Limit Sets and Their Applications

Christopher Stover
Florida State University

Topology Seminar
February 25, 2014

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Part I

Definitions

Definition.

A **Möbius Transformation** in $\mathbb{R}^n$ is a bijective conformal orientation-preserving map $\varphi : S^{n-1} \to S^{n-1}$.

Notation

$$E^n \quad = \quad n\text{-dimensional Euclidean space}$$

Notation

$$
\begin{aligned}
E^n &= n\text{-dimensional Euclidean space} \\
&= \mathbb{R}^n \text{ with the standard Euclidean metric}
\end{aligned}
$$

Notation

$$
\begin{aligned}
E^n &= n\text{-dimensional Euclidean space} \\
&= \mathbb{R}^n \text{ with the standard Euclidean metric} \\
\widehat{E^n} &= E^n \cup \{\infty\}
\end{aligned}
$$

Notation

$$
\begin{aligned}
E^n &= n\text{-dimensional Euclidean space} \\
&= \mathbb{R}^n \text{ with the standard Euclidean metric} \\
\widehat{E^n} &= E^n \cup \{\infty\} \\
B^n &= \{x \in \widehat{E^n} : x < 1\}
\end{aligned}
$$

Notation

$$
\begin{aligned}
E^n &= n\text{-dimensional Euclidean space} \\
&= \mathbb{R}^n \text{ with the standard Euclidean metric} \\
\widehat{E^n} &= E^n \cup \{\infty\} \\
B^n &= \{x \in \widehat{E^n} : x < 1\} \\
M(B^n) &= \text{collection of Möbius transformations of } B^n
\end{aligned}
$$

Notation

$$
\begin{aligned}
E^n &= n\text{-dimensional Euclidean space} \\
&= \mathbb{R}^n \text{ with the standard Euclidean metric} \\
\widehat{E^n} &= E^n \cup \{\infty\} \\
B^n &= \{x \in \widehat{E^n} : x < 1\} \\
M(B^n) &= \text{collection of Möbius transformations of } B^n \\
&= \text{collection of Möbius transformations of } \widehat{E^n} \text{ that} \\
&\quad\ \text{leave } B^n \text{ invariant}
\end{aligned}
$$

Notation

$$
\begin{aligned}
E^n &= n\text{-dimensional Euclidean space} \\
&= \mathbb{R}^n \text{ with the standard Euclidean metric} \\
\widehat{E^n} &= E^n \cup \{\infty\} \\
B^n &= \{x \in \widehat{E^n} : x < 1\} \\
M(B^n) &= \text{collection of Möbius transformations of } B^n \\
&= \text{collection of Möbius transformations of } \widehat{E^n} \text{ that} \\
&\quad\ \text{leave } B^n \text{ invariant}
\end{aligned}
$$

Remark.

Clearly, there is a natural group action $M(B^n) \times B^n \to B^n$ defined by $(\varphi, x) \mapsto \varphi(x)$.

(More) Definitions

Definition.
An element $\varphi \in M(B^n)$ is:

- **elliptic** if it fixes a unique point of B^n and fixes no point of S^{n-1}.

(More) Definitions

Definition.
An element $\varphi \in M(B^n)$ is:

- **elliptic** if it fixes a unique point of B^n and fixes no point of S^{n-1}.
- **parabolic** if it fixes no point of B^n and fixes a unique point of S^{n-1}.

(More) Definitions

Definition.
An element $\varphi \in M(B^n)$ is:

- **elliptic** if it fixes a unique point of B^n and fixes no point of S^{n-1}.
- **parabolic** if it fixes no point of B^n and fixes a unique point of S^{n-1}.
- **hyperbolic** if it fixes no point of B^n and fixes two points of S^{n-1}.

(More) Definitions

Definition.
An element $\varphi \in M(B^n)$ is:

- **elliptic** if it fixes a unique point of B^n and fixes no point of S^{n-1}.
- **parabolic** if it fixes no point of B^n and fixes a unique point of S^{n-1}.
- **hyperbolic** if it fixes no point of B^n and fixes two points of S^{n-1}.

