Limit Sets and Their Applications

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Topology Seminar
February 25, 2014
Outline

Introduction to Limit Sets
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Introduction to Limit Sets
Preliminaries
Outline

Introduction to Limit Sets
  Preliminaries
  A (More General) Example
Outline

Introduction to Limit Sets
  Preliminaries
  A (More General) Example
  Basic Results about Limit Points/Sets
Outline

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  Preliminaries
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Curt McMullen & LIM
Outline

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  A (More General) Example
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Curt McMullen & LIM
  Introduction to LIM
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- Preliminaries
- A (More General) Example
- Basic Results about Limit Points/Sets

Curt McMullen & LIM
- Introduction to LIM
- Some Technical Stuff
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  Preliminaries
  A (More General) Example
  Basic Results about Limit Points/Sets

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  Some Technical Stuff
  Examples and Output
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Part I

Introduction to Limit Sets

Preliminaries
A (More General) Example
Basic Results about Limit Points/Sets

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Applications & Future Work
Applications to Foliation Theory
Definitions

Definition.
A Möbius Transformation in $\mathbb{R}^n$ is a bijective conformal orientation-preserving map $\varphi : S^{n-1} \to S^{n-1}$. 
Notation

\[ E^n \quad = \quad n\text{-dimensional Euclidean space} \]
Limit Sets

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Notation

$E^n = n$-dimensional Euclidean space

$= \mathbb{R}^n$ with the standard Euclidean metric
Notation

\[ E^n = n\text{-dimensional Euclidean space} \]
\[ = \mathbb{R}^n \text{ with the standard Euclidean metric} \]
\[ \hat{E}^n = E^n \cup \{\infty\} \]
Notation

\[\begin{align*}
E^n &= n\text{-dimensional Euclidean space} \\
&= \mathbb{R}^n \text{ with the standard Euclidean metric} \\
\hat{E}^n &= E^n \cup \{\infty\} \\
B^n &= \{x \in \hat{E}^n : x < 1\}
\end{align*}\]
Limit Sets

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Notation

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M(B^n) &= \text{collection of Möbius transformations of } B^n
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\[ = \text{collection of Möbius transformations of } \widehat{E}^n \text{ that leave } B^n \text{ invariant} \]
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\begin{align*}
  E^n &= \text{n-dimensional Euclidean space} \\
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\end{align*}

**Remark.**

Clearly, there is a natural group action \( M(B^n) \times B^n \to B^n \) defined by \((\varphi, x) \mapsto \varphi(x)\).
(More) Definitions

Definition.
An element $\varphi \in M(B^n)$ is:

- **elliptic** if it fixes a unique point of $B^n$ and fixes no point of $S^{n-1}$.
(More) Definitions

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An element $\varphi \in M(B^n)$ is:

- **elliptic** if it fixes a unique point of $B^n$ and fixes no point of $S^{n-1}$.

- **parabolic** if it fixes no point of $B^n$ and fixes a unique point of $S^{n-1}$. 
(More) Definitions

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- **elliptic** if it fixes a unique point of $B^n$ and fixes no point of $S^{n-1}$.
- **parabolic** if it fixes no point of $B^n$ and fixes a unique point of $S^{n-1}$.
- **hyperbolic** if it fixes no point of $B^n$ and fixes two points of $S^{n-1}$. 
(More) Definitions

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- **hyperbolic** if it fixes no point of $B^n$ and fixes two points of $S^{n-1}$.

Definition.
A point $a \in S^{n-1}$ is a **limit point** of a subgroup $G \leq M(B^n)$ if there is a point $x \in B^n$ and a sequence $\{g_i\}_{i=1}^{\infty}$ of elements of $G$ such that $g_ix \to a$ as $i \to \infty$. 
(Even More) Definitions

Definition.
The limit set $L(G)$ of a subgroup $G \leq M(B^n)$ is the collection of all limit points of $G$. 
(Even More) Definitions

Definition.
The limit set $L(G)$ of a subgroup $G \leq M(B^n)$ is the collection of all limit points of $G$.

