Sutured 3-Manifolds, Finite-Depth Foliations, and Related Topics

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To my committee members, for their time, energy, and dedication.

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1 Introduction

The objective of this essay is twofold.

Overall, the aim is to present a number of key results from the literature which has served as the basis of the author's guided research thus far and to indicate clearly the intended incorporation of these results into the overall corpus of dissertation research to come.

On a much more subtle level, the goal is *also* to accomplish this feat in a manner that's both (somewhat) self-contained and (at least partially) succinct. In order to address both simultaneously, a number of assumptions—though hopefully none too liberal—have been utilized throughout.

Note first that the core of the current treatise lies in the intersection of low-dimensional manifold topology and geometry. As such, a number of basic background definitions fundamental to the understanding of these areas will be utilized without being stated. For example, unless otherwise noted, the author will refer to a space which is Hausdorff, second countable, and locally homeomorphic to $\mathbb{R}^n$ as a *manifold* of dimension n and will make no effort to define any of these adjectives; in the event that a subset of these adjectives needs to be dropped, however, the author will precisely indicate such. Other terms such as *foliations* fall somewhere in between being "background terms" and "specialist terms," and such terms will typically be defined within the paragraph they're introduced. Only for terms that appear well beyond the designation "background" will *formal* (numbered) definitions be given. Definitions may be clarified or made contextually specific throughout as warranted

Another conscious standard adhered to herein is the omission of certain fundamentally-important details which aren't contextually important. For example, when initially introducing rigorous definitions of foliations in section 2, there is no mention of smoothness or of differentiable structures. Indeed, despite existing at the crux of definitions in several important sources ([CN85], [CC00], [CC03], etc.), smoothness of manifolds will be left generally unmentioned in the current document unless specifically needed. When the adjective "smooth" *is* used, it will most often be assumed to mean "smoothness of a sufficient degree" unless the degree itself requires investigation. Similarly, the charts which comprise foliations (when viewed as foliated atlases) will likely remain unaddressed except when explicitly needed in a construction.

As a final remark, one should note that all of the areas upon which this manuscript centers are areas on which considerable work has been done. In particular, almost none are areas for which exhaustive—or, in some cases, even *thorough*—background may be included within a document of reasonable length. In light of this, the author will provide a discretionary combination of in-text details and outside references for the reader's convenience. Similarly, proofs of results presented throughout will be largely omitted unless deemed "worthwhile" by the author—a designation which will be reserved for proofs which satisfy an imprecise criterion of being enlightening, novel, foundationally-relevant, or unstated elsewhere. Again, the goal is to achieve a suitable combination of terseness and conceptual clarity though it should be noted that the ratio of inclusions to omissions is purely subjective.

2 Foliations

The purpose of this section is to collect some of the more elementary topological aspects needed henceforth.

Roughly speaking, a *foliation* is a topological tool which allows one the potential to study manifolds of a certain dimension by viewing them instead as a "nicely glued-together collection" of manifolds of smaller dimension. The literature on foliation theory is rather vast, as is the number of ways to make the above simplification precise. For the sake of this project, it is mostly sufficient to define a k-dimensional foliation of an n-dimensional manifold $M = M^n$ to be a disjoint union $\mathcal{F}$ of connected, properly embedded dimension-k submanifolds of M which is locally homeomorphic to the direct product decomposition of $\mathbb{R}^n$ into $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and whose union equals M. In this definition, k (resp. $n-k$) is said to be the *dimension* (resp. the *codimension*) of $\mathcal{F}$, the submanifolds which comprise $\mathcal{F}$ are called its *leaves*, and the collection of all leaves is known as the *leaf space* of $\mathcal{F}$ and will be denoted throughout this essay as $\Lambda = \Lambda_{\mathcal{F}}$.

A number of equivalent definitions may be found across the literature, though for some, the degree of technicality involved may blur the intuition. For example, one may define a k-dimensional foliation on a manifold[1] M^n to be an atlas $\mathcal{F}$ consisting of local charts (U, φ) (called *foliation charts*) which satisfy both[2] a local product decomposition[3]

$$\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$$

and whose coordinate-change maps satisfy a *compatibility condition*

$$(\psi \circ \varphi^{-1})(x, y) = (h_1(x, y), h_2(y)),\ (x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k},$$

on nonempty intersections $U \cap V \neq \varnothing$ of charts (U, φ) and (V, ψ). The leaves of a foliation presented in this way are defined as the equivalence classes of the equivalence relation R defined for $p, q \in M$ as follows: pRq if and only if there exists a sequence[4] $\alpha_1, \ldots, \alpha_k$ of *plaques* (that is, sets of the form $\varphi^{-1}(U_1 \times \{c\})$, $c \in U_2$) of $\mathcal{F}$ for which $\alpha_j \cap \alpha_{j+1} \neq \varnothing$, $p \in \alpha_1$, and $q \in \alpha_k$.

This latter definition presents a number of noteworthy points. First and foremost, one should note that it wasn't the definition mentioned initially; indeed, the level of specificity is typically greater than will be needed for talking points in this treatise. Even so, this definition is not without its benefits. For example, many applications and examples of foliations make use of the existence of smooth structures on both the ambient manifold M *and* on the submanifolds which make up $\mathcal{F}$; unsurprisingly, the expression of a foliated manifold in

[1] What follows is actually the definition of a foliation on a manifold without boundary; to see which items need tweaking in the case $\partial M \neq \varnothing$, see section 2.2 below.

[2] Some authors also require that the atlas $\mathcal{F}$ be maximal among all atlases satisfying these two conditions, though as pointed out in [CN85], this assumption is unnecessary due to the subtle fact that any atlas whose local charts satisfy these conditions is contained in a unique maximal chart which also satisfies them.

[3] The sets U_1, respectively U_2, are assumed to be open discs of $\mathbb{R}^k$, respectively $\mathbb{R}^{n-k}$.

[4] Such a sequence is called a *path of plaques*.

terms of atlases immediately presents the language necessary to require C^r differentiability on the coordinate-change maps $\psi \circ \varphi^{-1}$ for some desired r. As mentioned in section 1, however, this aspect will be essentially overlooked unless needed otherwise.

The theory of foliations has grown enormously in the last half-century, and indeed, there are huge numbers of "classical results" which should be stated here before moving on to the heart of the current investigation. First, consider the following examples, compiled with the intention to provide basic understanding, to present fundamental concepts which will be utilized later, and to demonstrate (among other things) a number of ways in which one can devise a foliation. Some of the examples will be stated in full generality while others will be specialized so as to be more directly applicable later.

2.1 Examples

Ex 1. By far the most trivial foliation is the decomposition of $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, i.e. the splitting of $\mathbb{R}^n$ into $\mathbb{R}^{n-k}$-many copies of the leaves (that is, of the hyperplanes $\mathbb{R}^k$). Clearly, this decomposition satisfies the conditions necessary to be a foliation: In particular, foliations require nothing more than a local product structure while the current example is globally a product. In the event that $n = 3$ and $k = 2$, the foliation in question is nothing more than decomposing $\mathbb{R}^3$ as a "stack" (of $\mathbb{R}$-many copies) of $\mathbb{R}^2$s as shown in Figure 1; one way to imagine this is to imagine the "decomposition" of a book into its (disjoint, 2-dimensional) pages.

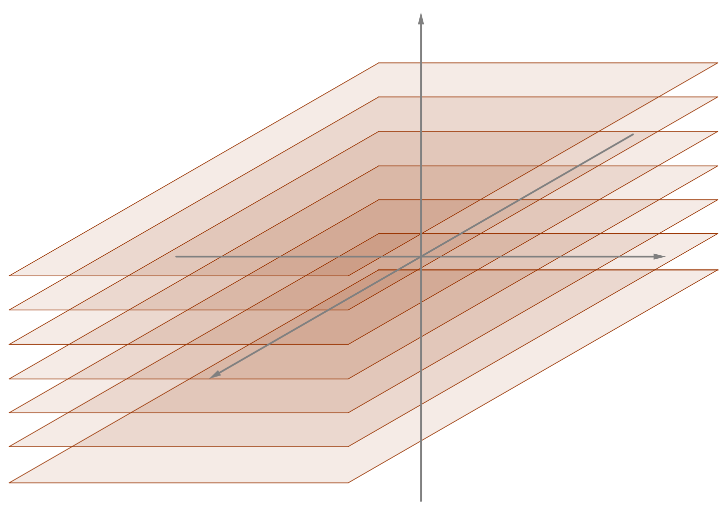

Figure 1

$\mathbb{R}^n$ *foliated by* $\mathbb{R}^{n-k}$*-many copies of* $\mathbb{R}^k$ *for* $n = 3$ *and* $k = 2$

Ex 2. A number of foliations of the torus T^2 can be constructed by viewing $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ as a quotient of the plane by the action of the integer lattice. In particular, any one-dimensional foliation of $\mathbb{R}^2$ by parallel lines of slope α will be (setwise) invariant by all (horizontal and vertical) translations of $\mathbb{R}^2$ and hence will induce a foliation $\mathcal{F}_\alpha$ on T^2 as shown in Figure 2.

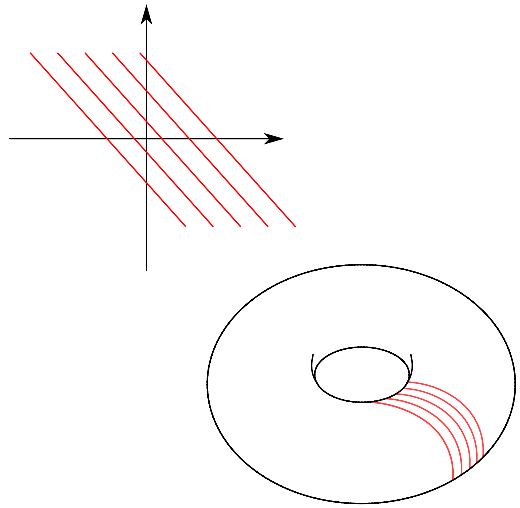

Figure 2

A codimension-1 foliation induced on T^2 from a foliation on $\mathbb{R}^2$

There exists a complete dichotomy of possible foliations $\mathcal{F}_\alpha$ of T^2 in terms of the slope α. In particular, if α is a rational number (or if $\alpha = \infty$ as in the case of the foliation of $\mathbb{R}^2$ by vertical lines), then each of the leaves of $\mathcal{F}_\alpha$ will be a circle and so $\mathcal{F}_\alpha$ will be a foliation of T^2 by circles[5]. On the other hand, when α is finite and irrational, each leaf of $\mathcal{F}_\alpha$ is a one-to-one immersion of $\mathbb{R}$ which is everywhere dense[6] in T^2.

Foliations of this type (i.e., foliations $\mathcal{F}_\alpha$ with α irrational) are known as *Kronecker foliations*, and a deep topological result states that Kronecker foliations $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ will be "unique" (i.e., topologically inequivalent) whenever α and β fail to be in the same $\mathrm{SL}(2,\mathbb{Z})$ orbit.

Ex 3. Arguably, one of the most important examples of a foliation is the Reeb foliation. Introduced in the 1950s by French mathematician Georges Reeb, the Reeb foliation is significant for a number of reasons and its importance throughout topology cannot be understated. In many ways, the Reeb foliation is the foundation upon which the current research—and the research upon which *that* is built—is framed.

First, consider the solid torus $V = D^2 \times S^1$. Intuitively, the Reeb foliation $\mathcal{F}_R$ of V can be defined in two distinct parts: (i) The boundary torus $\partial V = S^1 \times S^1$ will be a compact leaf $\mathcal{T}$ of $\mathcal{F}_R$, and (ii) the interior leaves of $\mathcal{F}_R$ will be topological planes which "spiral towards" $\mathcal{T}$. In particular, note that any leaf $\lambda \neq \mathcal{T}$ of $\mathcal{F}_R$ is non-compact; moreover, because of the overall limiting behavior of the leaves, the Reeb foliation is what's called *a depth-one foliation* (see Section 3.2 below for more information on depth).

[5]In this case, $\mathcal{F}_\alpha$ is actually a fiber bundle $\pi : T^2 \to S^1$ as noted in [CC00].

[6]Denseness is argued rigorously in [CC00].

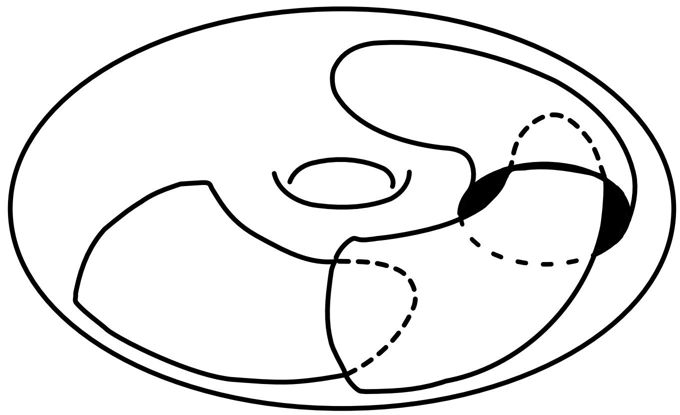

Figure 3

An illustration of the Reeb foliation of $V = D^2 \times S^1$. Note that the interior leaves are topological planes which are "bullet-shaped" and which continually spiral towards the boundary leaf $\mathcal{T}$ like a "snake eating itself."

The Reeb foliation generalizes to higher dimensional solid cylinders $M = D^n \times S^1$ and can be defined analytically in several ways. For example, [CC00] gives a sample construction obtained by restricting the submersion $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by

$$(r, z, t) \mapsto (r^2 - 1)e^t$$

to a submersion $\widetilde{f} : D^n \times \mathbb{R} \to \mathbb{H}^1$ (where D^n denotes the n-dimensional unit disc and $\mathbb{H}^n$ denotes the "upper half-hyperplane" $\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 \geq 0\}$ in $\mathbb{R}^n$) and by noting the invariance of the t-coordinate increment map $(r, z, t) \mapsto (r, z, t + k)$ for integers $k \in \mathbb{Z}$. In particular, the foliation $\mathcal{F}$ of $\mathbb{R}^{n+1}$ corresponding to level sets of f restricts to a foliation $\widetilde{\mathcal{F}}$ of

$$\widetilde{M} \overset{\text{def}}{=} D^n \times \mathbb{R}$$

by leaves which are "cup-shaped manifolds diffeomorphic to $\mathbb{R}^n$" [CC00], and $\widetilde{\mathcal{F}}$ induces a foliation $\mathcal{F}'$ on the quotient space $D^n \times (\mathbb{R}/\mathbb{Z}) = D^n \times S^1$ which has the form qualitatively described above.

The above-described Reeb foliation is related to a number of other similarly-named foliations as well. For example, the 3-sphere S^3 can be decomposed as a union of two solid tori V_1 and V_2 glued along their boundary tori; in particular, foliating each V_i with a Reeb foliation will induce a so-called Reeb foliation of S^3. This particular foliation is significant historically due to it being the first discovered non-singular foliation of S^3. In addition, the Reeb folation on V gives way to the so-called *Reeb component* in manifold topology which is defined to be a properly embedded (incompressible) solid torus V in a manifold $M = M^n$ which is endowed with a Reeb foliation. Foliations $\mathcal{F}$ on manifolds M are said to be *Reebless* provided they contain no Reeb components.

Ex 4. Let $\gamma : S^1 \to M$ be a closed embedding transversal to a C^k-smooth ($k \geq 1$) codimension-one foliated n-manifold ($M = M^n, \mathcal{F}$) which is "nicely orientable[7]" and let $N(\gamma)$ be a tubular neighborhood of S. Roughly speaking, one can define a foliation $\mathcal{F}_0$ on a small neighborhood of $N(\gamma) \cong S^1 \times D^{n-1}$ which agrees with $\mathcal{F}|N(S)$ on that neighborhood and which has the Reeb foliation $\mathcal{F}_R$ in its interior $\mathring{N}(S)$. This process of (somewhat artificially) introducing a Reeb component is called *turbulization* and in some ways represents the antithesis of the later goal of this research. This last point shall be talked on considerably in the pages which follow.

Ex 5. By the *Frobenius theorem*, a k-plane field is *completely integrable* if and only if it is *integrable* if and only if it is *involutive*. The plane field condition of complete integrability is equivalent to being tangent to a foliation, and because involutivity is a condition tied to differential forms, it follows that foliations can be defined in terms of such forms. More precisely, a k-plane field P defined on an open set U of a manifold $M = M^n$ by k linearly independent 1-forms $\omega^1, \ldots, \omega^k$ is completely integrable (i.e., tangent to a foliation) if and only if

$$d\omega^j \wedge \omega^1 \wedge \cdots \wedge \omega^k = 0$$

for every $j = 1, 2, \ldots, k$. This condition is sometimes useful in practice.

For example, the turbulization process discussed in the example above can be succinctly and precisely described in terms of differential forms. Given a leafwise- and transversely-oriented foliated n-manifold $(M, \mathcal{F})$ of codimension one and smoothness class C^k along with a closed transversal $S \subset M$ to $\mathcal{F}$, one can fix a tubular neighborhood $N(S) = D^{n-1} \times S^1$ with standard "cylindrical coordinates" (r, z, t) and can define the differential 1-form $\omega = \cos\lambda(r)\, dr + \sin\lambda(r)\, dt$ for a given smooth function $\lambda : [0, 1] \to [-\pi/2, \pi/2]$ which is strictly increasing on $[0, 3/4]$ and satisfies

$$\lambda(r) = \begin{cases} -\pi/2, & r = 0, \\ 0 & r = 2/3, \\ \pi/2 & 3/4 \leq r \leq 1 \end{cases}.$$

By requiring that $\lambda^{(m)}(0) = 0$ for all $m \geq 1$, it follows that ω is of class C^∞. Moreover, because ω is integrable—the identity $d\omega = \omega \wedge \lambda'(r)\, dt$ is easily verified—there is a foliation $\mathcal{F}_\omega$ associated to ω. Analyzing the behavior of the form ω as r approaches the values $r = 0$ and $r = 1$ illustrates the behavior of the corresponding foliation near the boundary and shows that $\mathcal{F}_\omega$ is precisely equal to $\mathcal{F}$ in a neighborhood of the boundary $\partial N(S)$; moreover, because ω is independent of z, it follows that $\mathcal{F}_\omega$ is rotationally symmetric. In particular, $\mathcal{F}_\omega$ is a Reeb component as stated in the previous example.