Definition.
A point $a \in S^{n-1}$ is a **limit point** of a subgroup $G \leq M(B^n)$ if there is a point $x \in B^n$ and a sequence $\{g_i\}_{i=1}^{\infty}$ of elements of G such that $g_i x \to a$ as $i \to \infty$.

(Even More) Definitions

Definition.
The **limit set** $L(G)$ of a subgroup $G \leq M(B^n)$ is the collection of all limit points of G.

(Even More) Definitions

Definition.
The **limit set** $L(G)$ of a subgroup $G \leq M(B^n)$ is the collection of all limit points of G.

Remark.
This is a specific notion of the more general term *limit set* appearing in the study of dynamical systems and defined to be "the state of a dynamical system after an infinite amount of time."

Example: Apollonian Gasket

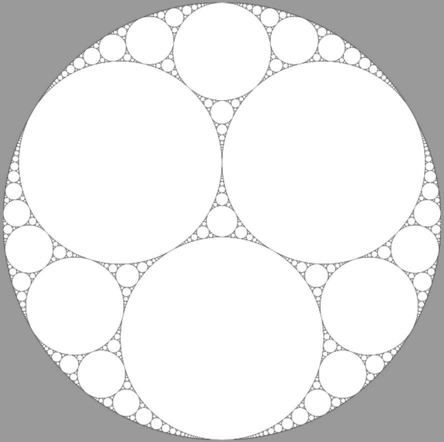

Figure 1
The Apollonian gasket is the limit of an iterated process

Background Results for $M(B^n)$ Sans Proof

Theorem 1.
If $a \in S^{n-1}$ is fixed by either a parabolic or hyperbolic element of a subgroup $G \leq M(B^n)$, then a is a limit point of G.

Background Results for $M(B^n)$ Sans Proof

Theorem 1.
If $a \in S^{n-1}$ is fixed by either a parabolic or hyperbolic element of a subgroup $G \leq M(B^n)$, then a is a limit point of G.

Theorem 2.
Let G be a subgroup of $M(B^n)$. Then $L(G)$ is empty if and only if G has a finite orbit in both B^n and $\overline{B^n}$.

Background Results for $M(B^n)$ Sans Proof

Theorem 1.
If $a \in S^{n-1}$ is fixed by either a parabolic or hyperbolic element of a subgroup $G \leq M(B^n)$, then a is a limit point of G.

Theorem 2.
Let G be a subgroup of $M(B^n)$. Then $L(G)$ is empty if and only if G has a finite orbit in both B^n and $\overline{B^n}$.

- Such a group G is said to be "elementary of elliptic type"—*elementary* because of $\overline{B^n}$ and *...of elliptic type* because of B^n.

Background Results for $M(B^n)$ Sans Proof

Theorem 1.
If $a \in S^{n-1}$ is fixed by either a parabolic or hyperbolic element of a subgroup $G \leq M(B^n)$, then a is a limit point of G.

Theorem 2.
Let G be a subgroup of $M(B^n)$. Then $L(G)$ is empty if and only if G has a finite orbit in both B^n and $\overline{B^n}$.

- Such a group G is said to be "elementary of elliptic type"—*elementary* because of $\overline{B^n}$ and *...of elliptic type* because of B^n.
- G has finite orbit in B^n if for some $x \in B^n$, the cardinality of the set $\{gx : g \in G\}$ is finite.

Background Results for $M(B^n)$ Sans Proof

Theorem 3.
Any subgroup $G \leq M(B^n)$ for which $L(G)$ is finite is elementary (i.e., it has a finite orbit in $\overline{B^n}$) and necessarily has at most two limit points.

Part II

The LIM Program

McMullen's description in the "Read Me" file:

The LIM Program

McMullen's description in the "Read Me" file:

Limit Sets of Kleinian Groups

The program lim draws the orbits of circles under the action of a group of Möbius transformations.

