& \text{such that } (\iota_X \xi)(X_1, \ldots, X_{k-1}) = \xi(X, X_1, \ldots, X_{k-1}) \\
d &= \text{exterior derivative} \\
\mathcal{L}_X &= \text{Lie derivative associated to } X \\
&= \iota_X d + d\iota_X \\
e^B &= \text{exponential map applied to } k\text{-form } B
\end{aligned}$$

Notation (Cont'd)

$$
\begin{aligned}
\iota_X &= \text{the interior product } \iota_X : \textstyle\bigwedge^k V \to \bigwedge^{k-1} V,\ \xi \mapsto (\iota_X \xi), \\
&\quad \text{such that } (\iota_X \xi)(X_1, \ldots, X_{k-1}) = \xi(X, X_1, \ldots, X_{k-1}) \\
d &= \text{exterior derivative} \\
\mathcal{L}_X &= \text{Lie derivative associated to } X \\
&= \iota_X d + d\iota_X \\
e^B &= \text{exponential map applied to } k\text{-form } B \\
&= \sum_{j=0}^{\infty} \frac{B^k}{k!} = I + B + \frac{B^2}{2} + \frac{B^3}{6} + \cdots + \frac{B^k}{k!}
\end{aligned}
$$

Notation (Cont'd)

$$
\begin{aligned}
\iota_X &= \text{the interior product } \iota_X : \textstyle\bigwedge^k V \to \bigwedge^{k-1} V,\ \xi \mapsto (\iota_X \xi), \\
&\quad \text{such that } (\iota_X \xi)(X_1, \ldots, X_{k-1}) = \xi(X, X_1, \ldots, X_{k-1}) \\
d &= \text{exterior derivative} \\
\mathcal{L}_X &= \text{Lie derivative associated to } X \\
&= \iota_X d + d \iota_X \\
e^B &= \text{exponential map applied to } k\text{-form } B \\
&= \sum_{j=0}^{\infty} \frac{B^k}{k!} = I + B + \frac{B^2}{2} + \frac{B^3}{6} + \cdots + \frac{B^k}{k!} \\
A^\dagger &= \text{conjugate transpose of } A.
\end{aligned}
$$

Part II

Inner Product

- Write $X + \xi$, $Y + \eta$ for elements of $V \oplus V^*$.

Inner Product

- Write $X + \xi$, $Y + \eta$ for elements of $V \oplus V^*$.
- Define two natural bilinear forms on $V \oplus V^*$:

Inner Product

- Write $X + \xi$, $Y + \eta$ for elements of $V \oplus V^*$.
- Define two natural bilinear forms on $V \oplus V^*$:

$$\langle X + \xi, Y + \eta \rangle_+ = \frac{1}{2}\left(\xi(Y) + \eta(X)\right)$$

Inner Product

- Write $X + \xi$, $Y + \eta$ for elements of $V \oplus V^*$.
- Define two natural bilinear forms on $V \oplus V^*$:

$$\langle X + \xi, Y + \eta \rangle_+ = \frac{1}{2} \left(\xi(Y) + \eta(X) \right)$$

$$\langle X + \xi, Y + \eta \rangle_- = \frac{1}{2} \left(\xi(Y) - \eta(X) \right)$$

Inner Product

- Write $X + \xi$, $Y + \eta$ for elements of $V \oplus V^*$.
- Define two natural bilinear forms on $V \oplus V^*$:
$$\langle X + \xi, Y + \eta \rangle_+ = \frac{1}{2}\left(\xi(Y) + \eta(X)\right)$$
$$\langle X + \xi, Y + \eta \rangle_- = \frac{1}{2}\left(\xi(Y) - \eta(X)\right)$$
These are non-degerate and are symmetric and anti-symmetric, respectively.