In order to get to the juicy part of the current exposition, a somewhat-broad-but-not-necessarily-deep knowledge of foliation theory will be beneficial. The above examples aid in

[7]Both leafwise and transversely orientable, the meanings of which are discussed in Section 2.2 below.

this process by giving explicit examples upon which to build intuition; the next step will be broaden the corpus of what's been stated thus far to accommodate for some of the more technical details which will spring to life in later parts.

2.2 More on Foliations and Related Terminology

There are a number of technical details related to foliation theory which were omitted from the introduction, several of which will be discussed presently either because they're intrinsically significant enough to warrant such treatment or because they'll be brought up later. This section will be nothing if not a potpourri of aspects of foliation theory which don't fit elsewhere.

The first issue in need of settling is that of orientability. Recall that a *fiber bundle* is a triple $\xi = (M, B, \pi)$ consisting of smooth manifolds M and B of dimensions $\dim M = n$ and $\dim B = n - k$ along with a smooth map $\pi : M \to B$ for which the following holds[8]: For each $x \in B$, there exists an open neighborhood U of x in B and a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[d, "p_1"] \\ U \arrow[r, "\mathrm{id}"'] & U \end{tikzcd}$$

with φ a diffeomorphism, $p_1 : (x, y) \mapsto x$ projection onto the first coordinate, and $\pi^{-1}(x) \stackrel{\text{def}}{=} F$ a smooth manifold of dimension $\dim F = k$ for all $x \in B$. Using this notation, the manifold F is called the *fiber*, B is called the *base space*, and M is called the *total space* of the bundle. For each $x \in B$, $F_x = \pi^{-1}(\{x\}) \cong F$ is called the *fiber of M over x* and the map π is called the *projection map* of the bundle. The sets $U \times F$ are called *local trivializations* of the bundle. Recall further that a *vector bundle* is a fiber bundle in which each section F_x is a vector space of dimension k so that the local trivializations are of the form $U \times \mathbb{R}^k$.

Definition 2.1.

(1) A vector bundle $\pi : M \to B$ is said to be *orientable* if there exists a vector bundle orientation, i.e. if there exists a covering by trivializations $U_i \times \mathbb{R}^k$ for which the transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}(k)$ are orientation-preserving as vector space maps.

(2) A foliation $\mathcal{F}$ is said to be *(leafwise) orientable* if the tangent bundle $T\mathcal{F}$ is orientable (as a vector bundle). Here, $T\mathcal{F}$ is defined to be the disjoint union of

[8]For simplicity, these manifolds are assumed to be without boundary. Also, the manifolds F, B, and M are assumed smooth with φ assumed a diffeomorphism; to be precise, the resulting structure is then a *smooth fiber bundle*. The same blurb holds true when F, B, and M are topological manifolds with φ a homeomorphism.

the tangent bundles $T\lambda = \sqcup_{x\in\lambda} T_x\lambda$ over all leaves λ of $\mathcal{F}$:

$$T\mathcal{F} = \bigsqcup_{\lambda\in\Lambda} \bigsqcup_{x\in\lambda} T_x\lambda.$$

(3) A foliation $\mathcal{F}$ is *transversely orientable* if its normal bundle $N\mathcal{F}$ is orientable (as a vector bundle). Here,

$$N\mathcal{F} = \bigsqcup_{\lambda\in\Lambda} \bigsqcup_{x\in\lambda} N_x\lambda$$

where $N_x\lambda$ is the collection of all vectors $\boldsymbol{v}$ in the orthogonal complement of $T_x\lambda$.

Worth noting is that definitions 2.1 above can also be framed in terms of k-plane fields or in terms of holonomy cocycles and determinants thereof. A variety of literature uses such definitions, e.g., [CN85] for the prior and [CC00] the latter.

Another issue which deserves mention is that of foliating manifolds with boundary. Notice that the definitions given above make no reference to the boundary of the foliated manifold though—as the Reeb foliation of $D^2 \times S^1$ shows—manifolds with boundary can clearly be foliated. This issue is made ultra-precise in [CC00], and while the current exposition doesn't require such extensive detail, it *does* deserve to be addressed briefly.

First, recall that a foliation $\mathcal{F}$ is said to be *transverse to* (respectively, *tangent to*) a smooth submanifold N of M if for each leaf λ of $\mathcal{F}$ and for each point $x \in \lambda \cap N$,

$$T_x\lambda + T_xN = T_xM$$

(respectively if either $\lambda \cap N = \varnothing$ or $\lambda \subseteq N$). These definitions are especially relevant when discussing manifolds with boundary because a manifold M for which $\partial M \neq 0$ is said to be *foliated by* $\mathcal{F}$ if $\mathcal{F}$ is defined as above and $\mathcal{F}$ is either transverse or tangent to every component N of ∂M. In the transverse case, one writes $\mathcal{F} \pitchfork N$ and notes that $\mathcal{F}|N$ is a naturally-defined foliation of N with the same codimension (relative to N) as $\mathcal{F}$ (relative to M).

As was the case of the original definition, the foliation theory on manifolds with boundary can also be expressed in terms of foliated charts as well. In this language, one denotes by $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{H}$ and, for the atlas (U, φ), looks at a decomposition of $\varphi(U)$ into subsets of the product $\mathbb{F}^k \times \mathbb{F}^{n-k}$ rather than restricting oneself to subsets of $\mathbb{R}^k \times \mathbb{R}^{n-k}$ as is done in the case of a manifold without boundary. Moreover, one requires that $\varphi : U \to B_\pitchfork \times B_\tau$ where $B_\pitchfork$ (read: "B-transversal") is a rectangular neighborhood of $\mathbb{F}^k$, B_τ (read: "B-tangential") is a rectangular neighborhood of $\mathbb{F}^{n-k}$, and where:

- The set $P_y = \varphi^{-1}(\{y\} \times B_\tau)$, $y \in B_\pitchfork$, is called a *plaque*.
- For each $x \in B_\tau$, the set $S_x = \varphi^{-1}(B_\pitchfork \times \{x\})$ is called a *transversal*.
- The set $\partial_\tau U = \varphi^{-1}(\partial B_\pitchfork \times B_\tau)$ is called the *tangential boundary* of U.
- The set $\partial_\pitchfork U = \varphi^{-1}(B_\pitchfork \times (\partial B_\tau))$ is called the *transverse boundary* of U.

The presence of foliations on manifolds with boundary will be largely inconsequential moving forward with the major exception coming as part of the technical details of the proof of Gabai's theorem 3.13. Even there, none of the above-mentioned technical details will be needed and a general acceptance of the existence of foliations on such manifolds will be sufficient.

One crucial aspect of foliation theory which *will* be of consequence and which still remains untreated is the link it provides between the topology and the geometry of a manifold. While this topic alone could serve as the focal point of an exposition thousands of pages long, only those details pertinent to later work will be brought to light herein. One very crucial aspect linking foliations to geometry is the notion of holonomy, and because this tool is used to whittle away at no fewer than four subcases within the proof of the main theorem 3.13, it will serve as the focus for the following section.

2.3 Holonomy

The theory of holonomy is robust and expansive in its own right so, as with other such topics, a comprehensive treatment is impossible. Both [CN85] and [CC00] contain worthwhile expositions on holonomy from the perspective of foliations. Note that the following information will come largely from [CN85]. Even so, an amazing motivation for the study of holonomy comes from [CC00]:

> *Intuitively, an inhabitant of a leaf L of $\mathcal{F}$ walks along a path s in L, keeping an eye on all of the nearby leaves. As he, she or it (hereafter denoted by $s(t)$) proceeds, some of these leaves may "peel away", getting out of visual range, others may suddenly come into range and approach L asymptotically, others may follow along in a more or less parallel fashion or wind around L laterally, etc. If s is a loop, then $s(t)$ repeatedly returns to the same point $s(t_0)$ as $t \uparrow \infty$ and each time more and more leaves may have spiraled into view or out of view, etc. This behavior, when appropriately formalized, is called the holonomy of $\mathcal{F}$.*

Throughout this section, $\mathcal{F}$ will denote a codimension-$(n-k)$ foliation of class C^r $(r \geq 1)$ of a manifold $M = M^n$ with the goal as stated above being to study the behavior of leaves λ_α of $\mathcal{F}$ near a fixed compact leaf F of $\mathcal{F}$. To begin, let $\gamma : I = [0,1] \to F$ be a continuous path and let Σ_0 and Σ_1 be two small transverse sections[9] to $\mathcal{F}$ of dimension $n-k$ passing through $p_0 = \gamma(0)$ and $p_1 = \gamma(1)$, respectively. The goal will be to define (local) map between Σ_0 and Σ_1 over the path γ which "follows along the leaves" of $\mathcal{F}$. The following figure demonstrates what will be accomplished.

[9] A *transverse section* to $\mathcal{F}$ is a submanifold Σ of the ambient manifold M which is transverse to $\mathcal{F}$ and for which $\dim(\Sigma) + \dim(\mathcal{F}) = \dim(M)$.

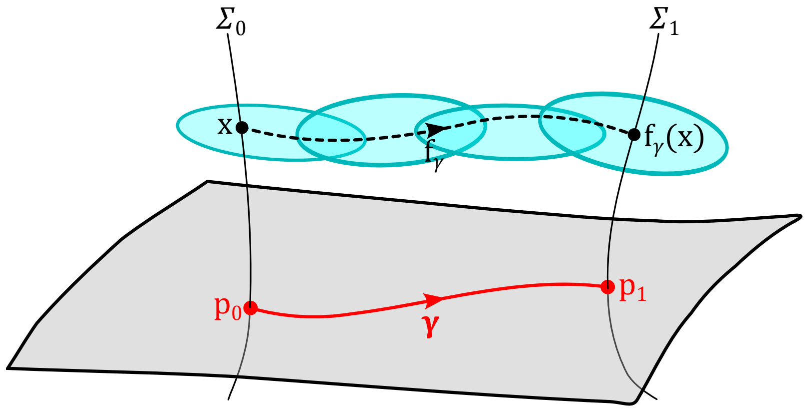

Figure 4

In the figure above, the teal-colored ovals above γ are plaques of a leaf L of $\mathcal{F}$. The map f_γ (dashed in the figure) follows along L, "above" γ, thereby tracing the leaf L. Notice the transversals Σ_0 and Σ_1 with points x, $f_\gamma(x)$ (over p_0, p_1, respectively).

Unsurprisingly, the map f_γ *does* exist, and the technical details of its construction can be found in [CN85]. The ideas motivating the construction are as follows: (i) Construct a sequence of local charts $(U_i)_{i=1}^k$ and a partition $0 = t_0 < \cdots < t_{k+1} = 1$ of I such that (U_i) is subordinated to γ[10]; (ii) define $D(t_0) = \Sigma_0$ and $D(t_{k+1}) = \Sigma_1$; (iii) for each $i = 1, 2, 3, \ldots, k$, fix a transverse section $D(t_i)$ to $\mathcal{F}$ which is homeomorphic to an n-dimensional disk, which passes through $\gamma(t_i)$, and for which $D(t_i) \subset U_{i-1}$; and (iv) finally, note that for each $x \in D(t_i)$ sufficiently near $\gamma(t_i)$, the plaque of U_i containing x meets $D(t_{i+1})$ in a unique point $f_i(x)$. Having noted these facts, one observes that for each i, the domain of f_i contains a disk $D_i \subset D(t_i)$ which contains $\gamma(t_i)$ and thus defines the map f_γ to be the composition

$$f_\gamma = f_k \circ f_{k-1} \circ \cdots \circ f_0,$$

noting throughout that such a function is well defined in some neighborhood V_0 of $p_0 \in \Sigma_0$.

Definition 2.2. The map f_γ is said to be the *holonomy map* associated to γ.

There are a number of desirable properties of the map f_γ which will be stated here sans proof. Recall that the *germ* of a function $f : V \subset X \to Y$ is the equivalence class of functions $g : V \subset X \to Y$ for which there exists a neighborhood W of $x \in V$ upon which $f|W = g|W$ and that, for a path $\beta : I \to M$, φ_β denotes the germ of the map f_β relative to a neighborhood of a point $\beta(0)$. Proofs and additional details can be found in [CN85].

Proposition 2.3.

1. f_γ is independent of both the choice of disks $D(t_i)$ and of the chain subordinated to γ.
2. $f_\gamma(p_0) = p_1$ and $\gamma^{-1}(t) = \gamma(1-t)$ implies that $f_{\gamma^{-1}} = (f_\gamma)^{-1}$.
3. f_γ is a C^r diffeomorphism provided that $\mathcal{F}$ is of class C^r $(r \geq 1)$.

[10] The sequence $(U_i)_{i=0}^k$ is *subordinated to* γ provided that (i) $U_i \cap U_j) \neq \varnothing$ implies that $U_i \cap U_j$ is contained in a local chart of $\mathcal{F}$ and (ii) $\gamma([t_i, t_{i+1}]) \subset U_i$ for all $i = 0, 1, \ldots, k$.

4. If γ' is a small perturbation (in the leaf F) of γ with endpoints fixed, then $f_{\gamma'} = f_\gamma$ in a neighborhood $p_0 \in \Sigma_0$.
5. Given different transversals $\Delta_0 \subset U_0$ and $\Delta_1 \subset U_k$ centered at p_0 and p_1, respectively, one can project along the plaques of U_0 and U_k, respectively, to produce C^r diffeomorphisms $\varphi_i : \Delta_i \to \Sigma_i$, $i \in \{0,1\}$. The result is that the new holonomy transformation $g_\gamma : \Delta_0' \subset \Delta_0 \to \Delta_1$ is conjugate to f_γ:

$$g_\gamma(x) = (\varphi_1^{-1} \circ f_\gamma \circ \varphi_0)(x) \text{ for all } x \in \Delta_0.$$

6. Given paths $\gamma_1, \gamma_2 : I \to M$ contained in a leaf F which satisfy $\gamma_i(j) = p_j$ for $j = 0,1$ and which are homotopic relative to $(0,1) \subset I$, then $\varphi_{\gamma_0} = \varphi_{\gamma_1}$ relative to a neighborhood of $p_0 = \gamma_i(0)$.

The last item on the above list is the foundation for a much stronger result in the case that γ is a *loop*.

Proposition 2.4. Suppose $p_0 = p_1$, $\Sigma_0 = \Sigma_1$, suppose that γ_1, γ_2 are loops in a leaf F of $\mathcal{F}$ satisfying the properties of the last item above, and denote by $G(X, x)$ the group of germs of local homeomorphisms which leave $x \in X$ fixed. Then the transformation $\gamma \mapsto \varphi_{\gamma^{-1}}$ induces a homomorphism

$$\Phi : \pi_1(F, p_0) \to G(\Sigma_0, p_0) \text{ defined by } \Phi : [\gamma] \mapsto \varphi_{\gamma^{-1}} \tag{2.3.1}$$

from the fundamental group of F at p_0 to the group of germs of C^r diffeomorphisms of Σ_0 which leave p_0 fixed.

And finally, the exposition has unveiled the machinery needed to accurately and precisely summarize the notion of holonomy hinted at by the analogy from [CC00] at the beginning of the section. Proposition 2.4 and the homomorphism Φ defined in (2.3.1) above yield the following definition.

Definition 2.5. The subgroup $\mathrm{Hol}(F, p_0) = \Phi(\pi_1(F, p_0))$ of $G(\Sigma_0, p_0)$ is called the *holonomy group* of F at p_0. Moreover, given any two points $p_0, p_1 \in F$ with a path $\alpha : I \to F$ connecting them, there is an induced isomorphism

$$\alpha^* : \mathrm{Hol}(F, p_0) \to \mathrm{Hol}(F, p_1)$$

defined by

$$\alpha^*(\Phi[\mu]) = \varphi_\alpha \circ \Phi[\mu] \circ \varphi_{\alpha^{-1}}$$

and so, in particular, it makes sense to talk about *the* holonomy group of F when referencing any group isomorphic to $\mathrm{Hol}(F, p_0)$.

Despite the considerable amount of legwork needed to state the properties of the holonomy map and to define the holonomy group of a compact leaf F of $\mathcal{F}$, the real power of holonomy from a foliations perspective is what it can be used for. For example, one can determine entirely the germ of a foliation $\mathcal{F}$ in a neighborhood of a compact leaf F of $\mathcal{F}$ based only on the holonomy of F, thereby providing understanding of the "global geometry" of the underlying manifold. The power of holonomy doesn't stop there, however, as a slew of significant results including the global trivialization lemma (Lemma 4.3.3 in [CN85]) and the local, (transverse) global, and (generalized) global stability theorems of Reeb (Theorems 4.4.3, 4.5.4, and 4.6.5 in [CN85]) provide an immense amount of information regarding (often large-scale) properties of foliations based solely upon properties of single leaves and/or of their holonomies. It's nearly impossible to overstate the significance of holonomy to the study of foliation theory.

The last subsection of this section is devoted to a somewhat more in-depth look at the Reeb foliation on the solid torus $V = D^2 \times S^1$. In light of that, perhaps it's best to include a holonomy example which bridges the gap between the two subsections and in a perfect world, that's precisely what would have gone here. For the sake of brevity, however, this example is omitted and instead, the diligent reader is encouraged to refer to the example at the top of [CN85, p.66], where the author explicitly exhibits the generators of the holonomy group corresponding to the compact toral leaf of the Reeb foliation of S^3.

2.4 The Reeb Foliation: The Good, The Bad, and The Ugly

A number of the above examples will be pertinent moving forward, though perhaps none more so than the Reeb foliation. "Pertinent" isn't always synonymous with *favorable*, however, and the Reeb foliation is a perfect example of that.

At its core, the Reeb foliation brings to light the unpleasant fact that foliations needn't be "nice," i.e. that the topology of the manifold needn't be reflected in the topology of its foliation and vice versa. To make this more precise, consider the Reeb foliation $\mathcal{F}_R$ on the solid torus $V = D^2 \times S^1$ and let $\mathcal{T}$ denote the boundary leaf $\mathcal{T} = S^1 \times S^1 = \partial V$ of $\mathcal{F}_R$. A trivial computation shows that $\pi_1(\mathcal{T}) = \mathbb{Z} \times \mathbb{Z}$ while $\pi_1(V) = \{0\}$. Intuitively, the fundamental group of a manifold is one (of many) tool(s) by which to measure (one facet of) the complexity of a manifold, and from that perspective, having non-trivial fundamental group indicates that the manifold is somehow "complex." However, the Reeb foliation paints an incoherent picture as far as topological complexity is concerned because $\mathcal{T}$ is somehow "complex" (i.e. it has non-trivial fundamental group) while being topologically embedded in and inheriting its topology from an ambient manifold V which is topologically "simple" (having trivial fundamental group).

