Sutured Manifold Hierarchies and Finite-Depth Foliations

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Florida State University

Topology Seminar
November 4, 2014

Outline

Part I

Definitions

Informally, a *dimension-k foliation* of a manifold $M = M^n$ is a decomposition of M into disjoint, connected submanifolds of dimension $k < n$ which, on a small scale, looks like the decomposition of $\mathbb{R}^n$ into $\mathbb{R}^k \times \mathbb{R}^{n-k}$.

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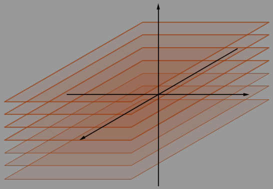

Figure 1
$\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ for $n = 3$ and $k = 2$

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Here, k is the *dimension* of $\mathcal{F}$, $n - k$ is its *codimension*, the submanifolds which comprise $\mathcal{F}$ are called its *leaves*, and the collection of all leaves is known as the *leaf space* of $\mathcal{F}$.

Examples

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3. Note that the foliation $\mathcal{F}_\alpha$ of $\mathbb{R}^2$ by parallel lines of slope α will be invariant (setwise) under horizontal and vertical translation.

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3. Note that the foliation $\mathcal{F}_\alpha$ of $\mathbb{R}^2$ by parallel lines of slope α will be invariant (setwise) under horizontal and vertical translation. In particular, $\mathcal{F}_\alpha$ is invariant by the $\mathbb{Z}^2$ action generated by these translations and hence descends to a foliation of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Examples

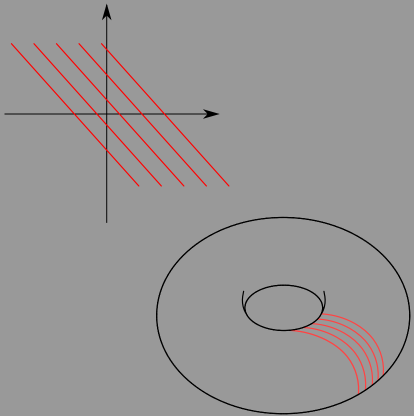

Figure 2
$\mathcal{F}_\alpha$ descends to foliation of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$

Depth

One consideration often made in the study of foliations is regarding the *depth* of the foliation and/or its leaves.

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Definition.
Under the same assumptions as above, $\mathcal{F}$ is said to be *depth* k if

$$k = \max\{\text{depth}(L) : L \text{ is a leaf of } \mathcal{F}\}.$$

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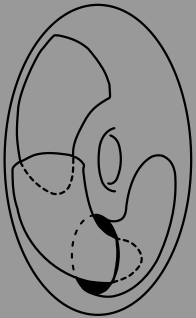

Figure 3
The Reeb foliation of $V^2 = D^2 \times S^1$

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- The boundary $\partial V^2 = T^2$ is a (compact) leaf.
- All other leaves are "interior leaves," all of which are topological planes which spiral towards the boundary torus. These are non-compact.

In particular, the toral boundary leaf is a depth-zero leaf and the interior leaves are all depth-one. Hence, the Reeb foliation is a depth-one foliation.

Constructing Higher-Depth Leaves

One can modify an existing finite-depth foliation to get a foliation of higher-depth. For example:

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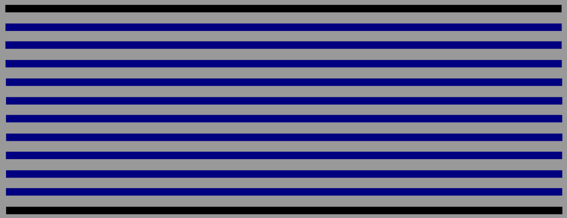
Start with a depth-zero foliation in two dimensions...



Constructing Higher-Depth Leaves

...identify a neighborhood of a curve γ...