The LIM Program

McMullen's description in the "Read Me" file:

Limit Sets of Kleinian Groups

The program lim draws the orbits of circles under the action of a group of Möbius transformations.

Remark.

The LIM Program

McMullen's description in the "Read Me" file:

Limit Sets of Kleinian Groups

The program lim draws the orbits of circles under the action of a group of Möbius transformations.

Remark.

- Here, we're in the case of $n = 2$.

The LIM Program

McMullen's description in the "Read Me" file:

Limit Sets of Kleinian Groups

The program lim draws the orbits of circles under the action of a group of Möbius transformations.

Remark.

- Here, we're in the case of $n = 2$.
- $M(\mathbb{C}) \cong \mathrm{PSL}(2, \mathbb{C})$

The LIM Program

McMullen's description in the "Read Me" file:

Limit Sets of Kleinian Groups

The program lim draws the orbits of circles under the action of a group of Möbius transformations.

Remark.

- Here, we're in the case of $n = 2$.
- $M(\mathbb{C}) \cong \mathrm{PSL}(2, \mathbb{C})$
- A **Kleinian Group** is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

How It Works—Short Version

Required Input

- Circles $c_1, \ldots, c_i$ known to be in the limit set

How It Works—Short Version

Required Input

- Circles $c_1, \ldots, c_i$ known to be in the limit set

Technical Input

- Threshold variables

How It Works—Short Version

Required Input

- Circles $c_1, \ldots, c_i$ known to be in the limit set

Technical Input

- Threshold variables
- Output style options

How It Works—Short Version

Required Input

- Circles $c_1, \ldots, c_i$ known to be in the limit set

Optional Input

- Circles $r_1, \ldots, r_k$ in which to define reflections for c_α

Technical Input

- Threshold variables
- Output style options

How It Works—Short Version

Required Input

- Circles $c_1, \ldots, c_i$ known to be in the limit set

Technical Input

- Threshold variables
- Output style options

Optional Input

- Circles $r_1, \ldots, r_k$ in which to define reflections for c_α
- Matrices $m_1, m_2, \ldots, m_j$, $t_1, \ldots, t_\ell \in \mathrm{PSL}(2, \mathbb{C})$ to be applied to the c_α and to the coordinate system, respectively

How It Works—Short Version

Required Input

- Circles $c_1, \ldots, c_i$ known to be in the limit set

Technical Input

- Threshold variables
- Output style options

Optional Input

- Circles $r_1, \ldots, r_k$ in which to define reflections for c_α
- Matrices $m_1, m_2, \ldots, m_j$, $t_1, \ldots, t_\ell \in \mathrm{PSL}(2, \mathbb{C})$ to be applied to the c_α and to the coordinate system, respectively
- Circles $u_1, \ldots, u_n$ in which to define reflections of the coordinate system

How It Works—Short Version

Behind the Scenes

- LIM applies the group $G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle$ to the collection $C = \{c_\alpha\}$.

How It Works—Short Version

Behind the Scenes

- LIM applies the group $G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle$ to the collection $C = \{c_\alpha\}$.
- It also applies the group $G' = \langle t_1, \ldots, t_\ell, u_1, \ldots, u_n \rangle$ to the coordinate system.

How It Works—Short Version

Behind the Scenes

- LIM applies the group
  $G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle$
  to the collection $C = \{c_\alpha\}$.
- It also applies the group
  $G' = \langle t_1, \ldots, t_\ell, u_1, \ldots, u_n \rangle$
  to the coordinate system.
- Iterates of these group actions are stored in stacks, parsed, sorted, and finalized.

How It Works—Short Version

Behind the Scenes

- LIM applies the group $G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle$ to the collection $C = \{c_\alpha\}$.
- It also applies the group $G' = \langle t_1, \ldots, t_\ell, u_1, \ldots, u_n \rangle$ to the coordinate system.
- Iterates of these group actions are stored in stacks, parsed, sorted, and finalized.
- The loop ends when the stacks are full or when optional user-input thresholds are reached.