Remark.
This is a specific notion of the more general term limit set appearing in the study of dynamical systems and defined to be “the state of a dynamical system after an infinite amount of time.”
Example: Apollonian Gasket

Figure 1
The Apollonian gasket is the limit of an iterated process
Background Results for $M(B^n)$ Sans Proof

**Theorem 1.**

If $a \in S^{n-1}$ is fixed by either a parabolic or hyperbolic element of a subgroup $G \leq M(B^n)$, then $a$ is a limit point of $G$. 
Background Results for $M(B^n)$ Sans Proof

**Theorem 1.**

*If $a \in S^{n-1}$ is fixed by either a parabolic or hyperbolic element of a subgroup $G \leq M(B^n)$, then $a$ is a limit point of $G$."

**Theorem 2.**

*Let $G$ be a subgroup of $M(B^n)$. Then $L(G)$ is empty if and only if $G$ has a finite orbit in both $B^n$ and $\overline{B^n}$.**
Background Results for $M(B^n)$ Sans Proof

**Theorem 1.**

If $a \in S^{n-1}$ is fixed by either a parabolic or hyperbolic element of a subgroup $G \leq M(B^n)$, then $a$ is a limit point of $G$.

**Theorem 2.**

Let $G$ be a subgroup of $M(B^n)$. Then $L(G)$ is empty if and only if $G$ has a finite orbit in both $B^n$ and $\overline{B^n}$.

- Such a group $G$ is said to be “elementary of elliptic type”—elementary because of $\overline{B^n}$ and ...of elliptic type because of $B^n$. 
Theorem 1.
If \( a \in S^{n-1} \) is fixed by either a parabolic or hyperbolic element of a subgroup \( G \leq M(B^n) \), then \( a \) is a limit point of \( G \).

Theorem 2.
Let \( G \) be a subgroup of \( M(B^n) \). Then \( L(G) \) is empty if and only if \( G \) has a finite orbit in both \( B^n \) and \( \overline{B^n} \).

- Such a group \( G \) is said to be “elementary of elliptic type”—elementary because of \( \overline{B^n} \) and ...of elliptic type because of \( B^n \).
- \( G \) has finite orbit in \( B^n \) if for some \( x \in B^n \), the cardinality of the set \( \{gx : g \in G\} \) is finite.
Background Results for $M(B^n)$ Sans Proof

**Theorem 3.**
Any subgroup $G \leq M(B^n)$ for which $L(G)$ is finite is elementary (i.e., it has a finite orbit in $\overline{B^n}$) and necessarily has at most two limit points.
Part II

Introduction to Limit Sets
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A (More General) Example
Basic Results about Limit Points/Sets

Curt McMullen & LIM
Introduction to LIM
Some Technical Stuff
Examples and Output

Applications & Future Work
Applications to Foliation Theory
The lim Program

McMullen’s description in the “Read Me” file:
The LIM Program

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*Limit Sets of Kleinian Groups*

The program lim draws the orbits of circles under the action of a group of Möbius transformations.
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Remark.
The LIM Program

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The program lim draws the orbits of circles under the action of a group of Möbius transformations.

Remark.

- Here, we’re in the case of $n = 2$. 
The LIM Program

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The program lim draws the orbits of circles under the action of a group of Möbius transformations.

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- Here, we’re in the case of $n = 2$.
- $M(\mathbb{C}) \cong \text{PSL}(2, \mathbb{C})$
The LIM Program

McMullen’s description in the “Read Me” file:

*Limit Sets of Kleinian Groups*

The program lim draws the orbits of circles under the action of a group of Möbius transformations.

**Remark.**

- Here, we’re in the case of $n = 2$.
- $M(\mathbb{C}) \cong \text{PSL}(2, \mathbb{C})$
- A **Kleinian Group** is a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. 
How It Works—Short Version

Required Input

- Circles $c_1, \ldots, c_i$ known to be in the limit set
How It Works—Short Version

Required Input

- Circles $c_1, \ldots, c_i$ known to be in the limit set

Technical Input

- Threshold variables
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Technical Input

- Threshold variables
- Output style options
How It Works—Short Version

Required Input

• Circles $c_1, \ldots, c_i$ known to be in the limit set

Optional Input

• Circles $r_1, \ldots, r_k$ in which to define reflections for $c_\alpha$

Technical Input

• Threshold variables
• Output style options
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- Circles $c_1, \ldots, c_i$ known to be in the limit set

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Technical Input

- Threshold variables
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Optional Input