Clearly, then, it would seem as if the Reeb foliation is bad. As above, however, it's worth noting that "incoherent" isn't always a synonym for *bad* and indeed, the Reeb foliation carries with it the track record of a reputable tool in topology. For example, it provides a non-trivial codimension-1 foliation of S^3 which is nonsingular, smooth, and of finite depth (see section 3.2 below), all of which are desirable properties to have on a space as fundamental as a sphere. In addition, the Reeb foliation makes surprise cameos in a number of significant results, e.g. in the proof of the Lickorish theorem[11] which constructs on any closed, orientable 3-manifold M a codimension-1 foliation ([Lic65], [Nov65]). By combining the process of turbulization (see the above examples) with Lickorish's theorem, it becomes apparent that the Reeb foliation is actually responsible for constructing infinite families of foliations on closed, orientable 3-manifolds. From this point of view, it seems as if the Reeb foliation is *not*...bad....

Clearly the issue isn't black and white.

Overall, the expert opinion is that the Reeb foliation does more harm than good; this fact is crucial to the motivation of the main component of section 3 which is, in no uncertain terms, the main component of this entire essay. It doesn't take much digging to witness the growth of the list of strikes against the Reeb foliation firsthand. For example, taken from [Fen02]:

> *Reebless foliations…are extremely useful in understanding the topology of 3-manifolds: Fundamental work of Novikov, and later Rosenberg, Palmeira showed that leaves inject in the fundamental group level (incompressible leaves), the manifold is irreducible (that is every embedded sphere bounds a ball), and the universal cover is homeomorphic to $\mathbb{R}^3$. Such foliations have excellent properties and they reflect the topology of the manifold. On the other hand Gabai constructed Reebless foliations in any irreducible, oriented, compact 3-manifold with non-trivial second homology and derived fundamental results in 3-manifold theory, such as property R and many other results. Roberts also constructed many Reebless foliations in large classes of 3-manifolds which are not Haken and jointly with Delman used this to prove property P for alternating knots. Notice that the Reebless property is crucial here since any closed 3-manifold admits a codimension one foliation, most of which are not useful for topology.*

That sentiment provides excellent closure to this section and a perfect segue into what lies ahead: The quest for Reebless foliations.

Remark. Lickorish's Theorem has a colorful history of expansions and generalizations since its original publication. The theorem is extended to the non-orientable case in [Woo69]. In [Law71], codimension-one foliations are constructed on spheres of dimension $2^k + 3$ for $k = 1, 2, 3, \dots$ and—as corollaries—on a handful of other manifolds. Later, [Dur72] and [Tam72] extend this construction to all odd-dimensional spheres and within a couple years, a number of other significant generalizations are proved[12]. All of these extensions were subsumed by the colossal work of Thurston [Thu76] which proves that a closed manifold $M = M^n$ has a C^∞ codimension-one foliation *if and only if* it has Euler characteristic zero. None of these results are detailed herein.

[11] Considerably more is written about this in a number of places including [Lic65], [Lic62], [Nov65], and [Wal60]

[12] Many of these are mentioned in the introduction to [Thu76].

3 The Quest for Reebless Foliations

Despite the "good" qualities afforded to the study of foliation theory by the existence of the Reeb foliation, Reeb foliations are, for all intents and purposes, *bad*. To quote one expert, the Reeb component is *the main villain* in 3-manifold foliation theory [Fen02] due to its failure to accurately convey the topological properties of the manifolds within which they're embedded. This failure, along with the role of the Reeb foliation in the construction of generic codimension-1 foliations on closed, orientable 3-manifolds, yields a relatively straightforward aspiration: To determine which 3-manifolds have codimension-1 Reebless foliations (a question which is extremely non-trivial as evidenced by [RRS03]) and to understand them as completely as possible.

The search for Reebless foliations isn't merely one founded on novelty. Indeed, such foliations have excellent properties—as was shown by ([Nov65]), Rosenberg ([Ros68]), and Palmeira ([Pal78])—and are known to reflect well the topology of the manifold [Fen02]. As such, much work has been done to understand foliations without Reeb components, whereby it follows that many examples of Reebless foliations such as *taut foliations* (foliations each of whose leaves meet a closed transversal contained in the underlying manifold), $\mathbb{R}$-*covered foliations* (foliations whose lifts to the universal cover have (lifted) leaf spaces which are Hausdorff), and foliations on manifolds admitting so-called *sutured manifold hierarchies* (see section 3.3 below) have been the center of a considerable amount of research, e.g. Gabai's work on sutured manifold hierarchies ([Gab83], [Gab87a], [Gab87b]).

Gabai's work—including a very small subset of the work upon which it was based—will be the crux of this section. While results contained herein may come from a variety of sources, the admitted "focal point" of the section is the presentation of a small subset Gabai's work including discussions on depth of foliations, sutured manifolds, and sutured manifold decompositions and hierarchies. Once the foundation has been lain, the task of proving Gabai's colossal theorem 3.13 will begin and will account for the majority of what's contained herein. The section ends with subsection 3.5 which presents (largely without proof) a small sample of the unbelievably-large corpus of corollaries and implications stemming from the proof of theorem 3.13.

3.1 Preliminaries

The quest for codimension-1 Reebless foliations in 3-manifolds is hardly a new one, and indeed, the literature comprising this quest is vast and multifaceted. Before delving into Gabai's work, several results, both foundational and related specifically to Reebless foliations, must be collected. The ordering of this material is essentially arbitrary except in instances where it's not (e.g., instances of material dependence, etc.). Unless otherwise noted, $(M, \mathcal{F})$ will denote a compact, connected, transversely-oriented foliated manifold of codimension one and L will denote a leaf of $\mathcal{F}$.

There are several significant building blocks upon which Gabai's work stands, not the least of which relates to the homology of the underlying manifold. One particular facet of homology that's needed is the notion of "norm-minimizing," a term which emerges naturally from Thurston's construction of a homological norm for surfaces embedded in 3-manifolds. Information pertaining thereto will be presented next.

3.1.1 A Norm on Homology

The purpose of this detour is to discuss the norm defined on the homology of 3-manifolds by Thurston. Much of the information presented is taken from a combination of [Thu86] and [Gab83]. Throughout, for R a properly embedded compact oriented surface in a compact oriented manifold M, the notation $[R]$ will be used to denote the homology class which R represents.

Definition 3.1.

(1) The *norm* of a compact oriented surface $S = \cup_{i=1}^{n} S_i$ (written as a union of its connected components S_i) is defined to be the function $x(S)$ of the form

$$x(S) = \sum_{i\,:\,\chi(S_i)<0} |\chi(S_i)| \tag{3.1.1}$$

where $\chi(S_i)$ denotes the Euler characteristic of the surface S_i and where $\chi(S) \overset{\text{def}}{=} 0$ for surfaces S which decompose into pieces S_i satifying $\chi(S_i) \geq 0$ for all i.

(2) Let M be a compact oriented 3-manifold with K a codimension-0 submanifold of ∂M. The *norm* of a class $z \in H_2(M, K)$ is defined to be

$$\begin{aligned} x(z) = \min\{x(S) \,:\, (S, \partial S) &\text{ is a properly embedded surface in } (M, K) \\ &\text{and } [S] = z \in H_2(M, K)\}. \end{aligned} \tag{3.1.2}$$

(3) Let S be a properly embedded oriented surface in the compact oriented 3-manifold M. Then S is said to be *norm-minimizing* in $H_2(M, K)$ if $\partial S \subset K$, S is incompressible, and $x(S) = x([S])$ for $[S] \in H_2(M, K)$.

There are several things to note about the above definitions. For example, intrinsic to equation (3.1.1) (taken from [Gab83]) is the fact that connected surfaces S for which $\chi(S) \geq 0$ are assigned the value $x(S) = 0$; this fact is more apparent using Thurston's original notation [Thu86] in which $x(S)$ is instead denoted by $\chi_-(S)$ and is defined to be $\chi_-(S) = \max\{0, -\chi(S)\}$. One important ramification of this is that the homological "norm" of (3.1.2) is then actually a *pseudonorm* due to the fact that having "norm" zero doesn't imply triviality: In particular, any surface S which is a disjoint union of spheres and tori will be minimal among its homology class and hence will have (homological) "norm" equal to zero. This caveat can be ignored by restricting oneself to, say, hyperbolic surfaces, in which case the above pseudonorm is actually a norm.

There are a number of key results pertaining to the homological pseudornorm above which will be important moving forward. These results are presented as a single lemma and are borrowed without proof from both [Thu86] and [Gab83].

Lemma 3.2.

(1) In an oriented 3-manifold M, every element $a \in H_2(M;\mathbb{Z})$ (or $H_2(M,\partial M;\mathbb{Z})$) is represented by an embedded oriented surface S. Moreover, if $k|a$, then S is the union of k components, each representing a/k.

(2) Let M be a compact oriented 3-manifold and let $\mathcal{F}$ be a codimension-1 transversely oriented foliation which is Reebless and transverse to ∂M. If R is a compact leaf of $\mathcal{F}$ then R is norm-minimizing as an element of $H_2(M,\partial M)$.

The second item of the above lemma provides precisely the link desired between Thurston's homological pseudonorm and the study of Reebless foliations. Even so, however, one would be remiss not to mention that the existence of the Thurston (pseudo-)norm has yielded a rather large collection of research in its own right, most notably the emergence of the so-called *fibered face theory* used to study 3-manifolds (and knots, and links) based on the geometry and topology of the unit ball of the Thurston norm (in the event that the pseudonorm in (3.1.2) is actually a norm). This remark is more for topical completeness than anything else and as such, the exposition on the Thurston norm is complete.

Next, the goal will be to finalize the foundational aspects necessary to proceed with results on sutured manifolds.

3.1.2 Partial Results

Having defined some of the elementary terminology needed to proceed, the focus shifts to stating a number of results which are "significant" (in the sense that they're important to the theory at hand) but also "elementary" (in the sense of being classical and motivating of the more recent work to be discussed). The first such result is presented as an expanded version of the result stated by Gabai in [Gab83] and combines a number of previous results including the titanic theorem of Novikov.

Theorem 3.3 (Novikov et al.'s Theorem)**.** Let M be a compact oriented 3-manifold with $\mathcal{F}$ a transversely-oriented codimension-1 foliation of M such that $\mathcal{F}$ is Reebless and transverse to ∂M (in the event that $\partial M \neq \varnothing$). Then:

(1) ∂M is a (possibly empty) union of tori.

(2) The fundamental group $\pi_1(M)$ of M is infinite.

(3) M is either *irreducible* (i.e., every embedded sphere in M bounds a ball) or $M = S^2 \times S^1$ with the product foliation.

(4) For every leaf L of $\mathcal{F}$, the map $\pi_1(L) \to \pi_1(M)$ is injective.

(5) Every closed curve in M which is transverse to $\mathcal{F}$ is homotopically nontrivial.

(6) If $\partial M = \varnothing$, then the universal cover $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$.

As noted in [Gab83], the first part of theorem 3.3 is classical; parts (2) through (6) are due to Novikov ([Nov65]) with the exception of parts (3) and (6); part (3) is actually due to Rosenberg ([Ros68]) and is a strengthening of Novikov's original observation that $\pi_2(M) = 0$, whereas part (6) is due to Palmeira ([Pal78]).

The ramifications of the results mentioned above are considerable. For example, because of item (2), any foliation $\mathcal{F}$ of S^3 *must* contain a Reeb component; similarly, any foliation of $V = D^2 \times S^1$ having $\partial V = T^2$ as a leaf must also include a Reeb component (by item (4)). Unmistakably, the results listed in theorem 3.3 provide a substantial framework for understanding the basic behavior of foliations without Reeb components.

Yet another fundamental result with significant ramifications, especially to the study of codimension-1 foliations, is Kopell's lemma, taken here from [CC00].

Lemma 3.4 (Kopell's Lemma)**.** Let $f \in \mathrm{Diff}^2_+[0,1]$ (that is, the collection of all C^2 diffeomorphisms on $[0,1]$ which preserve orientation) and let $h \in \mathrm{Homeo}_+[0,1]$ (i.e., the collection of all orientation-preserving homeomorphisms of $[0,1]$). Suppose that f and h commute and that f is a contraction $f : [0,1) \mapsto \{0\}$. If $h|[0,1)$ is a C^2 diffeomorphism of $[0,1)$ onto itself and fixes some point $x \in (0,1)$, then $h = \mathrm{id}_{[0,1]}$.

Although Kopell's lemma 3.4 may at first glance seem a bit esoteric relative to the current exposition, it actually plays a fundamental role in determining how smooth a foliation can be given properties of its host manifold. A palatable example of using Kopell's lemma in a foliations-related setting is included in the remark immediately following the statement of theorem 3.13.

Unsurprisingly, the list of results concerning Reebless foliations is vast in stature, even if one limits one's focal point to one of the many subclasses of foliations not having Reeb components (e.g., taut or $\mathbb{R}$-covered foliations). Many of these results are well beyond the scope of the current exposition, and indeed, any auxiliary results not stated elsewhere in the current essay are likely unnecessary to the goal of the paper and unwarranted from the perspective of brevity. Such results will be omitted.

The next section focuses on an important notion in the realm of foliation theory known as *depth*. This concept is yet another crucial piece of the puzzle that is Gabai's theorem 3.13 and having a working knowledge of it will be crucial moving forward.

3.2 Depth

First, the definitions.

Definition 3.5 (Depth of a Leaf)**.** A leaf L of a codimension-1 foliation $\mathcal{F}$ defined on a compact oriented 3-manifold M is said to be *depth zero* if L is compact. Higher depth leaves are defined inductively as follows: Having defined depth $j \leq k$ leaves of $\mathcal{F}$, a leaf L' is said to be *depth* $k+1$ if $\overline{L} - L$ is a union of depth j $(\leq k)$ leaves and contains at least one leaf of depth k.

Definition 3.6. A codimension-1 foliation $\mathcal{F}$ on a compact oriented 3-manifold M is said to be *depth* k if

$$k = \max\{\mathrm{depth}(L) : L \text{ is a leaf of } \mathcal{F}\}.$$

Note that, in general, the depth of a leaf (or of a foliation) may not be defined: In the case of a foliation $\mathcal{F}$ with a leaf λ which limits upon itself, for instance, the expression $\overline{L} - L$

is ambiguous. One extreme example is the class of *Anosov foliations* which are induced by Anosov flows: In particular, when the flow Φ associated to an Anosov foliation $\mathcal{F}_\Phi$ is transitive, the leaves of $\mathcal{F}_\Phi$ are all dense in the underlying manifold. Note, too, that despite the large corpus of work which focuses on behavior of finite-depth foliations, it makes perfect sense to define foliations of *infinite depth* to be those whose depth is unbounded (i.e., those foliations containing at least one leaf of depth-n for every integer n).

Also worth noting is that the notion of depth actually predates Gabai's work on the subject. As mentioned in [Gab83], for example, [CC81] is a good source on the subject even despite its authors having used different words to describe the same notions. This work is also unique because its terminology for depth-k (*totally proper at level* k) eliminates the possibility of leaves limiting on themselves.

Despite having gone undefined up to this point, depth is hardly a new concept; indeed, many of the intuitive pictures that come to mind when one thinks of foliations are likely finite-depth examples. For example, given a closed manifold S_g of genus g, the manifold $M = S_g \times I$ has a depth-zero foliation whose leaves are the $S_g \times \{t\}$ components; similarly, given a homeomorphism $f : S_g \to S_g$, the mapping torus

$$M_f = \frac{S_g \times I}{(x, 0) \sim (f(x), 1)}$$

is a 3-manifold with depth-zero foliation by leaves $S_g \times \{t\}$. It's also not hard to imagine examples of depth-one foliations since, e.g., the Reeb foliation of $V = D^2 \times S^1$ is depth-one: Indeed, the torus leaf $\mathcal{T} = \partial V$ is a depth-zero (i.e., compact) leaf while all other leaves are topological planes (non-compact, hence not depth-zero) which spiral (limit) towards $\mathcal{T}$. Thus, for an interior leaf L,

$$\begin{aligned} \overline{L} - L &= (L \cup \mathcal{T}) - L \\ &= \mathcal{T} \end{aligned} \tag{3.2.1}$$

is a union of depth-zero leaves and hence is depth-one. Note that (3.2.1) uses the fact that the only limiting done by an interior leaf L is upon the toral leaf $\mathcal{T}$, i.e. that no interior leaf limits on some other interior leaf L'.

One important property of depth—and one used throughout the proof of theorem 3.13—is that for k finite, existing depth-k foliations can be "modified" to yield new foliations of depth-$(k + 1)$. Due to the geometric underpinnings of this procedure, its best illustrations are pictorial; the subsequent subsections are devoted to illustrating this process.

3.2.1 Constructing Depth-One from Depth-Zero

The following figures will show how to construct a depth-one foliation from an existing depth-zero foliation. The first series of images will show the process with respect to one-dimensional manifolds embedded in two dimensions.

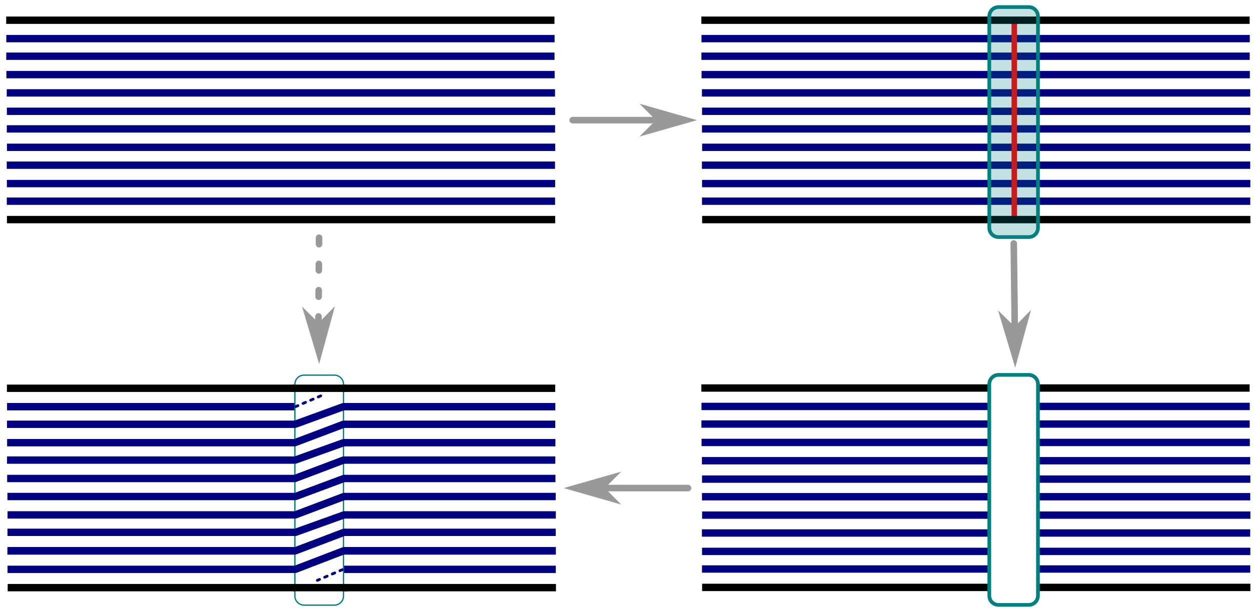

Figure 5

This image shows the process of turning the above-left depth-zero foliation into the below-left depth-one foliation.