How It Works—Short Version

Behind the Scenes

- LIM applies the group $G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle$ to the collection $C = \{c_\alpha\}$.
- It also applies the group $G' = \langle t_1, \ldots, t_\ell, u_1, \ldots, u_n \rangle$ to the coordinate system.
- Iterates of these group actions are stored in stacks, parsed, sorted, and finalized.
- The loop ends when the stacks are full or when optional user-input thresholds are reached.

Output

- The raw output is data in .ps format.

How It Works—Short Version

Behind the Scenes

- LIM applies the group $G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle$ to the collection $C = \{c_\alpha\}$.
- It also applies the group $G' = \langle t_1, \ldots, t_\ell, u_1, \ldots, u_n \rangle$ to the coordinate system.
- Iterates of these group actions are stored in stacks, parsed, sorted, and finalized.
- The loop ends when the stacks are full or when optional user-input thresholds are reached.

Output

- The raw output is data in .ps format.
- This can be converted to visual representations in .pdf format.

Example 1

hex.run

```
./lim -d 8 -s -h 3 <<eof > hex.ps
c 0.866025403784438 0.0 -0.5
c 0.25 0.433012701892219 -0.166666666666
c -0.25 0.433012701892219 -0.833333333333
r 0.866025403784438 0.0 -0.5
r 0.25 0.433012701892219 -0.166666666666
r -0.25 0.433012701892219 -0.833333333333
eof
```

Graph on sphere; omit to graph in plane
Output file name
Two different threshold variables

Example 1

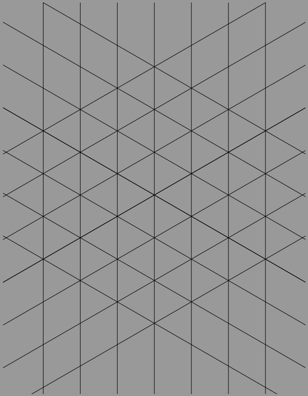

Figure 2
`hex.ps` without -s

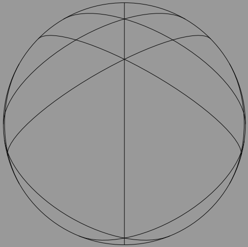

Figure 3
`hex.ps` with -s

Example 2

Example.run

```
./lim -s -d 60 -e 0.0001 <<eof > Example.ps
c 0.  0.  1
m 1 1 0 1 0 -1 1 -1
m 1 -1 0 -1 0 1 1 1
m 0.955 -0.025 0.045 0.025 -1.955 0.025 0.955 -0.025
m 0.955 -0.025 -0.045 -0.025 1.955 -0.025 0.955 -0.025
eof
```

A different threshold variable

Example 2

Example.run

```
./lim -s -d 60 -e 0.0001 <<eof > Example.ps
c 0.  0.  1
m 1 1 0 1 0 -1 1 -1
m 1 -1 0 -1 0 1 1 1
m 0.955 -0.025 0.045 0.025 -1.955 0.025 0.955 -0.025
m 0.955 -0.025 -0.045 -0.025 1.955 -0.025 0.955 -0.025
eof
```

Remark.

A different threshold variable

Example 2

Example.run

```
./lim -s -d 60 -e 0.0001 <<eof > Example.ps
c 0.  0.  1
m 1 1 0 1 0 -1 1 -1
m 1 -1 0 -1 0 1 1 1
m 0.955 -0.025 0.045 0.025 -1.955 0.025 0.955 -0.025
m 0.955 -0.025 -0.045 -0.025 1.955 -0.025 0.955 -0.025
eof
```

Remark.
According to McMullen: “This is a picture of the limit set of a Kleinian group on the boundary of Maskit’s embedding of the Teichmuller space of a once-punctured torus.”