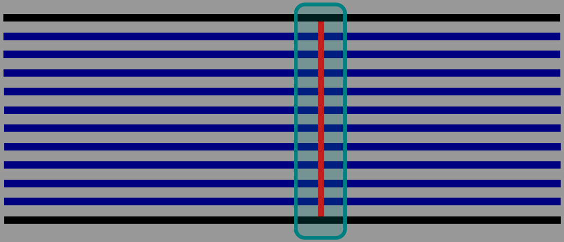
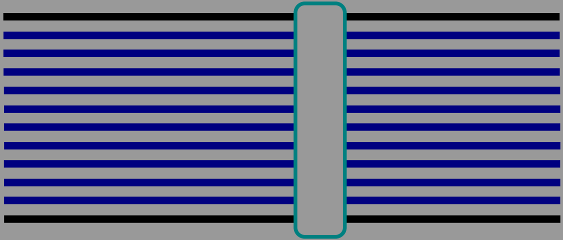


Figure 4
Neighorhood of the curve γ, which is shown in red

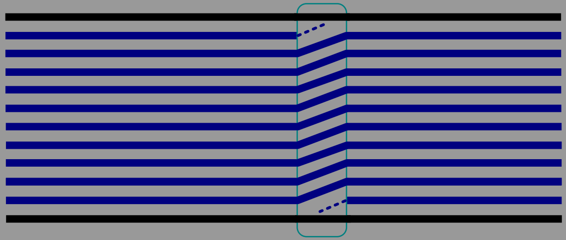
Constructing Higher-Depth Leaves

...delete it...



Constructing Higher-Depth Leaves

...and re-glue the interior leaves by a non-identity map while leaving the "boundary leaves" fixed, e.g.:



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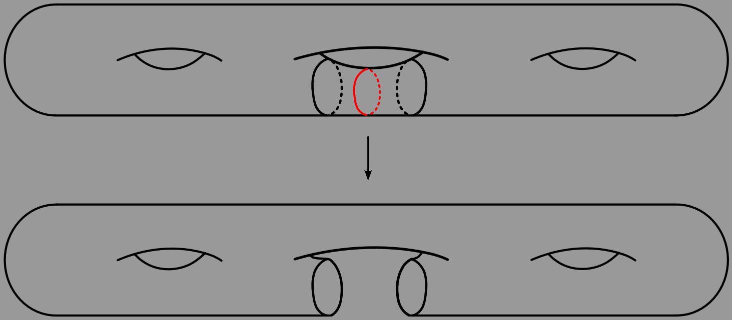

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...and gluing the resulting leaves via a non-identity map.

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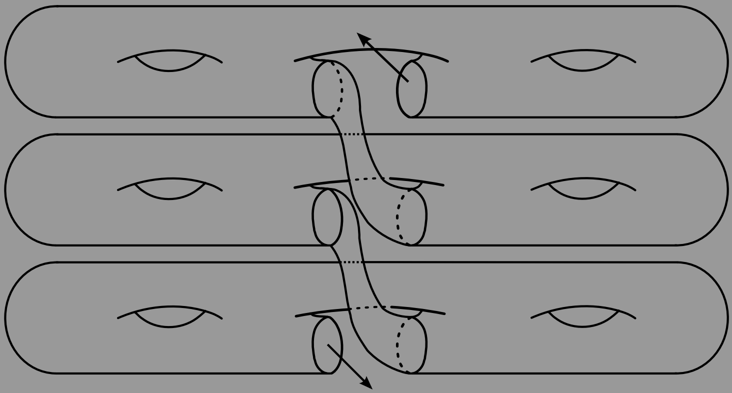

Constructing Higher-Depth Leaves

If this is done on “interior” depth-zero leaves while leaving some depth-zero “boundary leaf” fixed, the result will be a foliation with a depth-one leaf in the “interior” which has infinite genus and which limits on the fixed “boundary” depth-zero leaf.