Inner Product

- Write $X + \xi$, $Y + \eta$ for elements of $V \oplus V^*$.
- Define two natural bilinear forms on $V \oplus V^*$:
$$\langle X + \xi, Y + \eta \rangle_+ = \frac{1}{2}\left(\xi(Y) + \eta(X)\right)$$
$$\langle X + \xi, Y + \eta \rangle_- = \frac{1}{2}\left(\xi(Y) - \eta(X)\right)$$
These are non-degerate and are symmetric and anti-symmetric, respectively.
- Denote $\langle \cdot\, , \cdot \rangle_+$ as $\langle \cdot\, , \cdot \rangle$ and call it *the inner product* on $V \oplus V^*$.

Inner Product

- Write $X+\xi$, $Y+\eta$ for elements of $V \oplus V^*$.
- Define two natural bilinear forms on $V \oplus V^*$:

$$\langle X+\xi, Y+\eta\rangle_+ = \frac{1}{2}\left(\xi(Y)+\eta(X)\right)$$

$$\langle X+\xi, Y+\eta\rangle_- = \frac{1}{2}\left(\xi(Y)-\eta(X)\right)$$

  These are non-degerate and are symmetric and anti-symmetric, respectively.
- Denote $\langle \cdot\,,\cdot\rangle_+$ as $\langle \cdot\,,\cdot\rangle$ and call it *the inner product* on $V \oplus V^*$.
- Note that $\langle \cdot\,,\cdot\rangle$ is indefinite; it has signature (m,m).

Part II

Orientation-Preserving Tranformations

- Note that $V \oplus V^*$ has a canonical orientation.

Orientation-Preserving Tranformations

- Note that $V \oplus V^*$ has a canonical orientation.
- $\mathrm{SO}(V \oplus V^*) \cong \mathrm{SO}(m, m)$ preserves the inner product and canonical orientation on $V \oplus V^*$.

Orientation-Preserving Tranformations

- Note that $V \oplus V^*$ has a canonical orientation.
- $\mathrm{SO}(V \oplus V^*) \cong \mathrm{SO}(m, m)$ preserves the inner product and canonical orientation on $V \oplus V^*$.
- Define the associated Lie algebra $\mathfrak{so}(V \oplus V^*)$ by

$$\mathfrak{so}(V \oplus V^*) = \{T : \langle Tx, y\rangle + \langle x, Ty\rangle = 0\, \forall\, x, y \in V \oplus V^*\}.$$

Orientation-Preserving Tranformations

- Note that $V \oplus V^*$ has a canonical orientation.
- $\mathrm{SO}(V \oplus V^*) \cong \mathrm{SO}(m, m)$ preserves the inner product and canonical orientation on $V \oplus V^*$.
- Define the associated Lie algebra $\mathfrak{so}(V \oplus V^*)$ by
$$\mathfrak{so}(V \oplus V^*) = \{T : \langle Tx, y\rangle + \langle x, Ty\rangle = 0 \, \forall \, x, y \in V \oplus V^*\}.$$
- Splitting T into V-, V^*-parts yields that
$$T = \begin{pmatrix} A & \beta \\ B & -A^\dagger \end{pmatrix},$$
$A \in \mathrm{End}(V)$, $B \in \wedge^2 V^*$, $\beta \in \wedge^2 V$ with $B^\dagger = -B$, $\beta^\dagger = -\beta$.

Orientation-Preserving Tranformations

- Note that $V \oplus V^*$ has a canonical orientation.
- $\mathrm{SO}(V \oplus V^*) \cong \mathrm{SO}(m, m)$ preserves the inner product and canonical orientation on $V \oplus V^*$.
- Define the associated Lie algebra $\mathfrak{so}(V \oplus V^*)$ by

$$\mathfrak{so}(V \oplus V^*) = \{T : \langle Tx, y\rangle + \langle x, Ty\rangle = 0 \, \forall \, x, y \in V \oplus V^*\}.$$

- Splitting T into V-, V^*-parts yields that

$$T = \begin{pmatrix} A & \beta \\ B & -A^\dagger \end{pmatrix},$$

$A \in \mathrm{End}(V)$, $B \in \wedge^2 V^*$, $\beta \in \wedge^2 V$ with $B^\dagger = -B$, $\beta^\dagger = -\beta$.