To summarize figure 5, note that the process follows the (not-dashed) arrows. First (as shown in the top right), identify a transverse separating curve[13] (or arc) γ (shown in red) and a product neighborhood $N(\gamma)$ (shown in teal). Next (as shown in the bottom right), delete $N(\gamma)$. Finally (as shown in the bottom left), glue the two remaining components together by a map which fixes the "boundary leaves" (in black) and "shifts" the "interior leaves" (in navy) up by one. The result is a depth-one foliation in which the single depth-one leaf (bottom left, in navy) limits on the two depth-zero leaves (bottom left, in black). Note that the curve γ is called a *juncture* and is defined as suggested in the pictures above. Formalisms attached to junctures can be found in, e.g., [CC00].

The above procedure isn't limited to the two-dimensional case. As a matter of fact, it's actually impossible to define a depth-k foliation for $k \geq 2$ when looking at foliations of surfaces by one-manifolds. One can see this intuitively by observing that a depth-two foliation $\mathcal{F}_2$ would have at least one depth-two leaf L_2 which limits necessarily on a depth-one leaf L_1; moreover, by definition, L_1 would have to limit onto a depth-zero leaf L_0, thus forcing L_2 to limit on L_0 and hence to be depth-one as well. In two-dimensions, the same argument generalizes for $k > 2$ as well.

As such, the more interesting cases exist in higher dimensions. To that end, consider the following figures illustrating the higher-dimensional analogue of the procedure from figure 5 above, noting throughout that the "boundary leaves" (which remain fixed in the gluing) are omitted.

[13] As noted in Figure 6 below, if the curve (arc) γ is non-separating, no increase in depth will be achieved.

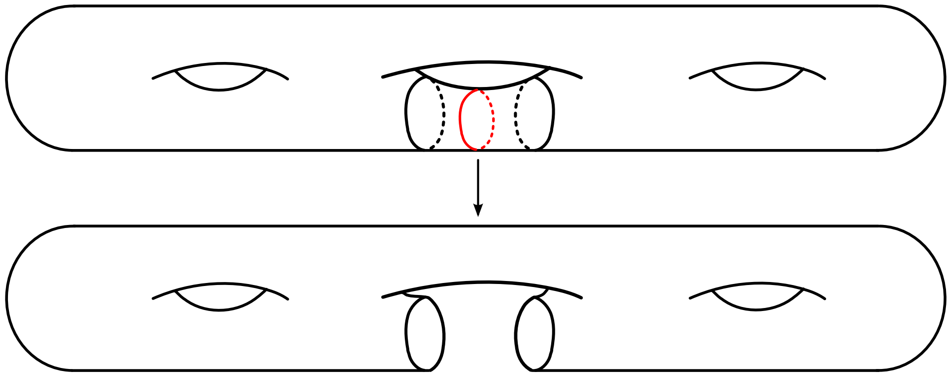

Figure 6

First, remove a neighborhood $N(\gamma)$ of some separating juncture γ (in red). The result is a surface with boundary.

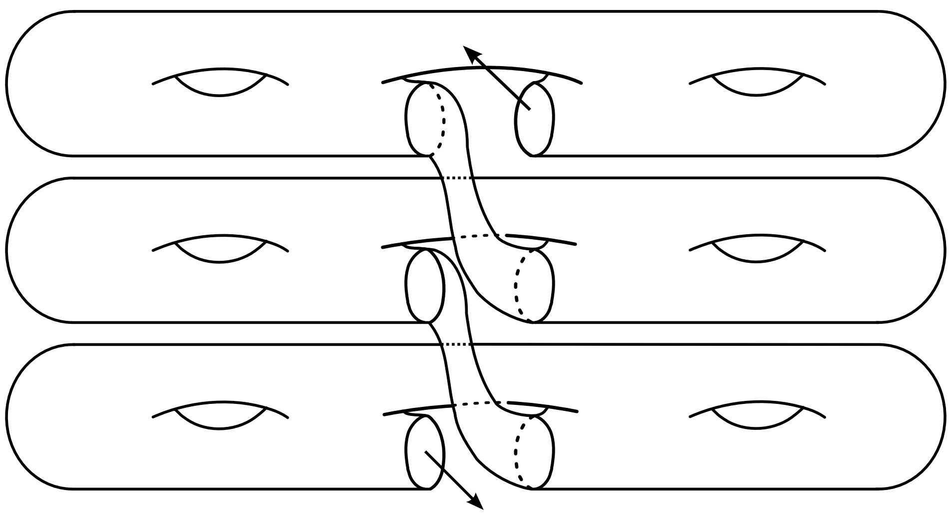

Figure 7

Next, glue the collection of surfaces with boundary using the "up by one" gluing pattern from figure 5.

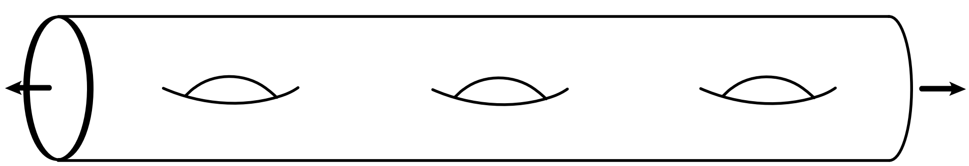

Figure 8

The result is a depth-one leaf (in "the interior") which is essentially an infinite-genus surface. Note that if γ is non-separating in figure 6, no increase in depth is achieved.

3.2.2 Constructing Depth-Two from Depth-One

Unsurprisingly, the procedure from subsection 3.2.1 above can also be generalized to arbitrarily high (finite-)depth. The following figures show the process repeated on depth-one leaves to yield a depth-two leaf.

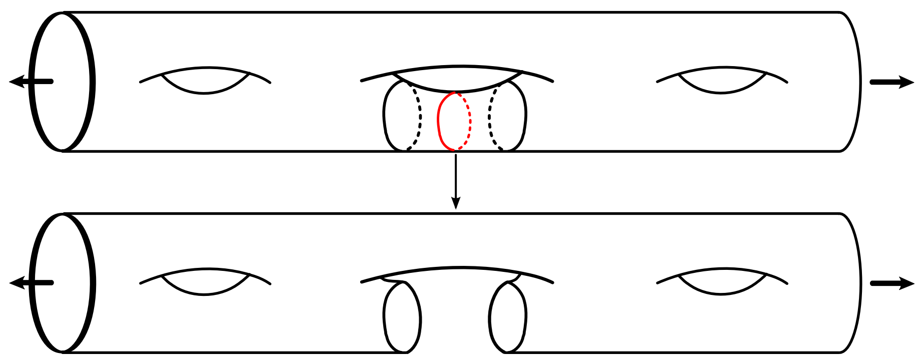

Figure 9

Again, remove a neighborhood $N(\gamma)$ of some (separating) juncture γ (in red).

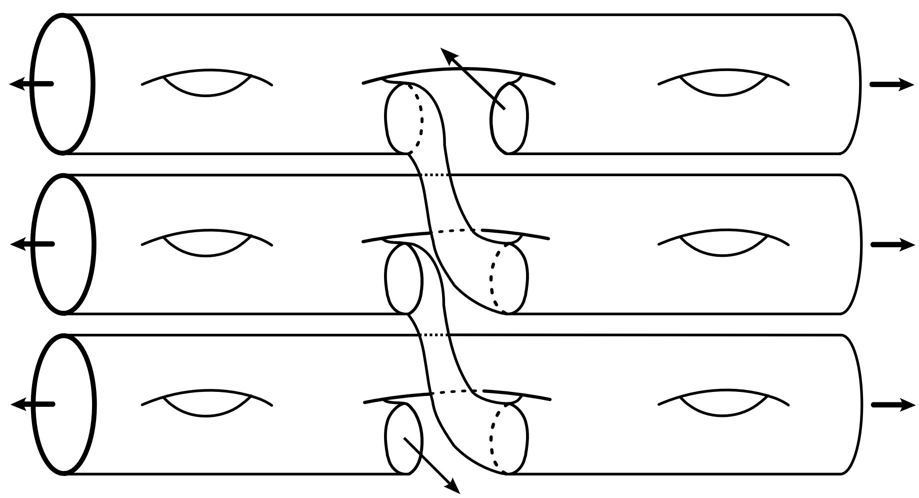

Figure 10

Now, use "one step up" gluing on the resulting manifolds.

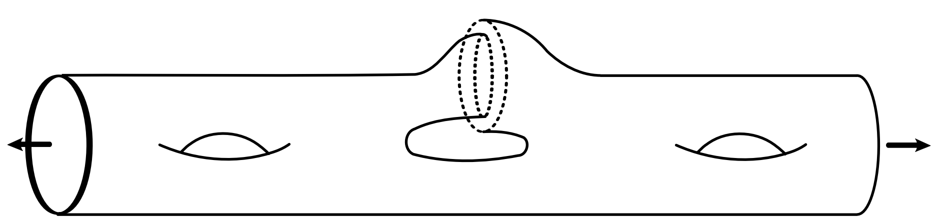

Figure 11

The result is a depth-two "interior leaf" which has "infinite genus around each of its (infinite) genus"

As mentioned above, these procedures will be relevant when constructing the foliations necessary to prove theorem 3.13. Otherwise, any further discussion of depth is unnecessary moving forward, meaning that now is a perfect time to shift gears and talk about sutured manifolds and topics related thereto.

3.3 Sutured Manifolds, Decompositions, and Hierarchies

The fundamental idea of Gabai's theorem 3.13 is that certain 3-manifolds which can be decomposed in a very precise way can be foliated "very nicely." Lying within this statement are layers of difficulty which must be whittled away one at a time, and easily the most fundamental difficulty lies in the aforementioned decomposition.

The motivation for the decomposition comes from the decomposition of surfaces into fundamental polygons by cutting along "well-behaved" codimension-one subobjects (namely, arcs and curves) and regluing in a way that is "coherent.". To mimic this procedure in a 3-manifold M, a considerable amount of effort must be made to ensure that the process is being completed "coherently." For example, to cut along an embedded surface S of M and/or to glue manifolds which have been cut in such a way requires the ability to keep track of various properties of M, both before and after the cut, including but not limited to the intersection of S with ∂M (when $\partial M \neq \varnothing$) and the effects of such procedures on the orientations of boundary components. Things are considerably more complicated, and the amount of data which must be maintained is significant.

One way to wrestle this *a priori* unmanageable construction into manageability is to impose additional structure on the host manifold M. One such structure—the structure at the heart of [Gab83], [Gab87a], and [Gab87b]—is that of a sutured manifold.

3.3.1 Sutured Manifolds

In this short subsection, the definition is given along with several examples to illustrate the nomenclature used therein.

Definition 3.7. A *sutured manifold* (M, γ) is a compact oriented 3-manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$ subject to the following conditions:

(1) Each component of $A(\gamma)$ contains a homologically nontrivial oriented simple closed curve called a *suture*. The set of all sutures in (M, γ) is denoted $s(\gamma)$.

(2) Every component of $R(\gamma) \stackrel{\text{def}}{=} \partial M - \mathring{\gamma}$ is oriented, and the orientations on $R(\gamma)$ are coherent with respect to $s(\gamma)$ so that if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture $\alpha \in s(\gamma)$.

In the above, one denotes by $R_+ = R_+(\gamma)$ and $R_- = R_-(\gamma)$ the components of $R(\gamma)$ whose normal vectors point out of and into M, respectively.

The best way to visualize sutured 3-manifolds is to look at some examples. To that end, consider the following figures.

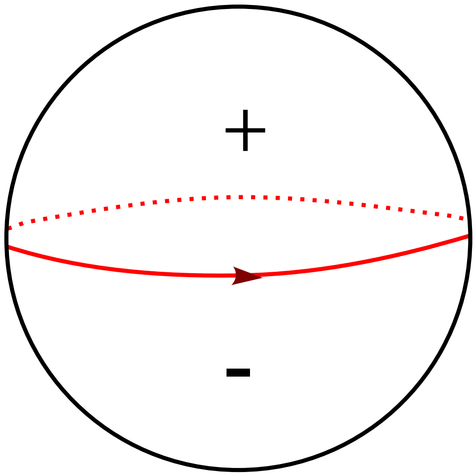

Figure 12
$M = B^3$ with one suture α at its equator. Here, "+" and "−" denote $R_+(\gamma)$ and $R_-(\gamma)$, respectively, and γ would consist of a single annulus $A(\alpha)$ containing α.

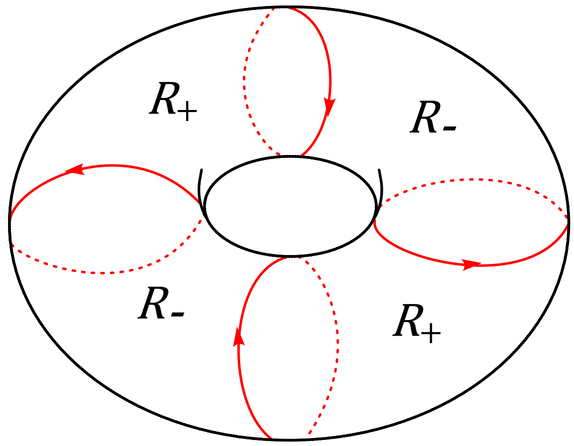

Figure 13
$M = D^2 \times S^1$ with four sutures, shown in red. Similar to figure 12, γ consists of the union of four annuli, one having each suture as its core.

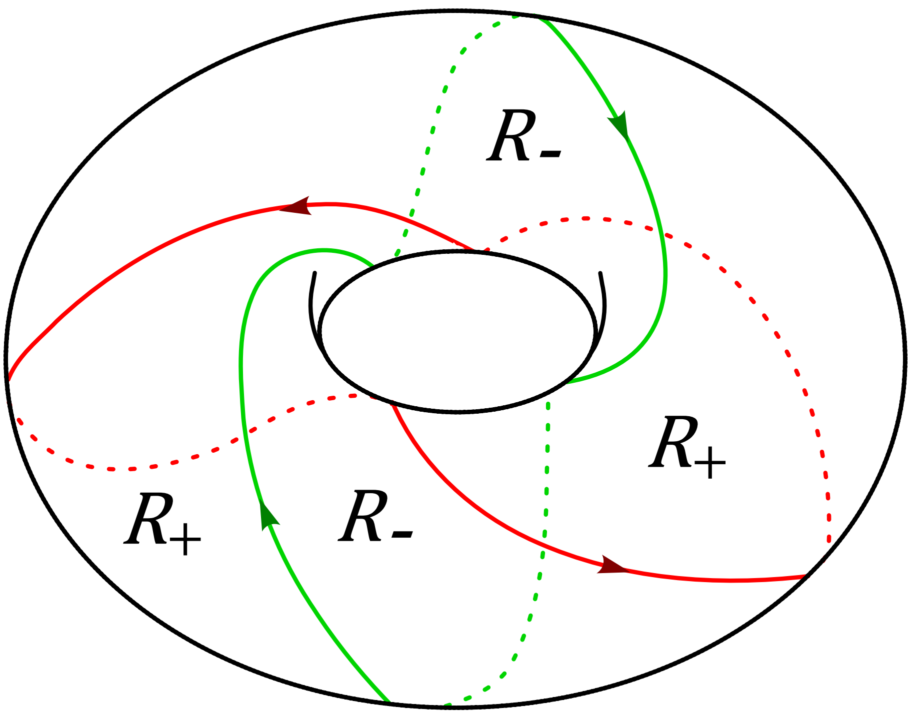

Figure 14
$M = B^3$ with two less-trivial sutures, one shown in red and one shown in green. Note that despite there being fewer sutures than in figure 13 above, ∂M is decomposed into the same number of R_+ and R_- regions, a fact attributed to the intersections of the sutures.

Remark. Condition (1) in definition 3.7 deserves a bit of explanation. Note that the homological nontriviality of a suture α is relative to the homology of its containing annulus $A(\alpha) \in A(\gamma)$ or torus $T(\alpha) \in T(\gamma)$. No other definition is possible, as the following example illustrates: Let (M, γ) consist of $M = B^3$ and $\gamma = A(\alpha)$ where α is the equator of M and $A(\alpha)$ is an annulus of the form $A(\alpha) = \alpha \pm \varepsilon$. This example is shown in figure 12 above. Note that this *is* a sutured manifold as it can be obtained as the end result of a *sutured manifold decomposition* as shown below, and due to the fact that the homologies of both B^3 and $\partial B^3 = S^2$ are trivial, α can only be homologically non-trivial relative to $A(\alpha)$.

3.3.2 Sutured Manifold Decompositions

While sutured manifolds have proven interesting in their own right (as the foundations for, e.g., sutured-Floer homology and in a number of works including [Sch90], [Juh10], etc.), their main role herein will be as a tool to derive significant results concerning Reebless foliations. To that end:

Definition 3.8 (Sutured Manifold Decomposition)**.** Let (M, γ) be a sutured manifold and let S be a *properly embedded surface*[14] in M such that (i) no component of ∂S bounds a disc in $R(\gamma)$, (ii) no component of S is a disc D with $\partial D \subset R(\gamma)$, and (iii) for every component λ of $S \cap \gamma$, one of the following holds:

(1) λ is a properly embedded nonseparating arc in γ.

(2) λ is a simple closed curve in an annular component A of γ which is in the same homology class as $A \cap s(\gamma)$.

(3) λ is a homotopically nontrivial curve in a toral component T of γ so that, if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $H_1(T)$.

Then, S defines a *sutured manifold decomposition*

$$(M, \gamma) \xrightarrow{S} (M', \gamma')$$

where:

- $M' = M - \mathring{N}(S)$ where $N(S)$ denotes a product neighborhood of S in M.
- S'_+ and S'_- denote the components of $\partial N(S) \cap M'$ whose normal vector points out of and into M', respectively.
- $\gamma' = (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma))$.
- $R_+(\gamma') = ((R_+(\gamma) \cap M') \cup S'_+) - \mathring{\gamma}'$.
- $R_-(\gamma') = ((R_-(\gamma) \cap M') \cup S'_-) - \mathring{\gamma}'$.