A different threshold variable

Example 2

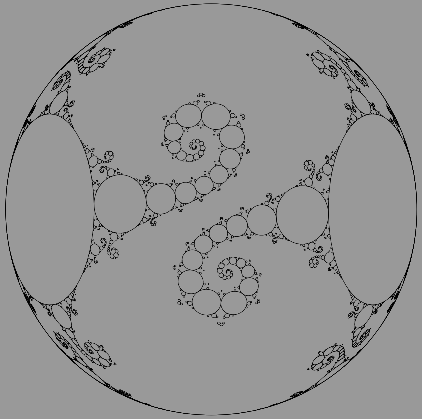

Figure 4
`Example.ps`

Example 3

Schottky.run

```
./lim -d 10 -e .001 <<eof > schottky2.ps
r 0 1 .7
r 0.866025 -.5 .8
r -0.866025 -.5 .8
c 0 1 .7
c 0.866025 -.5 .8
c -0.866025 -.5 .8
eof
```

Example 3

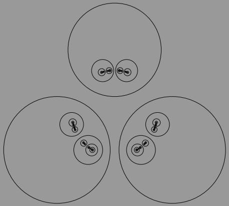

Figure 5
In the plane

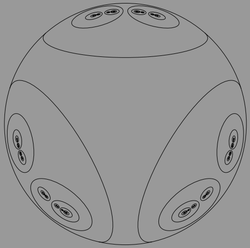

Figure 6
On the sphere

Example 4

ngon4.run

```
./lim -a 1000 -b -d 100 -e 0.001
-c 0 0 1 -w -1.1 -1.1 1.1 1.1
<<eof > ngon4.ps
r 1.553773974030037 0 1.189207115002721
c 1.553773974030037 0 1.189207115002721
r 0 1.553773974030037 1.189207115002721
c 0 1.553773974030037 1.189207115002721
r -1.553773974030037 0 1.189207115002721
c -1.553773974030037 0 1.189207115002721
r 0 -1.553773974030037 1.189207115002721
c 0 -1.553773974030037 1.189207115002721

eof
```

Optional style parameter
A *different* threshold variable

Example 4

ngon4.run

```
./lim -a 1000 -b -d 100 -e 0.001
-c 0 0 1 -w -1.1 -1.1 1.1 1.1
<<eof > ngon4.ps
r 1.553773974030037 0 1.189207115002721
c 1.553773974030037 0 1.189207115002721
r 0 1.553773974030037 1.189207115002721
c 0 1.553773974030037 1.189207115002721
r -1.553773974030037 0 1.189207115002721
c -1.553773974030037 0 1.189207115002721
r 0 -1.553773974030037 1.189207115002721
c 0 -1.553773974030037 1.189207115002721
eof
```

Remark.

Optional style parameter
A *different* threshold variable

Example 4

ngon4.run

```
./lim -a 1000 -b -d 100 -e 0.001
-c 0 0 1 -w -1.1 -1.1 1.1 1.1
<<eof > ngon4.ps
r 1.553773974030037 0 1.189207115002721
c 1.553773974030037 0 1.189207115002721
r 0 1.553773974030037 1.189207115002721
c 0 1.553773974030037 1.189207115002721
r -1.553773974030037 0 1.189207115002721
c -1.553773974030037 0 1.189207115002721
r 0 -1.553773974030037 1.189207115002721
c 0 -1.553773974030037 1.189207115002721

eof
```

Remark.

According to McMullen: "Tiling of H for torus with orbifold point of order 2."