- Hence, $\mathfrak{so}(V \oplus V^*) \cong \mathrm{End}(V) \oplus \wedge^2 V^* \oplus \wedge^2 V$.

B-fields and β-fields

Let $B : V \to V^*$, $\beta : V^* \to V$, viewed as 2-forms. There are two important orientation-preserving transformations of $T \oplus T^*$:

B-fields and β-fields

Let $B : V \to V^*$, $\beta : V^* \to V$, viewed as 2-forms. There are two important orientation-preserving transformations of $T \oplus T^*$:

Definition.

A B-field or B-transform is a transformation of the form

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : X + \xi \mapsto X + \xi + \iota_X B.$$

B-fields and β-fields

Let $B : V \to V^*$, $\beta : V^* \to V$, viewed as 2-forms. There are two important orientation-preserving transformations of $T \oplus T^*$:

Definition.
A B-field or B-transform is a transformation of the form

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : X + \xi \mapsto X + \xi + \iota_X B.$$

Definition.
A β-field or β-transform is a transform of the form

$$e^\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : X + \xi + \iota_\xi \beta.$$

B-fields and β-fields

- By definition,

$$e^B : X + \xi \mapsto X + \xi + \iota_X B$$

B-fields and β-fields

- By definition,

$$e^B : X + \xi \mapsto X + \xi + \iota_X B$$
$$\underbrace{X}_{T} + \underbrace{\xi + BX}_{T^*}$$

B-fields and β-fields

- By definition,

$$e^B : X + \xi \mapsto X + \xi + \iota_X B$$
$$\underbrace{X}_{T} + \underbrace{\xi + BX}_{T^*}$$

In particular, the B-transform is a shearing transformation which fixes projection onto T and shears in the "vertical" T^* direction.

B-fields and β-fields

- By definition,

$$e^B : X + \xi \mapsto X + \xi + \iota_X B$$
$$\underbrace{X}_{T} + \underbrace{\xi + BX}_{T^*}$$

In particular, the B-transform is a shearing transformation which fixes projection onto T and shears in the "vertical" T^* direction.

- Similarly,

$$e^\beta : X + \xi \mapsto \overbrace{X + \beta\xi}^{T} + \overbrace{\xi}^{T^*},$$

and so the β-transform fixes projection onto T^* and shears in the "horizontal" T direction.

Part II

Definitions

Definition.
A subspace $L < V \oplus V^*$ is *isotropic* when $\langle X, Y \rangle = 0$ for all $X, Y \in L$.

Definitions

Definition.
A subspace $L < V \oplus V^*$ is *isotropic* when $\langle X, Y \rangle = 0$ for all $X, Y \in L$.

Because $\langle \cdot\,, \cdot \rangle$ has signature (m, m), any isotropic subspace $L < V \oplus V^*$ has (real) dimension $\dim_{\mathbb{R}} L \leq m$.

Definitions

Definition.
A subspace $L < V \oplus V^*$ is *isotropic* when $\langle X, Y \rangle = 0$ for all $X, Y \in L$.

Because $\langle \cdot\,, \cdot \rangle$ has signature (m, m), any isotropic subspace $L < V \oplus V^*$ has (real) dimension $\dim_{\mathbb{R}} L \leq m$.

Definition.
An isotropic subspace $L < V \oplus V^*$ is *maximally isotropic* if $\dim_{\mathbb{R}} = m$.

Part III

Definition

Definition.
A *Lie algebroid* $(L, [\cdot\,,\,\cdot], a)$ is a vector bundle L on a smooth manifold M with Lie bracket $[\cdot\,,\,\cdot]$ on its module of C^∞ sections and a morphism $a : L \to T$ (called the *anchor*) inducing $\widetilde{a} : C^\infty(L) \to C^\infty(T)$ such that (i) $a([X, Y]) = [aX, aY]$ and (ii) $[X, fY] = f[X, Y] + (a(X)f)Y$ for all $X, Y \in C^\infty(L)$, $f \in C^\infty(M)$.

Examples

Ex 1. (Tangent Bundles)
Let $L = T$ with the usual Lie bracket of vector fields and the map $a = \mathrm{id}$.