Further, one defines the sets S_+ and S_- as follows:

$$S_+ = S'_+ \cap R_+(\gamma') \text{ and } S_- = S'_- \cap R_-(\gamma').$$

[14] S is a *properly embedded surface* in M provided that $S \subsetneq M$ and that the inclusion map $\iota : S \hookrightarrow M$ satisfies two conditions, namely that (i) $\iota(\partial S) = S \cap \partial M$, and (ii) S is transverse to ∂M in any point of ∂S.

Remark. Care must be exhibited when choosing and utilizing terminology related to decompositions and sutured manifolds. For example, one may use the notation $(M, \square) \xrightarrow{S} (M', \gamma')$ and refer to the *decomposition* of the pair $(M, \square)$ into the sutured manifold (M', γ') while not meaning that the operation in question is a *sutured manifold decomposition*. $(M, \square)$ may itself still be a sutured manifold and as the $\xrightarrow{S}$ indicates, the decomposition may still be obtained by removing the interior of a product neighborhood of the properly embedded surface S in M; the difference between such a decomposition and a "sutured manifold decomposition" should be assumed to lie in the failure of the object $\square$ in the (not *sutured manifold*) decomposition to satisfy the prerequisite hypotheses necessary to "upgrade" the operation to a *sutured manifold decomposition*. To avoid confusion, the prefix "sutured manifold" will generally be dropped from *sutured manifold decomposition* when describing this other kind of decomposition.

At face value, the definition of a sutured manifold decomposition may be hard to swallow. To assist in that regard, Gabai offers the following description: "The sutured manifold (M', γ') is constructed by splitting M along S, creating $R_+(\gamma')$ by adding S'_+ to what was left of $R_+(\gamma)$, and creating $R_(\gamma')$ by adding S'_- to what was left of $R_(\gamma)$. Finally, one creates the annuli of γ' by 'thickening' $R_+(\gamma') \cap R_-(\gamma')$" [Gab83, p. 451]. The figures below illustrate, step-by-step, a particular sutured manifold decomposition.

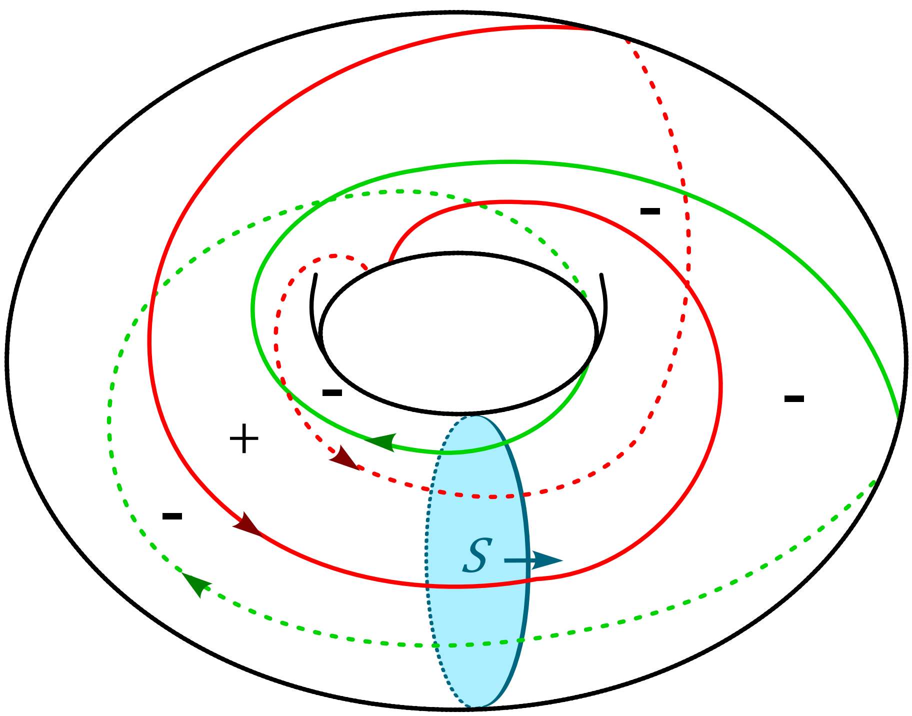

Figure 15

$M = D^2 \times S^1$ with two non-trivial sutures shown in red and green, respectively, and an embedded disk S shown in teal. S is oriented in a left-negative-to-right-positive fashion. The instances of "+" and "−" on the surface indicate R_+ and R_-.

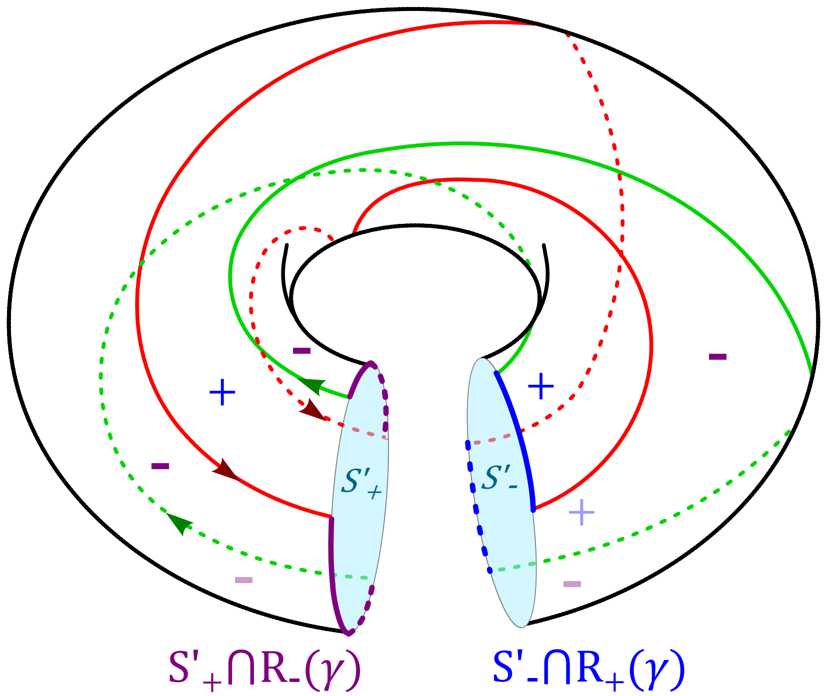

Figure 16

The first step is to remove $\mathring{N}(S)$ *to get* M'. $N(S'_{\pm} \cap R_{\mp}(\gamma))$ *is used to define* γ'*, and* $S'_{\pm} \cap R_{\mp}(\gamma)$ *are shown in purple, respectively blue, in the above figure.*

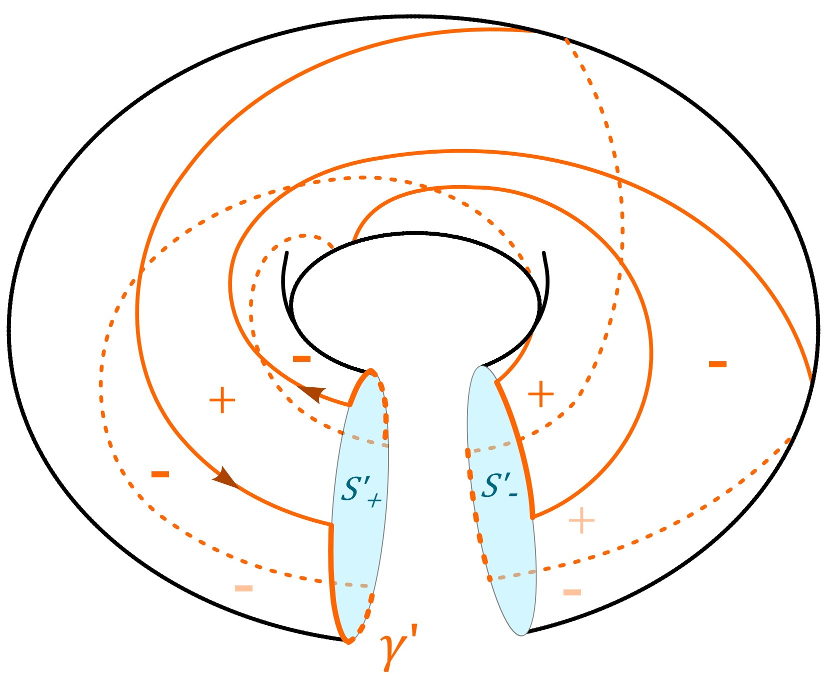

Figure 17

By defining $\gamma' = \gamma \cup N(S'_{\pm} \cap R_{\mp}(\gamma))$*, the original two sutures are reduced to one (shown here in orange). In particular, then,* (M', γ') *is a solid cylinder sutured by one simple closed curve...*

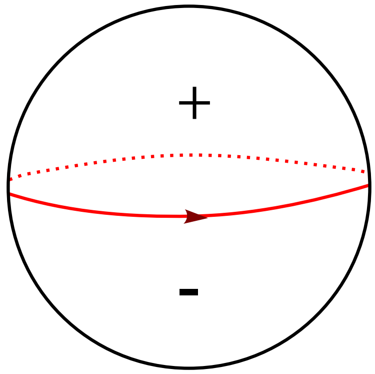

Figure 18

...and hence is equal to a 3-ball $M = B^3$ with a single suture (shown here at its equator) and the orientation shown. This is precisely the same picture as in figure 12 and verifies the claim made in the remark immediately following figure 14.

The above example is perfectly illustrative of the idea motivating the definition of the decomposition: Note that the original manifold was $M = D^2 \times S^1$—a manifold which has topology since its fundamental group is $\pi_1(M) = \mathbb{Z}$—and had two (somewhat complicated) sutures on it; in the end, the result was $M' = B^3$—a manifold with no topology of any kind—with a single (rather simple) suture. Moreover, the decomposition started with a sutured manifold and returned *another* sutured manifold; never once did the procedure "leave the sutured manifold world." These properties will be fundamental to the procedures used below.

Unsurprisingly, it makes sense to "compose" multiple sutured manifold decompositions as a tool for decreasing complexity even further. Ostensibly, it seems desirable to define such compositions in order to arrive at a sutured manifold which is "topologically the simplest" relative to the one with which the process began. This intuition is the motivation behind the next subsection.

3.3.3 Sutured Manifold Hierarchies

The name for the composition operation discussed above is a "hierarchy" as defined below. Note that the definition makes precise the above-used phrase, *topologically the simplest relative to the [sutured manifold] with which the process began.*

Definition 3.9. A *sutured manifold hierarchy* is a sequence of sutured manifold decompositions

$$(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} (M_2, \gamma_2) \longrightarrow \cdots \xrightarrow{S_n} (M_n, \gamma_n)$$

where $(M_n, \gamma_n) = (R \times I, \partial R \times I)$ and $R_+(\gamma_n) = R \times \{1\}$ for some surface R. Here, $I = [0, 1]$.

Ultimately, the result is to discuss the existence of "well-behaved" foliations on sutured manifolds which admit these sutured manifold hierarchies. When discussing foliations on sutured manifolds, a number of notions defined elsewhere must be reconsidered and manipulated in order to make sense of extra structure provided by the sutures. One such example is that of tautness.

Definitions 3.10.

(1) A sutured manifold (M, γ) is *taut* if M is irreducible and $R(\gamma)$ is norm minimizing in $H_2(M, \gamma)$.

(2) A transversely-oriented codimension-1 foliation $\mathcal{F}$ on (M, γ) is said to be *taut* if (i) $\mathcal{F}$ is transverse to γ, (ii) $\mathcal{F}$ is tangent to $R(\gamma)$ with the normal direction pointing inward (respectively, outward) along $R_-(\gamma)$ (respectively, $R_+(\gamma)$, (iii) $\mathcal{F}|\gamma$ has no Reeb components, and (iv) each leaf of $\mathcal{F}$ intersects a transverse curve or properly embedded arc.

At this point, the goal of reaching the proof of the main theorem 3.13 is *almost* within reach. The following two lemmas are related to tautness of sutured manifolds and foliations thereon. These lemmas will be used in the proof of the main result below and are taken in their current form from [Gab83]. The first—lemma 3.11—can be viewed as a generalization of the second part of lemma 3.2 stated above [Gab83].

Lemma 3.11. *Let M be oriented. If (M, γ) has a taut foliation $\mathcal{F}$, then either (i) (M, γ) is taut or (ii) $M \in \{S^2 \times S^1, S^2 \times I\}$ with $\mathcal{F}$ the product foliation on M.*

In addition, tautness of a sutured manifold gives a solid framework for understanding the existence of sutured manifold hierarchies.

Lemma 3.12. Provided that M is not a rational homology sphere containing no essential (i.e. incompressible) tori, every connected taut sutured manifold (M, γ) has a sutured manifold hierarchy

$$(M, \gamma) = (M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} (M_2, \gamma_2) \longrightarrow \cdots \xrightarrow{S_n} (M_n, \gamma_n)$$

satisfying (i) $S_i \cap \partial M_{i-1} \neq \varnothing$ if $\partial M_{i-1} \neq \varnothing$, and (ii) for every component V of $R(\gamma_i)$, $S_{i+1} \cap V$ is a union of $k \geq 0$ parallel oriented nonseparating simple closed curves or arcs.

And now, with all the machinery in place, the exposition can finally shift to the (main) work of Gabai. In what follows, the main theorem 3.13 will be stated and will be followed by a collection of the remaining lemmas needed for its proof; afterwards, the proof of the theorem will be outlined in order to allow for a broad overview of what follows. Finally, in sub-subsection 3.4.3, the proof is given in all its glory. The section itself ends with subsection 3.5 which collects a number of results following from theorem 3.13.

3.4 Main Theorem

Without further ado, the theorem towards which the exposition has been building:

Theorem 3.13. Suppose M is connected, and (M, γ) has a sutured manifold hierarchy

$$(M, \gamma) = (M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} (M_2, \gamma_2) \longrightarrow \cdots \xrightarrow{S_n} (M_n, \gamma_n)$$

so that no component of $R(\gamma_i)$ is a torus which is compressible. Then there exist transversely-oriented foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of M such that the following conditions hold:

(1) $\mathcal{F}_0$ and $\mathcal{F}_1$ are tangent to $R(\gamma)$.
(2) $\mathcal{F}_0$ and $\mathcal{F}_1$ are transverse to γ.
(3) If $H_2(M, \gamma) \neq 0$, then every leaf of $\mathcal{F}_0$ and $\mathcal{F}_1$ nontrivially intersects a transverse closed curve or a transverse arc with endpoints in $R(\gamma)$. However, if

$$\varnothing \neq \partial M \in \{R_+(\gamma), R_-(\gamma)\}, \tag{3.4.1}$$

then this holds only for interior leaves. The inclusion in (3.4.1) will sometimes be condensed as $\partial M \neq R_\pm(\gamma)$.
(4) There are no 2-dimensional Reeb components on $\mathcal{F}_i|\gamma$ for $i = 0, 1$.
(5) $\mathcal{F}_1$ is C^∞ except possibly along toral components of $R(\gamma)$ (if $\partial M \neq \varnothing$) or on S_1 (if $\partial M = \varnothing$).
(6) $\mathcal{F}_0$ is of finite depth.

Remark. The possible failure in (6) of $\mathcal{F}_1^i$ to be C^∞ along toral components of $R(\gamma)$ is a result of Kopell's Lemma 3.4 and can be illustrated with the following example taken from [CC00].

First, choose a C^∞ diffeomorphism $f : [0, 1] \to [0, 1]$ such that $f|[0, 1)$ is a contraction mapping to 0. Next, choose $t_0 \in (0, 1)$, define the points t_k so that $t_k = f^k(t_0)$ for $k \in \mathbb{Z}$, and let $h_0 : [t_1, t_0] \to [t_1, t_0]$ be a C^∞ diffeomorphism that is (i) a contraction of $[t_1, t_0)$ to t_1 and (ii) C^∞-tangent to the identity at both t_1 and t_0. By defining $h_k = f^k \circ h_0 \circ f^{-k}$ as well as a function $h : [0, 1] \to [0, 1]$ satisfying $h(0) = 0$, $h(1) = 1$, and $h(t) = h_k(t)$ for $t \in [t_{k+1}, t_k]$, it follows that $h(t_k) = t_k$ for all k, that $h \circ f = f \circ h$, and that $h|(0, 1)$ is a C^∞ diffeomorphism By Kopell's Lemma 3.4, it follows that h cannot be a C^2 diffeomorphism at $x = 0$.

This construction can be applied to foliations as follows: Let $T^2 = S^1 \times S^1$, and define a group homomorphism $\phi : \pi_1(T^2, x_0) \to \text{Homeo}_+[0, 1]$ which sends the generators α and β of $\pi_1(T^2, x_0) \cong \mathbb{Z} \oplus \mathbb{Z}$ to f and h, respectively. By suspension, there exists a $C^{\infty,0}$ foliation[15] $\mathcal{F} = \mathcal{F}(f, h)$ of $T^2 \times [0, 1]$ which is transverse to the interval fibers and which has precisely two compact leaves $T_i = T^2 \times \{i\}$, $i \in \{0, 1\}$, one cylindrical leaf L corresponding to the orbit $\{f^k(t_0)\}_{k=-\infty}^{\infty}$ and winding asymptotically on both T_0 and T_1, and all remaining leaves copies of $\mathbb{R}^2$ which wind around the cylinder L. By construction, the foliation $\mathcal{F}$ (and any foliation homeomorphic to it) has total holonomy generated by homeomorphisms f and h having the above-mentioned properties and hence Kopell's Lemma 3.4 ensures that such a foliation cannot be C^r for $r \geq 2$.

[15] A foliation $\mathcal{F}$ is of class $C^{r,k}$, $r > k \geq 0$, if the corresponding coherence class of foliated atlases contains a regular foliated atlas $\{U, x_\alpha, y_\alpha\}_{\alpha \in \mathfrak{A}}$ such that the change of coordinate formula $g_{\alpha\beta}(x_\beta, y_\beta) = (x_\alpha(x_\beta, y_\beta), y_\alpha(x_\beta))$ is of class C^k but x_α is of class C^r in the coordinates x_β and its mixed x_β partial derivatives of orders $\ell \leq r$ are C^k in the coordinates (x_β, y_β). Understanding this notation requires machinery not stated herein, though the concept is discussed several times in [CC00].

The proof of the above theorem is the main component of this paper and shall be presented in detail below. The proof requires a number of supplementary results from various sources and to avoid cluttering the presentation of the proof, the necessary framework will be collected first with the proof to follow.