- Matrices $m_1, m_2, \ldots, m_j$, $t_1, \ldots, t_\ell \in \text{PSL}(2, \mathbb{C})$ to be applied to the $c_\alpha$ and to the coordinate system, respectively
How It Works—Short Version

Required Input

- Circles \(c_1, \ldots, c_i\) known to be in the limit set

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Technical Input

- Threshold variables
- Output style options

Optional Input

- Matrices \(m_1, m_2, \ldots, m_j, t_1, \ldots, t_\ell \in \text{PSL}(2, \mathbb{C})\) to be applied to the \(c_\alpha\) and to the coordinate system, respectively
- Circles \(u_1, \ldots, u_n\) in which to define reflections of the coordinate system
How It Works—Short Version

Behind the Scenes

- LIM applies the group
  \[ G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle \]
  to the collection \( C = \{ c_\alpha \} \).
How It Works—Short Version

Behind the Scenes

- **LIM** applies the group
  \[ G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle \]
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- It also applies the group
  \[ G' = \langle t_1, \ldots, t_\ell, u_1, \ldots, u_n \rangle \]
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• Iterates of these group actions are stored in stacks, parsed, sorted, and finalized.
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- The loop ends when the stacks are full or when optional user-input thresholds are reached.
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Output

- The raw output is data in .ps format.
How It Works—Short Version

Behind the Scenes

- **LIM** applies the group $G = \langle m_1, \ldots, m_j, r_1, \ldots, r_k \rangle$ to the collection $C = \{c_\alpha\}$.
- It also applies the group $G' = \langle t_1, \ldots, t_\ell, u_1, \ldots, u_n \rangle$ to the coordinate system.
- Iterates of these group actions are stored in stacks, parsed, sorted, and finalized.
- The loop ends when the stacks are full or when optional user-input thresholds are reached.

Output

- The raw output is data in .ps format.
- This can be converted to visual representations in .pdf format.
Example 1

**hex.run**

```
./lim -d 8 -s -h 3 <<eof > hex.ps

c 0.866025403784438 0.0 -0.5

c 0.25 0.433012701892219 -0.166666666666

c -0.25 0.433012701892219 -0.833333333333

r 0.866025403784438 0.0 -0.5

r 0.25 0.433012701892219 -0.166666666666

r -0.25 0.433012701892219 -0.833333333333

eof
```

Graph on sphere; omit to graph in plane

Output file name

Two different threshold variables
Example 1

Figure 2
hex.ps without -s

Figure 3
hex.ps with -s
Example 2

Example.run

./lim -s -d 60 -e 0.0001 <<eof > Example.ps
c 0. 0. 1
m 1 1 0 1 0 -1 1 -1
m 1 -1 0 -1 0 1 1 1
m 0.955 -0.025 0.045 0.025 -1.955 0.025 0.955 -0.025
m 0.955 -0.025 -0.045 -0.025 1.955 -0.025 0.955 -0.025
eof

Remark. According to McMullen: "This is a picture of the limit set of a Kleinian group on the boundary of Maskit's embedding of the Teichmüller space of a once-punctured torus."
Example 2

Example.run

./lim -s -d 60 -e 0.0001 <<eof > Example.ps
c 0. 0. 1
m 1 1 0 1 0 -1 1 -1
m 1 -1 0 -1 0 1 1 1
m 0.955 -0.025 0.045 0.025 -1.955 0.025 0.955 -0.025
m 0.955 -0.025 -0.045 -0.025 1.955 -0.025 0.955 -0.025
eof

Remark.

A different threshold variable
Example 2

**Example.run**

```bash
./lim -s -d 60 -e 0.0001 <<eof > Example.ps
c 0. 0. 1
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m 0.955 -0.025 0.045 0.025 -1.955 0.025 0.955 -0.025
m 0.955 -0.025 -0.045 -0.025 1.955 -0.025 0.955 -0.025
eof
```

**Remark.**

According to McMullen: “This is a picture of the limit set of a Kleinian group on the boundary of Maskit’s embedding of the Teichmüller space of a once-punctured torus.”