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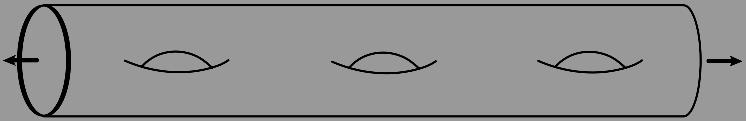

Figure 5
A depth-one leaf resulting from the above gluing process

Constructing Higher-Depth Leaves

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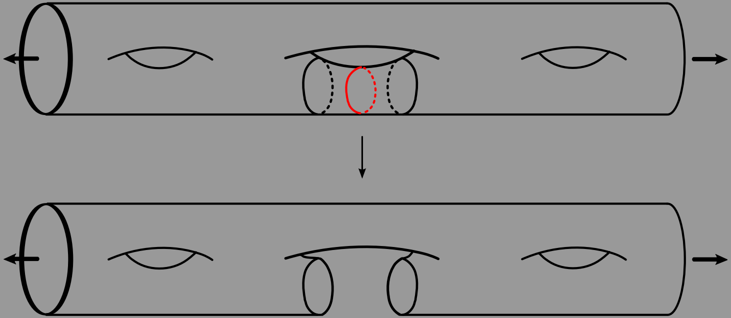

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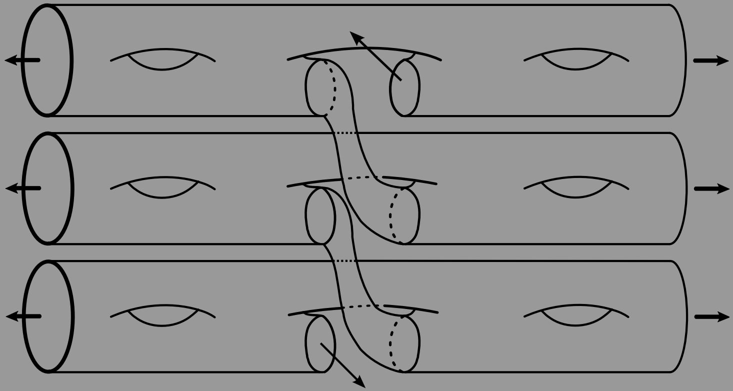

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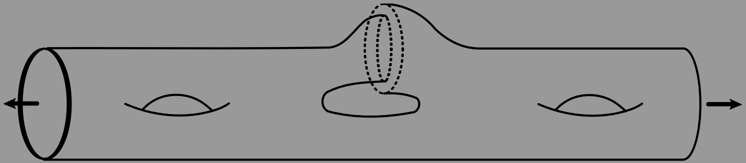

Figure 6
A depth-two leaf, as described by Candel and Conlon

Part II

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1. Each component of $A(\gamma)$ contains a homologically nontrivial oriented simple closed curve called a *suture*.
2. Every component of $R(\gamma) \stackrel{\text{def}}{=} \partial M - \mathring{\gamma}$ is oriented, and the orientations on $R(\gamma)$ must be "coherent" with respect to $s(\gamma)$.

Examples of Sutured Manifolds

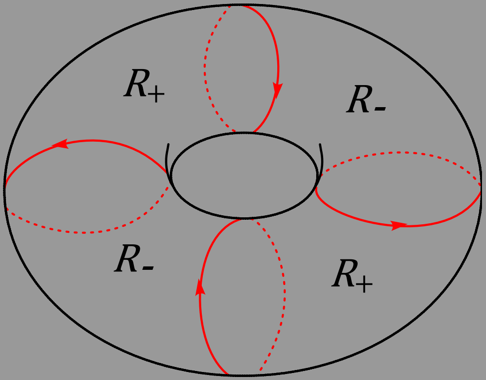

Figure 7
$M = D^2 \times S^1$ with four sutures, shown in red

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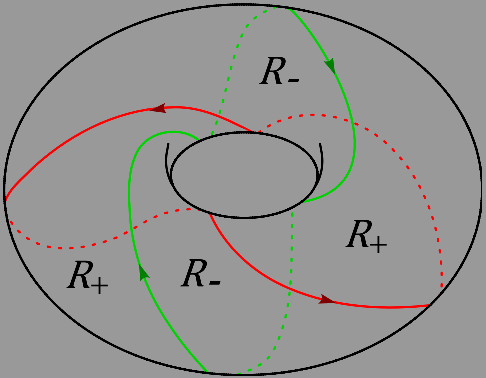

Figure 8

$M = D^2 \times S^1$ with two sutures, one red and one green

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3. λ is a homotopically nontrivial curve in a toral component T of γ so that, if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $H_1(T)$.