3.4.1 Necessary Lemmas

Lemma 3.14. Let $(M,\gamma) \xrightarrow{S} (M',\gamma')$ be a sutured manifold decomposition. Then there exists a commutative diagram of sutured manifold decompositions

$$\begin{tikzcd}
(M,\gamma) \arrow[rrrrr, "S"] \arrow[dr, "S_1"] & & & & & (M',\gamma') \\
& (M_1,\gamma_1) \arrow[r, "S_2"] & (M_2,\gamma_2) \arrow[r] & \cdots \arrow[r, "S_n"] & (M_n,\gamma_n) \arrow[ur, "S_{n+1}"] &
\end{tikzcd}$$

so that if V is a component of $R(\gamma_{i-1})$, then either (i) $S_i \cap V$ is a set of parallel nonseparating oriened simple closed curves or arcs, or (ii) $\partial V \neq \varnothing$ and $S_i \cap V$ is a set of oriented properly embedded arcs such that $|\lambda \cap S_i| = |\langle \lambda, S_i \rangle|$[16] for each component λ of ∂V. If S is a disc with $|S \cap s(\gamma)| = 2$, then the former holds for all i.

Proof. This proof can be found in [Gab83, pp 25–26]. □

The next lemma is utilized within the second case of the main construction.

Lemma 3.15 (Mather-Sergeraert-Thurston)**.** If $f : I \to I$ is a C^∞ diffeomorphism satisfying

$$\frac{d^n f}{dt^n}(\alpha) = \begin{cases} 1, & n = 1, \\ 0, & n > 1 \end{cases}$$

for $\alpha \in \{0,1\}$, then there exist C^∞ diffeomorphisms $c_i, b_i : I \to I$, $i = 1, 2, \ldots, k$, satisfying the above conditions so that

$$f \circ [c_1, b_1] \circ [c_2, b_2] \circ \cdots \circ [c_n, b_n] = \mathrm{id}$$

where $[g,h] = ghg^{-1}h^{-1}$ is the *commutator* of g and h.

The final lemma will be used *after* the construction to deduce properties related to the sutured manifolds themselves.

Lemma 3.16. Let (M,γ) be a taut sutured manifold such that $H_2(M,\partial M) \neq 0$. Then there exists a decomposition

$$(M,\gamma) \xrightarrow{S} (M',\gamma')$$

such that (M',γ') is taut, S is connected, and $0 \neq [\partial S] \in H_1(\partial M)$ if $\partial M \neq \varnothing$. Furthermore, for a component V of $R(\gamma)$, $S \cap V$ is a union of $k \geq 0$ parallel oriented nonseparating simple closed curves (if V is nonplanar) or arcs (if V is planar).

[16]Here, $\langle X, Y \rangle$ denotes the algebraic intersection number of two oriented submanifolds X, Y of M satisfying $\dim X + \dim Y = \dim M$.

3.4.2 Outline of the Proof

Before proceeding with the technical details of the proof, consider first the following outline of work to be done.

(O.I) First, the assumed hierarchy associated to (M,γ) will be "preprocessed" by applying Lemma 3.14 to each constituent decomposition

$$(M_i,\gamma_i) \xrightarrow{S_i} (M_{i+1},\gamma_{i+1})$$

to get a "better-behaved" hierarchy

$$(M,\gamma) = (M_0,\gamma_0) \xrightarrow{T_1} (M_1,\gamma_1) \xrightarrow{T_2} (M_2,\gamma_2) \longrightarrow \cdots \xrightarrow{T_k} (M_k,\gamma_k). \tag{3.4.2}$$

"Better-behaved" will be made precise below.

(O.II) By "backwards induction", foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ will be constructed on (M,γ) by first defining the product foliation on (M_k,γ_k) and constructing foliations $\mathcal{F}^i_\alpha$ on levels $i<k$, $\alpha\in\{0,1\}$. This is done by considering which decompositions of antecedent elements (M_{i-1},γ_{i-1}) could possibly result in subsequent levels (M_i,γ_i). Foliations constructed herein will satisfy items (1), (2), and (4) of the desired results and, under various circumstances, may also satisfy conditions (3), (5), and/or (6). More precisely:

(a) The foliations $\mathcal{F}_\alpha$, $\alpha\in\{0,1\}$, will satisfy condition (3) if there does not exist a subset X of $\cup_i S_i$ which is a union of tori for which $0=[X]\in H_2(M,\partial M)$.

(b) $\mathcal{F}_1$ will satisfy condition (5) if no S_i is a torus for either $i>1$ (if $\partial M=\varnothing$) or $i\geq 1$ (if $\partial M\neq\varnothing$).

(c) $\mathcal{F}_0$ will satisfy condition (6) if $V\cap S_i$ is a union of parallel oriented nonseparating simple curves for every component V of $R(\gamma_{i-1})$ satisfying $S_i\cap\partial V\neq\varnothing$

For this step, there are three main cases to consider, and the general idea is to (i) glue T_i^+ and T_i^- to one another, (ii) extend the foliation $\mathcal{F}^i_j$ to a new foliation $\mathcal{F}'_j$ on the intermediate manifold Q (defined later) for $j\in\{0,1\}$, and (iii) apply the second and third cases where appropriate to extend these foliations to M_{i-1}.

(O.III) Lemma 3.11 will be applied to conclude that $R_+(\gamma)$ and $R_-(\gamma)$ are norm-minimizing.

(O.IV) Lemmas 3.12 and 3.16 will be used to construct a new sutured manifold decomposition on (M,γ) (respectively, on (M_1,γ_1)) if $H_2(M,\gamma)\neq 0$ (respectively, if $H_2(M,\gamma)=0$) which is "even better-behaved" than the previous one. Here, better-behaved means the following: (i) No decomposing surface S_i is a torus unless $\partial M=\varnothing$, $H_2(M)$ is generated by tori, and $i=1$; and (ii) if V is a component of $R(\gamma_{i-1})$, then $V\cap S_i$ consists of k (≥ 0) parallel oriented nonseparating simple closed curves.

(O.V) The construction in (O.II) will then be applied to the decomposition yielded by (O.IV). The resulting foliations are the desired ones.

The point moving forward will be two-fold: The first will be to show rigorously that the claimed construction from (O.II) exists and yields the results desired, after which the task of sharpening some of the imprecise terminology referenced throughout will become a main focal point. First, however, the focus is on (O.II). As mentioned above, the proof of (O.II) is a highly non-trivial induction process involving multiple induction hypotheses and broken into several cases. This proof will round out the remainder of this section.

3.4.3 The Proof

Proof of Theorem 3.13. Let (M, γ) be a sutured 3-manifold, M connected, which admits a sutured manifold hierarchy of the form

$$(M, \gamma) = (M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} (M_2, \gamma_2) \longrightarrow \cdots \xrightarrow{S_n} (M_n, \gamma_n) \tag{3.4.3}$$

for which no component of $R(\gamma_i)$ is a compressing torus, $i = 1, 2, \ldots, n$, and let (3.4.2) be the hierarchy obtained from preprocessing the given hierarchy (3.4.3). By Lemma 3.14, the modified hierarchy (3.4.2) is such that, for any component V of $R(\gamma_{i-1})$, either (i) $T_i \cap V$ is a set of parallel nonseparating oriented simple closed curves or arcs[17]; or (ii) $\partial V \neq \varnothing$ and $T_i \cap V$ is a set of oriented properly embedded arcs such that $|\lambda \cap T_i| = |\langle \lambda, T_i \rangle|$ for each component λ of ∂V. These properties constitute the adjective "better-behaved" in (O.I) and will be fundamental when formalizing the spiraling procedure for constructing the foliation $\mathcal{F}_0^{i-1}$ in the second case discussed below.

By definition, (M_k, γ_k) in (3.4.2) has the form $(M_k, \gamma_k) = (S \times I, \partial S \times I)$ with

$$R_+(\gamma_k) = \partial S \times \{1\}$$

for some surface S. On this manifold, define $\mathcal{F}_0^k = \mathcal{F}_1^k$ to be the product foliation. Now it's time to induct (in a backwards fashion) on i with the intention of defining foliations $\mathcal{F}_0^i$, $\mathcal{F}_1^i$ for $i = k-1, k-2, \ldots, 1, 0$. The proof will proceed with the following (induction) hypotheses in place:

(H1) Foliations $\mathcal{F}_\alpha^i$, $\alpha \in \{0, 1\}$, have been constructed on (M_i, γ_i) satisfying the results of Theorem 3.13 except possibly conclusions (3), (5), and (6) (see item (O.II) above).

(H2) $\mathcal{F}_0^i$ and $\mathcal{F}_1^i$ satisfy (3) if $\partial M_j \notin \{R_+(\gamma_j), R_-(\gamma_j)\}$ for $j \geq i$. As a corollary, if $\cup_{j=i+1}^k T_j$ contains no tori and $\partial M_i \neq R_\pm(\gamma_j)$, then (3) is satisfied in its entirety.

(H3) $\mathcal{F}_1^i$ is C^∞ except possibly along toral components of $\cup_{j=i+1}^k T_j \cup R(\gamma_i)$.

(H4) If δ is a curve on a nontoral component of $R(\gamma_i)$ and if $f : [0, \alpha) \to [0, b)$ is a representative of the germ of the holonomy map around δ for the foliation $\mathcal{F}_1^i$, then[18]

$$\frac{d^n f}{dt^n}(0) = \begin{cases} 1, & i = 1 \\ 0, & i > 1 \end{cases}$$

[17] Note that when V satisfies this condition, $\mathcal{F}_0^{i-1}$ will be of finite depth by (O.II) part (c)

[18] In particular, this condition implies that the holonomy is C^∞ tangent to the identity at $x = 0$.

(H5) $\mathcal{F}_0^i$ is of finite depth if, for all $j \geq i$, $V \cap T_{j-1}$ is a union of parallel oriented simple curves for each component V of $R(\gamma_j)$ with $T_{j-1} \cap \partial V \neq \varnothing$.

(H6) $\mathcal{F}_\alpha^i$ has no Reeb components for $\alpha \in \{0, 1\}$.

Having stated the induction hypotheses, the genesis of the desired foliations can begin. As mentioned above, the overall idea of the construction is to "build" the foliations $\mathcal{F}_\alpha^{i-1}$ by gluing T_i^+ and T_i^- to one another and "extending" the existing foliations $\mathcal{F}_\alpha^i$. The corpus of this process lies in analyzing the extension process for all possible types of gluings, ensuring throughout that each gluing type is formalized in a way that meets the induction hypotheses above. To that end, there are three main cases to consider.

Case 1: $\boldsymbol{\partial T_i \cap s(\gamma_{i-1}) = \varnothing}$

This is the simplest case. In this scenario, M_{i-1} is the direct result of gluing T_i^+ and T_i^- and the gluing "preserves" the foliations $\mathcal{F}_0^i$ and $\mathcal{F}_1^i$ present on the unglued manifold (M_i, γ_i). As a result, the manifold (M_{i-1}, γ_{i-1}) has a pair of foliations naturally induced on it by the gluing. Define $\mathcal{F}_0^{i-1}$ and $\mathcal{F}_1^{i-1}$ to be the foliations induced by $\mathcal{F}_0^i$ and $\mathcal{F}_1^i$, respectively. This is shown in the figure below.

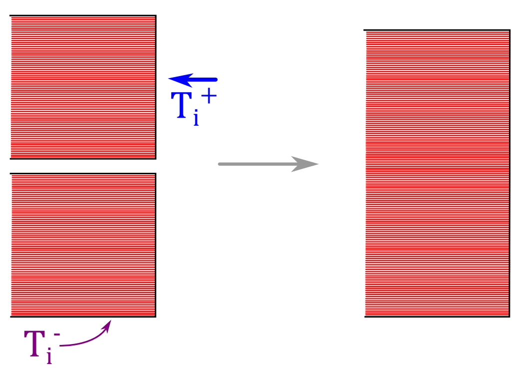

Figure 19

M_{i-1}, obtained by gluing T_i^+ to T_i^-, and its naturally-induced foliation(s)

Due to triviality of the gluing, one easily verifies that $\mathcal{F}_{0,1}^{i-1}$ satisfy the same desired properties (1)–(6) of Theorem 3.13 satisfied by $\mathcal{F}_{0,1}^i$. $\boxed{\text{Case 1}}$

> **Remark.** The simplicity of case 1 is because (i) gluings are done along convex regions (because of the condition on ∂T_i) and (ii) no gluings change the "compatibility" of previously-existing leaves (because $s(\gamma_{i-1})$ is undisturbed by ∂T_i).

Case 2: $\boldsymbol{\partial T_i}$ is contained in a component $\boldsymbol{V}$ of $\boldsymbol{R(\gamma_{i-1})}$

Without loss of generality[19], it can be supposed that ∂T_i is connected and contained in $R_-(\gamma_{i-1})$. In particular, $T_i \cap V$ is a single oriented simple closed curve and the foliation $\mathcal{F}_0^i$

[19] In the event that ∂T_i is disconnected, these constructions may be repeated on each of its components; the constructions may also be repeated verbatim in the event that ∂T_i is contained in $R_+(\gamma_{i-1})$. In the event that ∂T_i has nonempty intersection with but fails to be a subset of $R_\pm(\gamma_{i-1})$, then either ∂T_i can be isotoped to be contained in $R_\pm(\gamma_{i-1})$ or the situation in question is actually a matter of case 3 below.

has finite depth. This case consists of two separate, completely unrelated constructions due to the drastic difference in smoothness requirements for $\mathcal{F}_0^{i-1}$ and $\mathcal{F}_1^{i-1}$.

— *Constructing* $\mathcal{F}_0^{i-1}$:
Let Q be the manifold obtained by gluing $T_i^+ \subset R_+(\gamma_i)$ to $T_i^- \subset R_-(\gamma)$ as in the figures below.

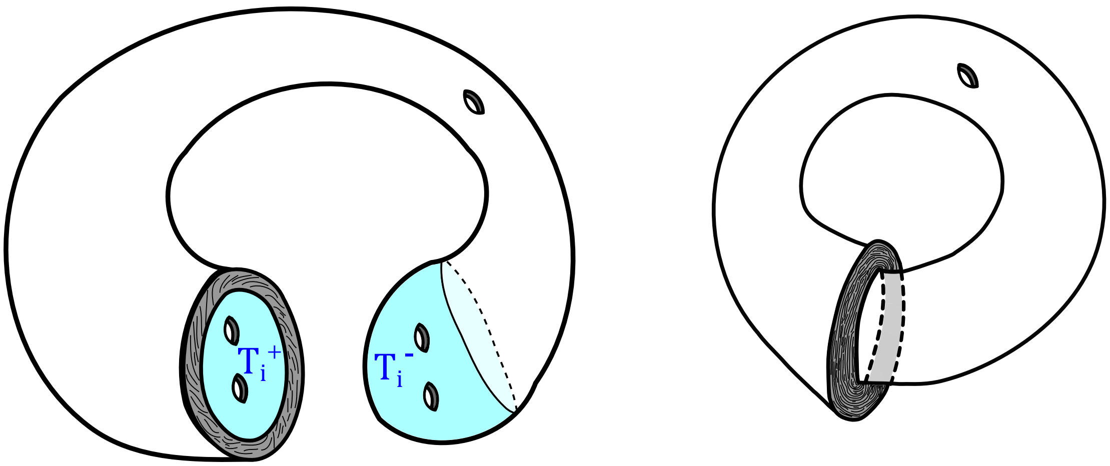

Figure 20
The leftmost image shows the part of ∂M_i containing $T_i^+ \cup T_i^-$. The manifold on the right is Q and is obtained by gluing T_i^+ to T_i^-.

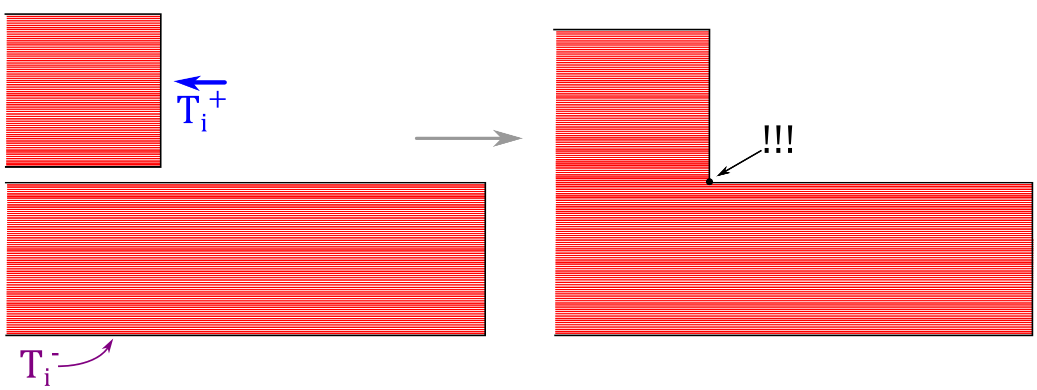

Figure 21
This view better illustrates the technicalities unseen in Figure 20 above. The leftmost viewpoint shows the foliations present on $R_\pm(\gamma_i)$ in red with $T_i^\pm$ highlighted. Notice that both regions are convex pre-gluing though the emphasized point (on the right) becomes a point yielding non-convexity.

Because the action of the decomposition

$$(M_{i-1}, \gamma_{i-1}) \xrightarrow{T_i} (M_i, \gamma_i),$$

on the manifold M_{i-1} can be viewed as merely removing the product neighborhood $N(T_i)$, it follows that Q is naturally homeomorphic to M_{i-1} so that the difference $M_{i-1} - Q$ is "small" and looks roughly like the complement of Q in a "smoothed out and extended" version of Q (see the below figure). Moreover, because V is a component of $R(\gamma_{i-1})$ by hypothesis, $V \subset \partial M_{i-1} - \mathring{\gamma}_{i-1}$, a fact which implies that V is contained in the surface with boundary obtained by removing the interiors of the annular components $A(\gamma_{i-1})$ from ∂M_{i-1}. In particular, by (homeomorphically) shrinking the widths of these annular components to be arbitrarily small, the product neighborhood $N(V)$ can be viewed as a 3-manifold as nothing more than a "thickened" version of V which has all of its removed annular components filled in. Combining these observations, it follows that Q can be visualized as being embedded in M_{i-1} in such a way that $M_{i-1} - Q \subset N(V)$.