Examples

Ex 1. (Tangent Bundles)
Let $L = T$ with the usual Lie bracket of vector fields and the map $a = \mathrm{id}$.

Ex 2. (Foliations)
A foliation $\mathcal{F}$ of M is an integrable subbundle of T. It's also a Lie algebroid with $L = \mathcal{F}$, the usual Lie bracket, and $a : \mathcal{F} \hookrightarrow T$ the usual inclusion map.

Examples

Ex 3. (Complex Structures)

A complex structure on a smooth manifold M^{2n} is an integrable endomorphism $J : T \to T$ such that $J^2 = -1$. In particular, J has eigenvectors of $\pm i$. Consider the subspace $L = T^{1,0} < T \otimes \mathbb{C}$ defined by

$$T^{1,0} = \{v \in T \,:\, Jv = iv\}.$$

This L is a complex bundle, is closed under the usual Lie bracket, with anchor map $a : L \hookrightarrow T$ the usual inclusion.

Some Structures on Lie Algebroids

Other generalized structures defined on Lie algebroids include:

Some Structures on Lie Algebroids

Other generalized structures defined on Lie algebroids include:

- Exterior derivative $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$.

Some Structures on Lie Algebroids

Other generalized structures defined on Lie algebroids include:

- Exterior derivative $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$.
- Interior product ι_X.

Some Structures on Lie Algebroids

Other generalized structures defined on Lie algebroids include:

- Exterior derivative $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$.
- Interior product ι_X.
- Lie derivative $\mathcal{L}_X^L = d_L \iota_X + \iota_X d_L$.

Some Structures on Lie Algebroids

Other generalized structures defined on Lie algebroids include:

- Exterior derivative $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$.
- Interior product ι_X.
- Lie derivative $\mathcal{L}_X^L = d_L \iota_X + \iota_X d_L$.
- Lie Algebroid connection

Some Structures on Lie Algebroids

Other generalized structures defined on Lie algebroids include:

- Exterior derivative $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$.
- Interior product ι_X.
- Lie derivative $\mathcal{L}_X^L = d_L \iota_X + \iota_X d_L$.
- Lie Algebroid connection
- Generalized foliations.

Some Structures on Lie Algebroids

Other generalized structures defined on Lie algebroids include:

- Exterior derivative $d_L : C^\infty(\wedge^k L^*) \to C^\infty(\wedge^{k+1} L^*)$.
- Interior product ι_X.
- Lie derivative $\mathcal{L}_X^L = d_L \iota_X + \iota_X d_L$.
- Lie Algebroid connection
- Generalized foliations.
- The so-called "Schouten bracket."

Part III

Definition

Definition.
The *Courant bracket* is the skew symmetric bracket on smooth sections of $T \oplus T^*$ given by

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d\left(\iota_X\eta - \iota_Y\xi\right).$$

Definition

Definition.
The *Courant bracket* is the skew symmetric bracket on smooth sections of $T \oplus T^*$ given by

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d\left(\iota_X\eta - \iota_Y\xi\right).$$

Remark.

1. If $\xi, \eta = 0$, then the Courant bracket is simply the Lie bracket. Also, $\pi = \pi_T : T \oplus T^* \to T$ satisfies $[\pi(A), \pi(B)] = \pi[A,B]$ for all $A, B \in C^\infty(T \oplus T^*)$.

Definition

Definition.
The *Courant bracket* is the skew symmetric bracket on smooth sections of $T \oplus T^*$ given by

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d\left(\iota_X\eta - \iota_Y\xi\right).$$

Remark.

1. If $\xi, \eta = 0$, then the Courant bracket is simply the Lie bracket. Also, $\pi = \pi_T : T \oplus T^* \to T$ satisfies $[\pi(A), \pi(B)] = \pi[A,B]$ for all $A, B \in C^\infty(T \oplus T^*)$.
2. If $X, Y = 0$, Courant bracket vanishes.

Relation to Lie Algebroids

- The first remark shows that π satisfies the first "anchor property" of Lie algebroids.