---

A different threshold variable
Example 2

Figure 4

Example.ps
Example 3

*Schottky.run*

```bash
./lim -d 10 -e .001 <<eof > schottky2.ps
r 0 1 .7
r 0.866025 -.5 .8
r -0.866025 -.5 .8
c 0 1 .7
c 0.866025 -.5 .8
c -0.866025 -.5 .8
eof
```
Example 3

Figure 5
In the plane

Figure 6
On the sphere
Example 4

**ngon4.run**

```bash
./lim -a 1000 -b -d 100 -e 0.001
-c 0 0 1 -w -1.1 -1.1 1.1 1.1
<<eof > ngon4.ps
r 1.553773974030037 0 1.189207115002721
c 1.553773974030037 0 1.189207115002721
r 0 1.553773974030037 1.189207115002721
c 0 1.553773974030037 1.189207115002721
r -1.553773974030037 0 1.189207115002721
c -1.553773974030037 0 1.189207115002721
r 0 -1.553773974030037 1.189207115002721
c 0 -1.553773974030037 1.189207115002721

eof
```

Remark. According to McMullen: "Tiling of $H$ for torus with orbifold point of order 2.

Optional style parameter

A different threshold variable
Example 4

*ngon4.run*

```
./lim -a 1000 -b -d 100 -e 0.001
-c 0 0 1 -w -1.1 -1.1 1.1 1.1
<<eof > ngon4.ps
r 1.553773974030037 0 1.189207115002721
c 1.553773974030037 0 1.189207115002721
r 0 1.553773974030037 1.189207115002721
c 0 1.553773974030037 1.189207115002721
r -1.553773974030037 0 1.189207115002721
c -1.553773974030037 0 1.189207115002721
r 0 -1.553773974030037 1.189207115002721
c 0 -1.553773974030037 1.189207115002721
eof
```

Remark.

Optional style parameter

*A different* threshold variable
Example 4

`ngon4.run`

```plaintext
./lim -a 1000 -b -d 100 -e 0.001 -c 0 0 1 -w -1.1 -1.1 1.1 1.1 <<eof > ngon4.ps
r 1.553773974030037 0 1.189207115002721
c 1.553773974030037 0 1.189207115002721
r 0 1.553773974030037 1.189207115002721
c 0 1.553773974030037 1.189207115002721
r -1.553773974030037 0 1.189207115002721
c -1.553773974030037 0 1.189207115002721
r 0 -1.553773974030037 1.189207115002721
c 0 -1.553773974030037 1.189207115002721
eof
```

**Remark.**

According to McMullen: “Tiling of $H$ for torus with orbifold point of order 2.”

Optional style parameter

*A different* threshold variable
Example 4

Figure 7

ngon4.ps + a box because of -b
Example 5

```
lattice.run

./lim -s -d 10 <<eof > lattice.ps
  c 0 0 0.0
  c 0 0 -.5
  m 1 0 1 0 0 0 1 0
  m 1 0 0 1 0 0 1 0
  m 1 1 0 0 0 0 1 0
  u .3 .4 2
  eof
```
Figure 8
In the plane

Figure 9
On the sphere
Part III

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Applications & Future Work
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For the sake of brevity, some background will be assumed. In particular, I won’t take the time to define the following (important) terms:
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- Leaf / Leaf Space of a foliation
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- Depth of a leaf within finite-depth foliations
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- Depth of a leaf within finite-depth foliations
- Fibers / bundle theory
Limit Sets
- Applications & Future Work
  - Applications to Foliation Theory

---

**Big Idea**

Is it possible to modify the above results in order to get decent pictures of the limit sets of the lifts of finite-depth foliations to the universal cover of hyperbolic 3-manifolds?
Context-Specific Things

- There are several notions of *limit set* in this context.
Limit Sets

Applications & Future Work

Applications to Foliation Theory

Context-Specific Things

• There are several notions of limit set in this context.

1. On a manifold $M^3$ which is hyperbolic, the universal cover $\tilde{M}$ is homeomorphic to the hyperbolic space $H^3$. In this case, it makes sense to talk about the limit set $L(G)$ of a group $G$ of Möbius transformations acting on $H^3$ (or $S^2$).
There are several notions of limit set in this context.

1. On a manifold $M^3$ which is hyperbolic, the universal cover $\tilde{M}$ is homeomorphic to the hyperbolic space $H^3$. In this case, it makes sense to talk about the limit set $L(G)$ of a group $G$ of Möbius transformations acting on $H^3$ (or $S^2$).