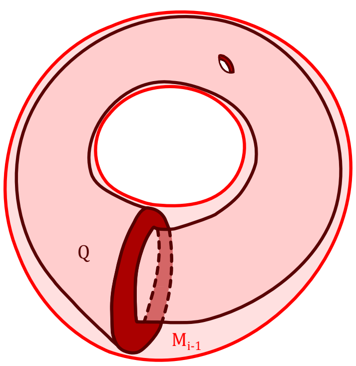

Figure 22
Q embedded into M_{i-1}

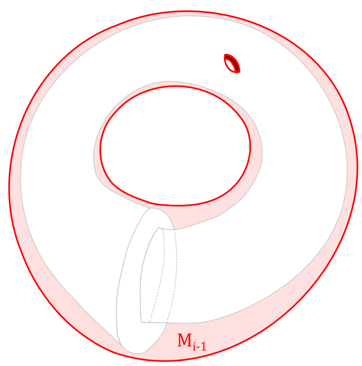

Figure 23
$M_{i-1} - Q$

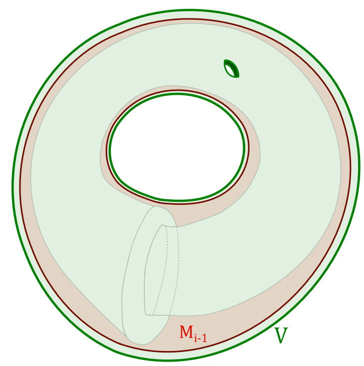

Figure 24
Roughly, $M_{i-1} - Q$ contained in $N(V)$

To construct the foliation $\mathcal{F}_0^{i-1}$, the idea will be to extend $\mathcal{F}_0^i$ to M_{i-1} by "spiraling" the boundary leaves towards the surface $V \subset \partial M_{i-1}$. A fair amount of complex machinery is needed to formalize this notion of spiraling, though as noted in [Gab83], one can imagine the process being analogous to the extension to an entire annulus of a dimension-1 foliation defined only on a subset of that annulus. The following figure shows this annulus analogy. Note that the spiraling will be constructed so that depth $\mathcal{F}_0^{i-1} =$ depth $\mathcal{F}_0^i + 1$ and hence will ensure finiteness of depth for foliations $\mathcal{F}_0^j$ for $j = i-2, i-2, \ldots, 0$.

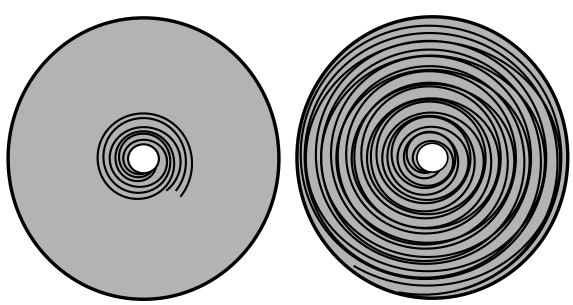

Figure 25
Extension of a codimension-1 foliation to a whole annulus by spiraling.

To begin, let A be the annular component of γ_i for which $\partial T_i^- \subset A$ and let $f : I \to I$ be the holonomy map of $\mathcal{F}_0^i|A$. Further, let δ be the simple closed curve $\delta \overset{\text{def}}{=} \partial T_i \cap V$ and let λ be a simple closed cirve in V whose geometric intersection number with δ is 1. These objects are highlighted in the following figure.

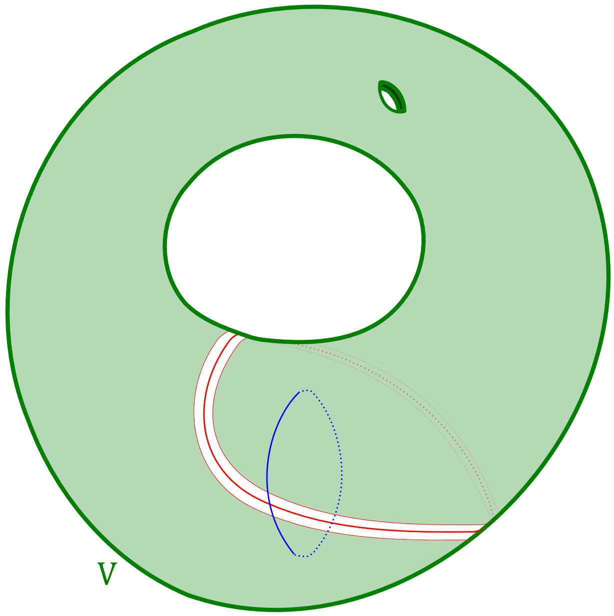

Figure 26

V with the simple closed curves δ (in blue) and λ (in red). Not shown is the annular component A of γ_i containing ∂T_i^- as shown in Figure 24 above, though the deleted neighborhood $\lambda \times (0,1)$ (utilized in (S1) *below)* ***is*** *shown.*

Let $\lambda \times I$ be a tubular neighborhood of λ in V for some closed interval I. The first goal is to construct a foliation on $V \times I \cong V \times [-\infty, \infty]$.

(S1) First, give $(V - (\lambda \times (0,1))) \times [-\infty, \infty]$ the product foliation $\mathcal{F}^0$.

(S2) $\mathcal{F}^0$ induces a foliation $\mathcal{F}^1$ on $V \times [-\infty, \infty]$ by identifying points of the form $(\lambda, 0, t)$ to points of the form $(\lambda, 1, \lfloor t \rfloor + f(t - \lfloor t \rfloor))$ where $\lfloor \cdot \rfloor$ denotes the floor (or greatest integer) function. This identification is shown schematically in the following figure and can be thought of as "pushing the transverse holonomy f across the gap" created by removing $\lambda \times (0,1)$ from V.

(S3) Define $\mathcal{F}^2$ to be the restriction $\mathcal{F}^1|V'$ where $V' = (V - \delta \times (0,1)) \times [-\infty, \infty]$.

(S4) Extend $\mathcal{F}^2$ to a foliation $\mathcal{F}^3$ on $V \times [-\infty, \infty]$ by identifying points of the form $(\delta, 0, t)$ on V' with those of the form $(\delta, 1, t+1)$. This identification is shown schematically in the following figure and is responsible for increasing by one the depth of the resultant foliation (whenever depth is defined), thus justifying the previously-claimed identity $\operatorname{depth} \mathcal{F}_0^{i-1} = \operatorname{depth} \mathcal{F}_0^i + 1$. This identification mimics the procedure from Section 3.2 in which existing finite-depth foliations were cut along junctures and re-glued to increase the depth.

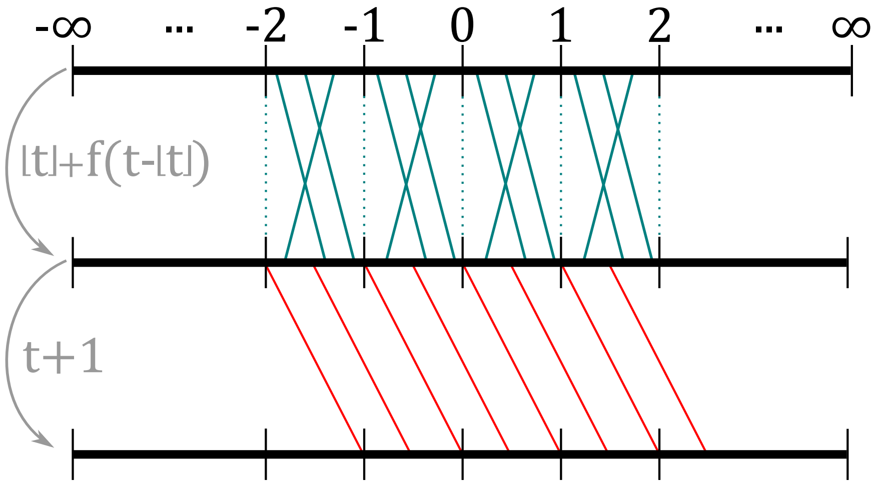

Figure 27

This is a schematic representation of the identifications $(\lambda, 0, t) \sim (\lambda, 1, \lfloor t \rfloor + f(t - \lfloor t \rfloor))$ *and* $(\delta, 0, t) \sim (\delta, 1, t+1)$*. Note that the first identification maps any interval* $[n, n+1]$*,* $n \in \mathbb{Z}$*, to itself (setwise) while fixing the endpoints. This can be thought of as "pushing the holonomy across the gap"* $\lambda \times (0,1)$*. The second identification shifts every* t*-value upward by one unit, thereby increasing the depth of the resultant foliation (whenever depth is defined) by one and justifying the previously-claimed identity* $\operatorname{depth} \mathcal{F}_0^{i-1} = \operatorname{depth} \mathcal{F}_0^i + 1$

Next, the goal is to show that the foliation $\mathcal{F}^3$ constructed on $V \times [-\infty, \infty]$ yields a suitable foliation on the complement $M_{i-1} - \mathring{Q}$, whereby the combination of this with the foliation already on Q will yield the desired foliation $\mathcal{F}_0^{i-1}$ on M_{i-1}.

To that end, let μ be the circle $(\delta, 0, 0)$ in $V \times [-\infty, \infty]$. Because of the gluing utilized in the spiraling process above, the leaf L of $\mathcal{F}^3$ containing μ is homeomorphic to the infinite ladder shown below and its ends limit towards $V \times \pm\infty$.

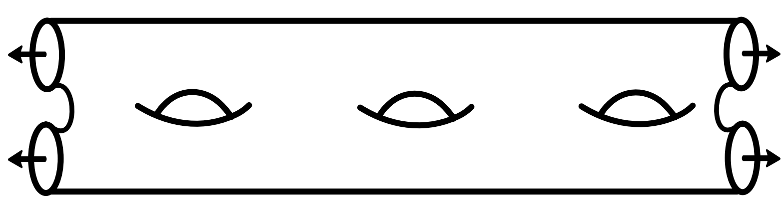

Figure 28

The infinite ladder, homeomorphic to the leaf L *of* $\mathcal{F}^3$ *containing* μ*.*

Now, define the sets $\widetilde{L}$ and $Z \subset V \times [-\infty, \infty]$, respectively, as

$$\begin{aligned} \widetilde{L} &= \mu \cup \{\text{points of } L \text{ lying on the } + \text{ side of } \mu\}, \text{ and} \\ Z &= \widetilde{L} \cup \{(x,t) : \text{the normal ray } (x,(t,\infty]) \text{ intersects } \widetilde{L} \text{ nontrivially}\}, \end{aligned}$$

respectively. From these definitions, two things are true: First, Z is topologically equivalent to $V \times I$. Moreover, Z is *diffeomorphic* to $M_{i-1} - \mathring{Q}$ where $V \times \{0\}$ is the unique compact leaf of $\mathcal{F}^3|Z$ and $V \times I$ the union of a twice-punctured surface contained in $\widetilde{L}$ and an annulus transverse to $\mathcal{F}^3|Z$, the holonomy along which is identically f and hence is equal to the holonomy of $\mathcal{F}_0^i$ along the transverse annulus A. As a result, the foliated space $(Z, \mathcal{F}_3|Z)$ (being viewed throughout as $M_{i-1} - \mathring{Q}$) can be glued to Q to obtain a manifold diffeomorphic to M_{i-1} and foliated in a "compatible" way by a foliation defined to be $\mathcal{F}_0^{i-1}$. This process is sketched in the figure below, whereby the construction of $\mathcal{F}_0^{i-1}$ is complete.

— *Constructing* $\mathcal{F}_1^{i-1}$:

As above, let Q be the manifold which results from gluing T_i^+ to T_i^- and denote by $\mathcal{F}^1$ the foliation on Q obtained by extending $\mathcal{F}_1^i$ to Q. Let f be the holonomy of $\mathcal{F}^1$ along the transverse annulus. There are a number of different cases to consider depending on the properties of f and V.

(C1) In the event that $f = \mathrm{id}$, the procedure performed above for $\mathcal{F}_0^{i-1}$ can be duplicated on $\mathcal{F}^1$ to yield $\mathcal{F}_1^{i-1}$ with all the desired smoothness properties. In many cases, this procedure fails to be adequate and produces only a C^0 foliation in general. More on this potential failure is discussed in the remark following the below construction.

(C2) If $f \neq \mathrm{id}$ and $\partial V \neq \varnothing$, then the decomposition actually falls into case (C1). This can be seen by *pushing the holonomy to the boundary* [Gab83] as follows:

(a) Construct a codimension-1 foliation $\mathcal{F}'$ on $(S^1 \times I) \times I$ so that the leaves of $\mathcal{F}'$ are transverse to $S^1 \times I \times t$ surfaces and the associated holonomy to $\mathcal{F}'$ is given by f^{-1}.

(b) Let Q' be the manifold obtained by gluing $(S^1 \times I) \times \{0\}$ to a *collar neighborhood* of a component W of ∂V (i.e., a neighborhood of W of the form $W \times [0, 1)$).

(c) Finally, attach a band $(I \times I) \times I$ with the product foliation to Q' so that $I \times I \times \{0\}$ glues to V, $\{0\} \times I \times I$ glues to the transverse annulus, and $\{1\} \times I \times I$ glues to $(S^1 \times I) \times I$.

A depiction of this construction is shown in figure 29 and can be considered a specialized case of the more general construction shown in figure 30 below.

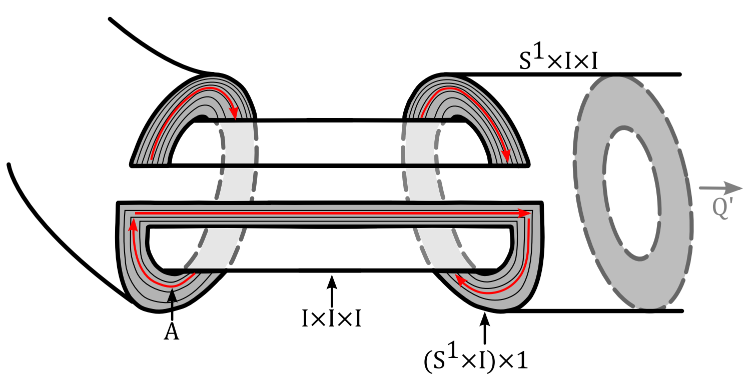

Figure 29

$I \times I \times I$ glued to the transverse annulus A and to $(S^1 \times I) \times \{1\}$. Q' is not pictured. The red arrows indicate the holonomies: f for A and f^{-1} for $S^1 \times I \times I$.

(C3) If $f \neq \mathrm{id}$, $\partial V = \varnothing$, and V is a torus, things are less advantageous: In particular, the construction for $\mathcal{F}_0^{i-1}$ above must be mimicked and nothing more than C^0 smoothness can be ensured for the holonomy maps of $\mathcal{F}_1^{i-1}$ along V. As mentioned previously, this limitation is a result of Kopell's Lemma 3.4 (see the remark immediately preceding case 1 for details).

(C4) The only remaining case is that $V = S_g$ is a surface of genus $g > 1$ and $f \neq \mathrm{id}$. Much like (C2) above, the decomposition again can be deformed into case (C1), this time thanks to the Lemma 3.15 of Mather, Sergeraert, and Thurston. To make this more precise: Let Q_1 be the manifold obtained by attaching thickened bands B_1 and C_1 on ∂Q as shown in the figure below. This transforms the existing transverse annulus on Q into a new transverse annulus on Q_1.

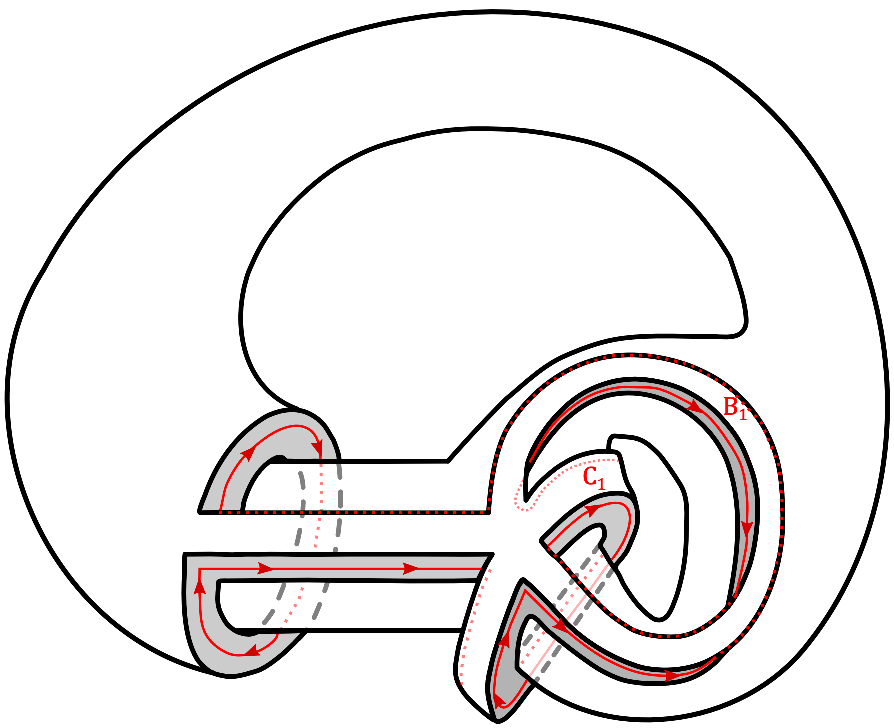

Figure 30

The manifold Q_1 with its thickened bands B_1 and C_1 added and labeled. Note the arrows showing the holonomy of the (modified) transverse annulus as it winds around Q_1 and satisfies (3.4.4) *below. Note that the foliations on the transverse annuli* ***are*** *present but are omitted from this figure for clarity; see figure 29 above for comparison.*

Consider on Q_1 the foliations $\mathcal{F}^1$ and $\mathcal{F}^2$ defined in the construction of $\mathcal{F}_0^{i-1}$ above and extend $\mathcal{F}^1$ and $\mathcal{F}^2$ by foliating the thickened bands so that the holonomy along B_1 (respectively, C_1) is b_1 (respectively, c_1). With these assumptions, one notes that the holonomy along the new transverse annulus (on Q_1) is none other than

$$f c_1 b_1 c_1^{-1} b_1^{-1} = f \circ [c_1, b_1]. \tag{3.4.4}$$

Repeating this procedure n times yields a foliation $\mathcal{F}^{n+1}$ on Q^n whose holonomy along the resulting transverse annulus has the form

$$f \circ [c_1, b_1] \circ [c_2, b_2] \circ \cdots \circ [c_n, b_n]. \tag{3.4.5}$$

By Lemma 3.15, (3.4.5) is precisely the identity whereby it follows that the foliation $\mathcal{F}^{n+1}$ on Q^n has trivial holonomy along the transverse annulus of Q^n. This results in case (C1) as claimed. $\boxed{\text{Case 2}}$

Remark. Note that considerably more work is required to create a C^∞ foliation $\mathcal{F}_1^{i-1}$ than to create it's C^0 counterpart $\mathcal{F}_0^{i-1}$. Indeed, obtaining increased levels of differentiability requires a number of intrinsic factors to be considered. For example, in (C1) above, $f = \text{id}$ allows the "smoothing" process used in the construction of $\mathcal{F}_0^{i-1}$ to be applied to $\mathcal{F}_1^i$ and yields a C^∞ foliation $\mathcal{F}_1^{i-1}$. However, for $f \neq \text{id}$, attempting to replicate the "smoothing" procedure will result in a foliation which in general cannot be smoother than C^0. Succinctly, this is because nontrivial holonomy is perhaps the most significant hurdle to the creation of well-behaved foliations on surfaces with nonzero genus. This fact ensures that extra effort be required in cases (C2), (C3), and (C4), and the failure of (C2) and (C4) methods to alleviate the crisis in (C3) is a testament to the further fact that nontrivial holonomy is especially bad when tori are involved.