Relation to Lie Algebroids

- The first remark shows that π satisfies the first "anchor property" of Lie algebroids.
- Even so, $(T \oplus T^*, [\cdot, \cdot], \pi)$ fails to be a Lie algebroid.

Relation to Lie Algebroids

- The first remark shows that π satisfies the first "anchor property" of Lie algebroids.
- Even so, $(T \oplus T^*, [\cdot\,,\,\cdot], \pi)$ fails to be a Lie algebroid.
- This is because $[\cdot\,,\,\cdot]$ fails to satisfy the Jacobi identity.

Relation to Lie Algebroids

- The first remark shows that π satisfies the first "anchor property" of Lie algebroids.
- Even so, $(T \oplus T^*, [\cdot\,,\,\cdot], \pi)$ fails to be a Lie algebroid.
- This is because $[\cdot\,,\,\cdot]$ fails to satisfy the Jacobi identity.
- This failure can be made formal by introducing the $\mathrm{Jac}(\cdot\,,\,\cdot\,,\,\cdot)$ and $\mathrm{Nij}(\cdot\,,\,\cdot\,,\,\cdot)$ morphisms, and one can show that the Courant bracket satisfies
$$[A, fB] = f[A,B] + (\pi(A)f)B - \langle A, B\rangle df$$
for all $A, B \in T \oplus T^*$, $f \in C^\infty(M)$. Hence, it fails the second "anchor property."

Symmetries of the Courant Bracket

Motivation

Symmetries of the Courant Bracket

Motivation

The only symmetries of T preserving the usual Lie bracket are diffeomorphisms. We want the situation for $T \oplus T^*$.

Symmetries of the Courant Bracket

Motivation

The only symmetries of T preserving the usual Lie bracket are diffeomorphisms. We want the situation for $T \oplus T^*$.

Facts (Sans Proof)

Symmetries of the Courant Bracket

Motivation

The only symmetries of T preserving the usual Lie bracket are diffeomorphisms. We want the situation for $T \oplus T^*$.

Facts (Sans Proof)

- Both the Courant bracket and the inner product on $T \oplus T^*$ are invariant under diffeomorphism.

Symmetries of the Courant Bracket

Motivation

The only symmetries of T preserving the usual Lie bracket are diffeomorphisms. We want the situation for $T \oplus T^*$.

Facts (Sans Proof)

- Both the Courant bracket and the inner product on $T \oplus T^*$ are invariant under diffeomorphism.
- The B-field e^B is an automorphism preserving the Courant bracket if and only if $dB = 0$.

Symmetries of the Courant Bracket

Motivation

The only symmetries of T preserving the usual Lie bracket are diffeomorphisms. We want the situation for $T \oplus T^*$.

Facts (Sans Proof)

- Both the Courant bracket and the inner product on $T \oplus T^*$ are invariant under diffeomorphism.
- The B-field e^B is an automorphism preserving the Courant bracket if and only if $dB = 0$.
- In fact, the collection $\mathrm{Aut}_C(T \oplus T^*)$ of automorphisms on $T \oplus T^*$ preserving this Courant bracket is exactly

$$\mathrm{Aut}_C(T \oplus T^*) = \mathrm{Diff}(M) \ltimes \Omega^2_{\mathrm{closed}}(M).$$

Part III

Definition

Definitions.

Definition

Definitions.

1. A real, maximal isotropic subbundle $L < T \oplus T^*$ is an *almost-Dirac structure.*

Definition

Definitions.

1. A real, maximal isotropic subbundle $L < T \oplus T^*$ is an *almost-Dirac structure.*
2. If L is also closed under the Courant bracket (i.e., is *involutive*), then L is *integrable* and is said to be a *Diract structure.*

Examples

Ex 1. (Symplectic Geometry)
T is maximal, isotropic, and involutive with respect to the Courant bracket. Therefore, T is a Dirac structure. Moreover, applying a non-degenerate closed 2-form $\omega \in \Omega^2_{\text{closed}}(M)$ to T yields another Dirac structure.