2. If $M^3$ is a hyperbolic manifold and $\mathcal{F}$ is a codimension-one Reebless foliation on $M$, the lift $\tilde{\mathcal{F}}$ is a foliation of $H^3$ and the leaves $\mathcal{L}$ of $\mathcal{F}$ are planes. In particular, $\mathcal{L}$ is non-compact and so it makes sense to talk about the limit set of $\mathcal{L}$ as the collection of accumulation points of $\mathcal{L}$ in the sphere at infinity $S_\infty^2$. 
Context-Specific Things

- There are several notions of limit set in this context.
  
  1. On a manifold $M^3$ which is hyperbolic, the universal cover $\tilde{M}$ is homeomorphic to the hyperbolic space $H^3$. In this case, it makes sense to talk about the limit set $L(G)$ of a group $G$ of Möbius transformations acting on $H^3$ (or $S^2$).

  2. If $M^3$ is a hyperbolic manifold and $\mathcal{F}$ is a codimension-one Reebless foliation on $M$, the lift $\tilde{\mathcal{F}}$ is a foliation of $H^3$ and the leaves $\mathcal{L}$ of $\mathcal{F}$ are planes. In particular, $\mathcal{L}$ is non-compact and so it makes sense to talk about the limit set of $\mathcal{L}$ as the collection of accumulation points of $\mathcal{L}$ in the sphere at infinity $S^2_\infty$.

- Both of these ideas may be relevant when talking about the limit sets of finite depth foliations.
Finite-Depth Foliations & Limit Sets of Their Lifts

- By definition, any depth-zero leaf $L_0$ of $\mathcal{F}$ is compact. From results of Thurston, Bonahon, and Marden, it follows that either:

  1. $L_0$ is (up to finite covers) a fiber of $M$ over the circle. In this case, the limit set of the lift $\tilde{L}_0$ is all of $S^2$.
  2. $L_0$ corresponds to a quasi-Fuchsian subgroup of $\pi_1(M)$ and the limit set of the lift $\tilde{L}_0$ is a quasicircle $C_0$ of Hausdorff dimension less than 2.
Finite-Depth Foliations & Limit Sets of Their Lifts

- By definition, any depth-zero leaf $\mathcal{L}_0$ of $\mathcal{F}$ is compact. From results of Thurston, Bonahon, and Marden, it follows that either:

  1. $\mathcal{L}_0$ is (up to finite covers) a fiber of $M$ over the circle. In this case, the limit set of the lift $\widetilde{\mathcal{L}}_0$ is all of $S^2_\infty$. 

- Let $D_0$ be the collection of depth-zero leaves, let $M_1$ be the closure of a component of $M - D_0$ such that $\mathcal{L}_0 \in \partial M_1$, and let $\mathcal{L}_1$ be a depth-one leaf in $M_1$. Then:

  1. Iteratively applying elements $g \in \pi_1(M_1)$ to $\widetilde{\mathcal{L}}$ yields an element in the limit set of $\widetilde{\mathcal{L}}_1$.

  2. The union of all such iterates is dense therein.
Finite-Depth Foliations & Limit Sets of Their Lifts

- By definition, any depth-zero leaf $\mathcal{L}_0$ of $\mathcal{F}$ is compact. From results of Thurston, Bonahon, and Marden, it follows that either:
  
  1. $\mathcal{L}_0$ is (up to finite covers) a fiber of $M$ over the circle. In this case, the limit set of the lift $\widetilde{\mathcal{L}}_0$ is all of $S^2_\infty$.
  
  2. $\mathcal{L}_0$ corresponds to a quasi-Fuchsian subgroup of $\pi_1(M)$ and the limit set of the lift $\widetilde{\mathcal{L}}_0$ is a quasicircle $\mathcal{C}_0$ of Hausdorff dimension less than 2.
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What’s Next?

My tentative work plan moving forward is to:

• Spend (a considerable amount of) time learning the theory of limit sets and foliations.
• Investigate ways to code quasiconformal mappings (or approximations thereof) using finitely-much data.
• Work on understanding McMullen’s $\lim$ well enough to modify its functionality to this context.
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Limit Sets

Applications & Future Work

Applications to Foliation Theory

Thank you!