Definition (Cont'd)

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$$(M, \gamma) \xrightarrow{S} (M', \gamma')$$

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- $\gamma' = (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma))$.

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- $R_-(\gamma') = ((R_-(\gamma) \cap M') \cup S'_-) - \mathring{\gamma}'$.

Example

Begin with $M = D^2 \times S^1$ with two sutures, one red and one green, and a properly embedded disk S, oriented as shown.

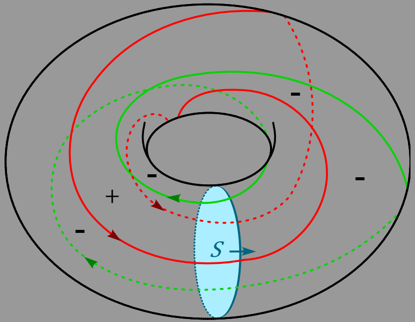

Figure 9
(M, γ) as described above

Example (Cont'd)

Next, obtain M' by removing a product neighborhood $N(S)$ of S in M as shown below. Note the labeling of S'_+, S'_-, and the intersections used to define γ'.

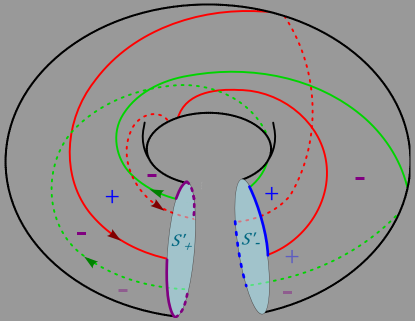

Figure 10
(M, γ) as described above

Example (Cont'd)

Defining γ' as above, one sees that the manifold M' now has only one suture, shown below in orange.

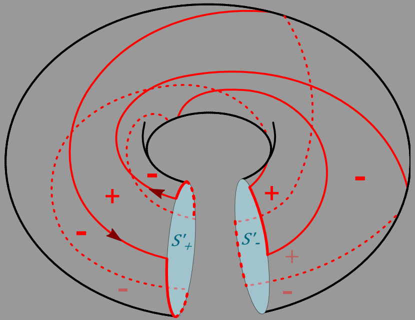

Figure 11
(M', γ') as described above

Example (Cont'd)

And finally, note that (M', γ') deformation retracts onto the manifold $M = B^3$ with one suture at the equator.

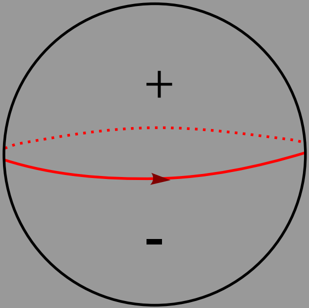

Figure 12
The final result of the decomposition

Definition

A *sutured manifold hierarchy* is a sequence of sutured manifold decompositions

$$(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} (M_2, \gamma_2) \longrightarrow \cdots \xrightarrow{S_n} (M_n, \gamma_n)$$

where $(M_n, \gamma_n) = (R \times I, \partial R \times I)$ and $R_+(\gamma_n) = R \times \{1\}$ for some surface R. Here, $I = [0, 1]$ and R is some surface.

Ginormous Main Theorem of Awesome Non-Triviality and Awesomeness

The following is a fundamental result of David Gabai:

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Theorem.
Suppose M is connected, and (M, γ) has a sutured manifold hierarchy

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so that no component of $R(\gamma_i)$ is a compressing torus. Then there exist transversely oriented foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of M which are "well-behaved" and where $\mathcal{F}_0$ is of finite-depth.

To Be Continued...

To see more, come to my ATE talk on December 1!!

Thank you!