Optional style parameter

A *different* threshold variable

Example 4

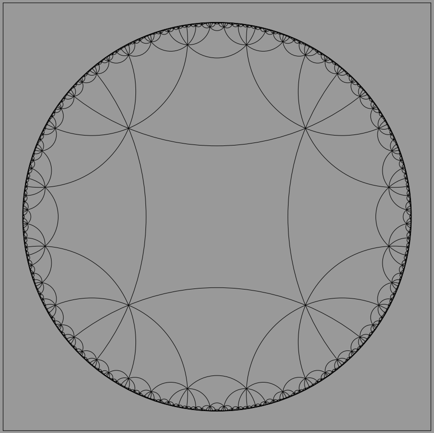

Figure 7

`ngon4.ps` + a box because of -b

Example 5

lattice.run

```
./lim -s -d 10 <<eof > lattice.ps
c 0 0 0.0
c 0 0 -.5
m 1 0 1 0 0 0 1 0
m 1 0 0 1 0 0 1 0
m 1 1 0 0 0 0 1 0
u .3 .4 2
eof
```

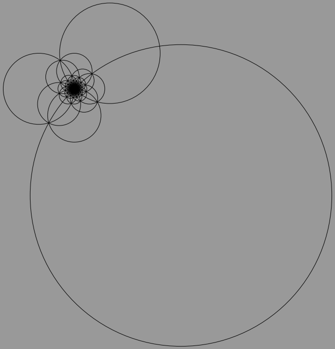

Figure 8
In the plane

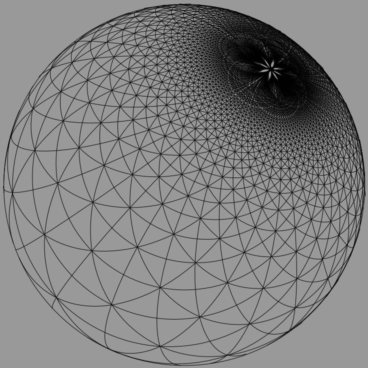

Figure 9
On the sphere

Part III

Introduction

For the sake of brevity, some background will be assumed. In particular, I won't take the time to define the following (important) terms:

Introduction

For the sake of brevity, some background will be assumed. In particular, I won't take the time to define the following (important) terms:

- Foliation / Foliated Manifold

Introduction

For the sake of brevity, some background will be assumed. In particular, I won't take the time to define the following (important) terms:

- Foliation / Foliated Manifold
- Leaf / Leaf Space of a foliation

Introduction

For the sake of brevity, some background will be assumed. In particular, I won't take the time to define the following (important) terms:

- Foliation / Foliated Manifold
- Leaf / Leaf Space of a foliation
- Dimension / Codimension of foliation

Introduction

For the sake of brevity, some background will be assumed. In particular, I won't take the time to define the following (important) terms:

- Foliation / Foliated Manifold
- Leaf / Leaf Space of a foliation
- Dimension / Codimension of foliation
- Reeb components

Introduction

For the sake of brevity, some background will be assumed. In particular, I won't take the time to define the following (important) terms:

- Foliation / Foliated Manifold
- Leaf / Leaf Space of a foliation
- Dimension / Codimension of foliation
- Reeb components
- Finite-depth foliations

Introduction

For the sake of brevity, some background will be assumed. In particular, I won't take the time to define the following (important) terms:

- Foliation / Foliated Manifold
- Leaf / Leaf Space of a foliation
- Dimension / Codimension of foliation
- Reeb components
- Finite-depth foliations
- Depth of a leaf within finite-depth foliations

Introduction

For the sake of brevity, some background will be assumed. In particular, I won't take the time to define the following (important) terms:

- Foliation / Foliated Manifold
- Leaf / Leaf Space of a foliation
- Dimension / Codimension of foliation
- Reeb components
- Finite-depth foliations
- Depth of a leaf within finite-depth foliations
- Fibers / bundle theory

Big Idea

Is it possible to modify the above results in order to get decent pictures of the limit sets of the lifts of finite-depth foliations to the universal cover of hyperbolic 3-manifolds?

Context-Specific Things

- There are several notions of *limit set* in this context.

Context-Specific Things

- There are several notions of *limit set* in this context.
  1. On a manifold M^3 which is hyperbolic, the universal cover $\widetilde{M}$ is homeomorphic to the hyperbolic space H^3. In this case, it makes sense to talk about the limit set $L(G)$ of a group G of Möbius transformations acting on H^3 (or S^2).