Examples

Ex 1. (Symplectic Geometry)
T is maximal, isotropic, and involutive with respect to the Courant bracket. Therefore, T is a Dirac structure. Moreover, applying a non-degenerate closed 2-form $\omega \in \Omega^2_{\text{closed}}(M)$ to T yields another Dirac structure.

Ex 2. (Foliated Geometry)
For $\Delta < T$ a smooth distribution of constant rank, $\Delta \oplus \text{Ann}(\Delta) < T \oplus T^*$ is almost-Dirac. To be Dirac, Δ must be integrable, which occurs if and only if M has a foliation induced by Δ.

Examples

Ex 3. Let $J \in \text{End}(T)$ be an almost-complex structure with $T^{0,1} < T \otimes \mathbb{C}$ the $(-i)$-eigenspace. Form the maximal isotropic subspace

$$L_J = T^{0,1} \oplus \text{Ann}(T^{0,1})$$

Examples

Ex 3. Let $J \in \mathrm{End}(T)$ be an almost-complex structure with $T^{0,1} < T \otimes \mathbb{C}$ the $(-i)$-eigenspace. Form the maximal isotropic subspace

$$L_J = T^{0,1} \oplus \mathrm{Ann}(T^{0,1})$$
$$= T^{0,1} \oplus \left(T^{1,0}\right)^*$$

Examples

Ex 3. Let $J \in \mathrm{End}(T)$ be an almost-complex structure with $T^{0,1} < T \otimes \mathbb{C}$ the $(-i)$-eigenspace. Form the maximal isotropic subspace

$$\begin{aligned} L_J &= T^{0,1} \oplus \mathrm{Ann}(T^{0,1}) \\ &= T^{0,1} \oplus \left(T^{1,0}\right)^* < (T \oplus T^*) \otimes \mathbb{C}, \end{aligned}$$

Examples

Ex 3. Let $J \in \mathrm{End}(T)$ be an almost-complex structure with $T^{0,1} < T \otimes \mathbb{C}$ the $(-i)$-eigenspace. Form the maximal isotropic subspace

$$\begin{aligned} L_J &= T^{0,1} \oplus \mathrm{Ann}(T^{0,1}) \\ &= T^{0,1} \oplus \left(T^{1,0}\right)^* < (T \oplus T^*) \otimes \mathbb{C}, \end{aligned}$$

which can be proven to be involuted if and only if J is integrable. Hence, complex structures are complex Dirac structures.

Part III

Definition

Definition.
A *generalized complex structure* is an endomorphism $J \in \mathrm{End}(T \oplus T^*)$ such that (i) $J^2 = -1$, (ii) $\langle JX, Y \rangle = \langle -X, JY \rangle$, and (iii) $T^{1,0}$ is involutive with respect to the Courant bracket.

Remark.
This can also be defined as an isotropic subbundle $E < (T \oplus T^*) \otimes \mathbb{C}$ which satisfies $E \oplus \overline{E} = (T \oplus T^*) \otimes \mathbb{C}$ and whose space of sections is closed under the Courant bracket.

Examples (Sans Justification)

Here are some examples of objects admitting generalized complex structures.

Examples (Sans Justification)

Here are some examples of objects admitting generalized complex structures.

- Complex manifolds.

Examples (Sans Justification)

Here are some examples of objects admitting generalized complex structures.

- Complex manifolds.
- Symplectic manifolds.

Examples (Sans Justification)

Here are some examples of objects admitting generalized complex structures.

- Complex manifolds.
- Symplectic manifolds.
- Holomorphic Poisson manifolds.

Examples (Sans Justification)

Here are some examples of objects admitting generalized complex structures.

- Complex manifolds.
- Symplectic manifolds.
- Holomorphic Poisson manifolds.
- 5 classes of "exotic" nilmanifolds.

References

- Marco Gualtieri, *Generalized Complex Geometry*.
- Nigel Hitchin, *Lectures on Generalized Geometry*.
- Nigel Hitchin *Generalized Calabi-Yau Manifolds*.

Thank you!