Context-Specific Things

- There are several notions of *limit set* in this context.
    1. On a manifold M^3 which is hyperbolic, the universal cover $\widetilde{M}$ is homeomorphic to the hyperbolic space H^3. In this case, it makes sense to talk about the limit set $L(G)$ of a group G of Möbius transformations acting on H^3 (or S^2).
    2. If M^3 is a hyperbolic manifold and $\mathcal{F}$ is a codimension-one Reebless foliation on M, the lift $\widetilde{\mathcal{F}}$ is a foliation of H^3 and the leaves $\mathcal{L}$ of $\mathcal{F}$ are planes. In particular, $\mathcal{L}$ is non-compact and so it makes sense to talk about the limit set of $\mathcal{L}$ as the collection of accumulation points of $\mathcal{L}$ in the sphere at infinity S^2_∞.

Context-Specific Things

- There are several notions of *limit set* in this context.
  1. On a manifold M^3 which is hyperbolic, the universal cover $\widetilde{M}$ is homeomorphic to the hyperbolic space H^3. In this case, it makes sense to talk about the limit set $L(G)$ of a group G of Möbius transformations acting on H^3 (or S^2).
  2. If M^3 is a hyperbolic manifold and $\mathcal{F}$ is a codimension-one Reebless foliation on M, the lift $\widetilde{\mathcal{F}}$ is a foliation of H^3 and the leaves $\mathcal{L}$ of $\mathcal{F}$ are planes. In particular, $\mathcal{L}$ is non-compact and so it makes sense to talk about the limit set of $\mathcal{L}$ as the collection of accumulation points of $\mathcal{L}$ in the sphere at infinity S^2_∞.
- Both of these ideas may be relevant when talking about the limit sets of finite depth foliations.

Finite-Depth Foliations & Limit Sets of Their Lifts

- By definition, any depth-zero leaf $\mathcal{L}_0$ of $\mathcal{F}$ is compact. From results of Thurston, Bonahon, and Marden, it follows that either:

Finite-Depth Foliations & Limit Sets of Their Lifts

- By definition, any depth-zero leaf $\mathcal{L}_0$ of $\mathcal{F}$ is compact. From results of Thurston, Bonahon, and Marden, it follows that either:
    1. $\mathcal{L}_0$ is (up to finite covers) a fiber of M over the circle. In this case, the limit set of the lift $\widetilde{\mathcal{L}_0}$ is all of S^2_∞.

Finite-Depth Foliations & Limit Sets of Their Lifts

- By definition, any depth-zero leaf $\mathcal{L}_0$ of $\mathcal{F}$ is compact. From results of Thurston, Bonahon, and Marden, it follows that either:
  1. $\mathcal{L}_0$ is (up to finite covers) a fiber of M over the circle. In this case, the limit set of the lift $\widetilde{\mathcal{L}_0}$ is all of S^2_∞.
  2. $\mathcal{L}_0$ corresponds to a quasi-Fuchsian subgroup of $\pi_1(M)$ and the limit set of the lift $\widetilde{\mathcal{L}_0}$ is a quasicircle $\mathcal{C}_0$ of Hausdorff dimension less than 2.

Finite-Depth Foliations & Limit Sets of Their Lifts

- By definition, any depth-zero leaf $\mathcal{L}_0$ of $\mathcal{F}$ is compact. From results of Thurston, Bonahon, and Marden, it follows that either:
  1. $\mathcal{L}_0$ is (up to finite covers) a fiber of M over the circle. In this case, the limit set of the lift $\widetilde{\mathcal{L}_0}$ is all of S^2_∞.
  2. $\mathcal{L}_0$ corresponds to a quasi-Fuchsian subgroup of $\pi_1(M)$ and the limit set of the lift $\widetilde{\mathcal{L}_0}$ is a quasicircle $\mathcal{C}_0$ of Hausdorff dimension less than 2.
- Let $\mathcal{D}_0$ be the collection of depth-zero leaves, let M_1 be the closure of a component of $\overline{M - \mathcal{D}_0}$ such that $\mathcal{L}_0 \in \partial M_1$, and let $\mathcal{L}_1$ be a depth-one leaf in M_1. Then:

Finite-Depth Foliations & Limit Sets of Their Lifts

- By definition, any depth-zero leaf $\mathcal{L}_0$ of $\mathcal{F}$ is compact. From results of Thurston, Bonahon, and Marden, it follows that either:
  1. $\mathcal{L}_0$ is (up to finite covers) a fiber of M over the circle. In this case, the limit set of the lift $\widetilde{\mathcal{L}_0}$ is all of S^2_∞.
  2. $\mathcal{L}_0$ corresponds to a quasi-Fuchsian subgroup of $\pi_1(M)$ and the limit set of the lift $\widetilde{\mathcal{L}_0}$ is a quasicircle $\mathcal{C}_0$ of Hausdorff dimension less than 2.
- Let $\mathcal{D}_0$ be the collection of depth-zero leaves, let M_1 be the closure of a component of $\overline{M - \mathcal{D}_0}$ such that $\mathcal{L}_0 \in \partial M_1$, and let $\mathcal{L}_1$ be a depth-one leaf in M_1. Then:
  1. Iteratively applying elements $g \in \pi_1(M_1)$ to $L(\widetilde{\mathcal{L}_0})$ yields an element in the limit set of $\widetilde{\mathcal{L}_1}$.

Finite-Depth Foliations & Limit Sets of Their Lifts

- By definition, any depth-zero leaf $\mathcal{L}_0$ of $\mathcal{F}$ is compact. From results of Thurston, Bonahon, and Marden, it follows that either:
  1. $\mathcal{L}_0$ is (up to finite covers) a fiber of M over the circle. In this case, the limit set of the lift $\widetilde{\mathcal{L}_0}$ is all of S^2_∞.
  2. $\mathcal{L}_0$ corresponds to a quasi-Fuchsian subgroup of $\pi_1(M)$ and the limit set of the lift $\widetilde{\mathcal{L}_0}$ is a quasicircle $\mathcal{C}_0$ of Hausdorff dimension less than 2.
- Let $\mathcal{D}_0$ be the collection of depth-zero leaves, let M_1 be the closure of a component of $\overline{M - \mathcal{D}_0}$ such that $\mathcal{L}_0 \in \partial M_1$, and let $\mathcal{L}_1$ be a depth-one leaf in M_1. Then:
  1. Iteratively applying elements $g \in \pi_1(M_1)$ to $L(\widetilde{\mathcal{L}_0})$ yields an element in the limit set of $\widetilde{\mathcal{L}_1}$.
  2. The union of all such iterates is dense therein.

What's Next?

My tentative work plan moving forward is to:

What's Next?

My tentative work plan moving forward is to:

- Spend (a considerable amount of) time learning the theory of limit sets and foliations.

What's Next?

My tentative work plan moving forward is to:

- Spend (a considerable amount of) time learning the theory of limit sets and foliations.
- Investigate ways to code quasiconformal mappings (or approximations thereof) using finitely-much data.

What's Next?

My tentative work plan moving forward is to:

- Spend (a considerable amount of) time learning the theory of limit sets and foliations.
- Investigate ways to code quasiconformal mappings (or approximations thereof) using finitely-much data.
- Work on understanding McMullen's LIM well enough to modify its functionality to this context.

What's Next?

My tentative work plan moving forward is to:

- Spend (a considerable amount of) time learning the theory of limit sets and foliations.
- Investigate ways to code quasiconformal mappings (or approximations thereof) using finitely-much data.
- Work on understanding McMullen's LIM well enough to modify its functionality to this context.
- Win a Fields medal. :)

Thank you!