Case 3: $\partial T_i \cap \gamma_{i-1} \neq \varnothing$, ∂T_i Connected

For this particular case, the procedures required are a bit more tedious. In particular, the process will be presented in steps: First, foliations $\mathcal{F}'_\alpha$ will be defined on the manifold $Q \subset M_{i-1} - N(R(\gamma_{i-1}))$ obtained by gluing T_i^+ to T_i^-; afterward, special attention will be paid to developing a procedure for extending these foliations near individual components V of $R(\gamma_{i-1})$, at which point the aforementioned procedures can be applied to the remaining components of $R(\gamma_{i-1})$, thus yielding the desired foliations $\mathcal{F}_\alpha^{i-1}$ on M_{i-1}. Defining these foliations will complete case 3 and will therefore mark the end of this part of the proof (corresponding to outline item (O.II)).

As shown in 31 below, ∂T_i^+ (respectively, ∂T_i^-) consists of the union of arcs contained in $\partial\gamma_i$ and of arcs which are properly embedded in $R_+(\gamma_i)$ (respectively, in $R_-(\gamma_i)$). This will be the starting point for the construction that follows.

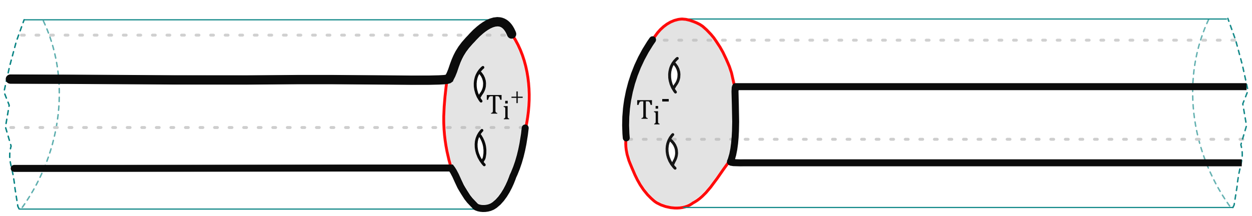

Figure 31

A view of M_i, particularly of $\partial T_i^\pm$ as the union of arcs in $\partial\gamma_i$ (thickened, in black) and arcs which are properly embedded in $R_\pm(\gamma_i)$ (in red). Note that all of $\partial\gamma_i$ is thickened and black (hence the thickened bands continuing along the sides of M_i) though the induced foliation on γ_i ***isn't*** *shown (compare with Figures 33 and 34 below).*

(1) First, perform the diffeomorphism on M_i obtained by "stretching" (or "extending") those pieces of γ_i which contain $\partial T_i^+ \cup \partial T_i^-$ as shown in the figure below.

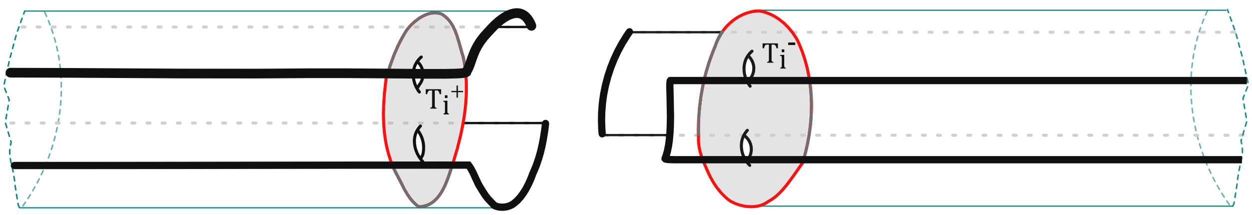

Figure 32

*The result obtained by "stretching" (or "extending") the pieces of γ_i which contain $\partial T_i^+ \cup \partial T_i^-$. As noted in the figure above, the induced foliation on γ_i **isn't** shown (compare with Figures 33 and 34 below).*

Next, glue T_i^+ to T_i^- to create the manifold Q which—like in Case 2 above—is homeomorphic to M_{i-1} and should be thought of as lying in $M_{i-1} - N(R(\gamma_{i-1}))$. According to [Gab83], this gluing process can be visualized analogously to stacking chairs atop one another. Note that the transverse holonomy is trivial in this case (as it lies along the vertical arc segments in figure 33).

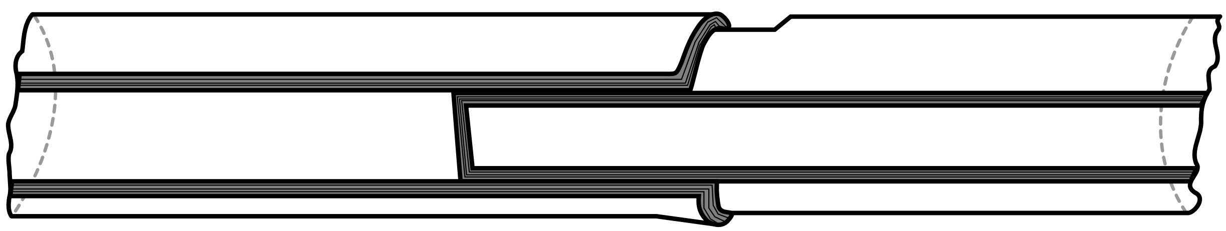

Figure 33

Gluing T_i^+ to T_i^- in the "stretched" version of M_i yields the manifold Q, as before.

Finally, for $\alpha \in \{0, 1\}$, define $\mathcal{F}'_\alpha$ to be equal to the foliation $\mathcal{F}^i_\alpha$ extended to Q. The goal (shown in the figure 34) is to define $\mathcal{F}^{i-1}_\alpha$ to be the extension of $\mathcal{F}'_\alpha$ from Q to all of M_{i-1}, and per the remark immediately preceding the beginning of case 3 above, this extension process should be qualitatively similar to the "smoothing" process utilized in the construction of $\mathcal{F}^{i-1}_0$ in case 2. Note, however, that the foliations $\mathcal{F}'_\alpha$ defined up to this point have been constructed on $Q \subset M_{i-1} - N(R(\gamma_{i-1}))$ and **not** on all of M_{i-1} as desired.

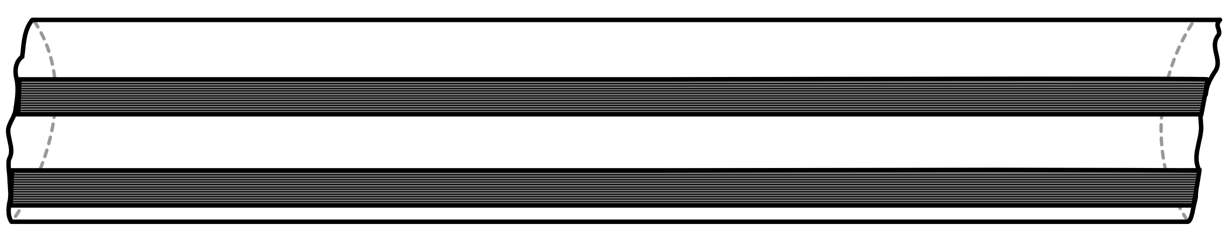

Figure 34

A visual representation of the desired result: Foliations $\mathcal{F}^{i-1}_\alpha$ resulting from extending $\mathcal{F}'_\alpha$ to all of M_{i-1}.

(2) Next, the goal is to extend the above construction to all of M_{i-1} by first defining foliations $\mathcal{F}''_\alpha$ on $N(R(\gamma_{i-1}))$ which can be glued, coherently, to the foliations $\mathcal{F}'_\alpha$ defined on Q above. Once this is done, performing the appropriate gluings will "connect" $\mathcal{F}'_\alpha$ to $\mathcal{F}''_\alpha$ and the result will be the pair of foliations $\mathcal{F}^{i-1}_\alpha$ on all of M_{i-1} as desired. This goal is achieved by first constructing intermediate foliations $\mathcal{F}''_\alpha(V)$ relative to single components V of $R(\gamma_{i-1})$ and devising gluing procedures to connect these coherently to the $\mathcal{F}'_\alpha$ defined on Q above, at which point this process is repeated for the remaining components of $R(\gamma_{i-1})$.

To that end, let V be a component of $R(\gamma_{i-1})$ for which $\partial T_i \cap V \neq \varnothing$ and define the manifold P to be $P \overset{\text{def}}{=} N(V) \cap Q$ as shown in the following figure.

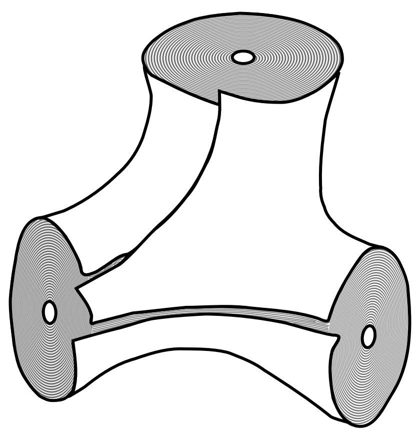

Figure 35
$P = N(V) \cap Q$

Note that $P \cong V \times I$ and that $V \times \{1\}$ has the form

$$V \times \{1\} = J \cup (\mu_1 \times I) \cup \cdots \cup (\mu_n \times I)$$

where J is tangent to $\mathcal{F}'_\alpha$ for $\alpha \in \{0,1\}$ and where, for all $m = 1, 2, \ldots, n$ and for $\alpha \in \{0,1\}$, (i) $\mu_m \times \{0\}$ is properly embedded in both $V \times \{1\}$ and in the leaf L of $\mathcal{F}'_\alpha$ which contains J, (ii) $\mu_m \times \{1\} \subset \partial L$ is embedded in $V \times \{1\}$, and (iii) $\mathcal{F}'_\alpha|(\mu_m \times I)$ has the product foliation. This is illustrated in the figure below.

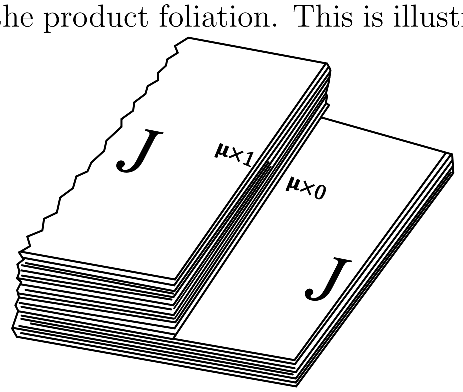

Figure 36
The product foliation on $\mathcal{F}'_\alpha|(\mu_m \times I)$

(3) Define $J' = J - N(\cup_{m=1}^{n}(\mu_m \times \{0\}))$ and let Q_1 be the union along $J' \times \{0\}$ of Q with $J' \times I$:

$$Q_1 = Q \bigcup_{J' \times 0} (J' \times I)\,.$$

Intuitively, Q_1 looks like (M_{i-1}, γ_{i-1}) with "ditches" $\beta_m \times \mathring{I} \times \mathring{I}$ drilled out where here, $\beta_m \times I \times \{0\} \subset L$ and $\beta_m \times \{0\} \times \{0\}$ is identified with $\mu_m \times \{0\}$ for $m = 1, 2, \ldots, n$. Q_1 is shown in the figure below. Note that (the Q part of) Q_1 inherits the foliations $\mathcal{F}'_\alpha$ previously constructed on Q above and that both the foliations on Q and those on (the Q part of) Q_1 will be referred to as $\mathcal{F}'_\alpha$ moving forward. The goal now is to foliate the $J' \times I$ component of Q_1 in a coherent fashion—a goal equivalent to foliating the ditches $\beta_m \times \mathring{I} \times \mathring{I}$ coherently—whereby, upon gluing, the process of constructing $\mathcal{F}_\alpha^{i-1}$ on M_{i-1} will be complete.

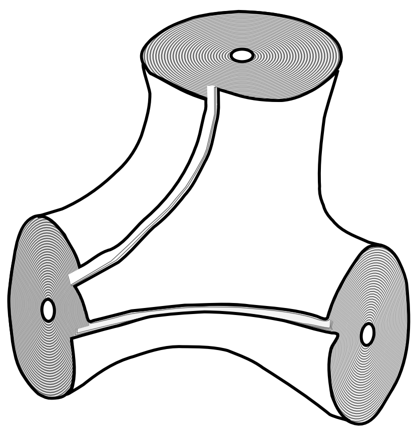

Figure 37

Q_1, which looks like (M_{i-1}, γ_{i-1}) with "ditches" drilled out

First, give $B_m \overset{\text{def}}{=} (\beta_m \times I) \times I$ the product foliation $\mathcal{F}_0^\sharp = \mathcal{F}_1^\sharp$. Because Q_1 can be thought of as (M_{i-1}, γ_{i-1}) minus the ditches discussed above, the procedure will be complete assuming B_m can be glued to Q_1 in a way which is coherent with respect to the foliations $\mathcal{F}'_\alpha$ and $\mathcal{F}^\sharp_\alpha$ on Q_1 and B_m, respectively. Half of this process is a technicality and is immediate; the other requires a bit more creativity. The end result of the process is the foliated manifold (M_{i-1}, γ_{i-1}) shown in figure 38 below.

(a) Because of how $\mathcal{F}'_\alpha$ and $\mathcal{F}^\sharp_\alpha$ were constructed, any smooth gluing of B_m into Q_1 will result in a foliation which is C^∞ and which satisfies the remaining properties claimed by the statement of the theorem. This foliation is defined to be $\mathcal{F}''_1(V)$.

(b) A bit more work is required to construct $\mathcal{F}''_0(V)$ due to the requirement that it glue to F'_0 to result in a foliation that's finite-depth. In this case, write $J' \times I$ as $J' \times [1, \infty]$ and $\beta_m \times I \times I$ as $\beta_m \times I \times [0, \infty]$, and glue $\beta_m \times \{0\} \times [0, \infty]$ into Q_1 by identifying $\beta_m \times \{0\} \times [0, 1]$ with

$\mu_m \times [0,1]$ and $\beta_m \times \{0\} \times [1,\infty]$ with $(\mu_m \times \{1\}) \times [1,\infty]$ via a map which is the identity on the $[1,\infty]$ coordinate. Next, define

$$\alpha_m \stackrel{\text{def}}{=} \mu_m \times \{1\} \subset J',$$

$$\alpha'_m \stackrel{\text{def}}{=} (N(\mu_m \times \{0\}) \cap J') - \alpha_m,$$

and glue $\beta_m \times \{1\} \times [0,\infty]$ into Q_1 by identifying $\beta_m \times \{1\} \times [0,\infty]$ with $\alpha'_m \times [1,\infty]$ via a map f for which $f : x \mapsto x+1$ on the second factors. Let $\mathcal{F}''_0(V)$ be the result of this process.

These gluings yield the desired properties in most cases. In particular, whenever the μ_m are parallel, the gluing f ensures finiteness of depth and guarantees that depth $\mathcal{F}''_0(V) =$ depth $\mathcal{F}^i_0 + 1$. The drawback, however, is that this process fails in general when the μ_m *aren't* parallel: Indeed, in this case, there's no way to glue $\beta \times I \times I$ in a way that ensures $\mathcal{F}''_0(V)$ has finite depth regardless of what one knows about the depth of $\mathcal{F}^i_0$.

(4) Note that the process in steps (1), (2), and (3) yields the desired foliations $\mathcal{F}''_\alpha(V)$ relative to a single component V of $R(\gamma_{i-1})$, extending $\mathcal{F}'_\alpha$ by focusing on

$$P = P(V) = N(V) \cap Q$$

for $Q \subset M_{i-1} - N(R(\gamma_{i-1}))$. As it happens, $R(\gamma_{i-1})$ may have many components V_κ, and by repeating steps (1) through (3) on each individual component, one gets foliations on all of M_{i-1} which satisfy the desired properties. Call the resulting foliations $\mathcal{F}^{i-1}_0$ and $\mathcal{F}^{i-1}_1$, whereby the construction is complete. $\boxed{\text{Case 3}}$

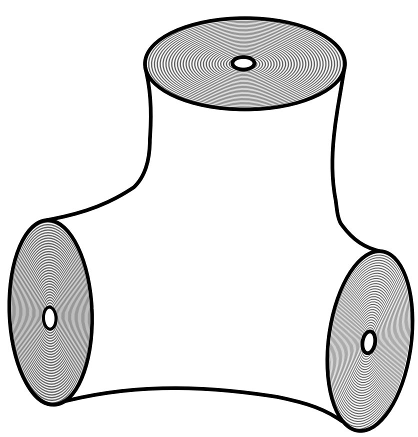

Figure 38
Q_1 after gluing in $(\beta_m \times I) \times I$

The above constructions verify the legitimacy of outline item (O.II), though as noted in that item, they're only required in general to satisfy items (1), (2), and (4) from the statement of theorem 3.13. To complete the proof, all that's left is to address the outline items (O.III), (O.IV), and (O.V), whereby foliations will be produced on (M, γ) satisfying all of items (1) through (6) of theorem 3.13, thus completing the proof.

To that end, note that the foliations $\mathcal{F}_\alpha^{i-1}$ are *taut foliations* by definition 3.10 above and so, by lemma 3.14, either (M_{i-1}, γ_{i-1}) is a sutured manifold or it's one of the products $S^2 \times S^1$, $S^2 \times I$ with $\mathcal{F}^{i-1}$ the product foliation. By definition of taut sutured manifold, this result implies that M_{i-1} is irreducible and that $R\pm(\gamma_{i-1})$ are norm-minimizing in $H_2(M_{i-1}, \gamma_{i-1})$. This confirms item (O.III).

Following the induction from (O.II) yields tautness of $(M_0, \gamma_0) = (M, \gamma)$. In particular, the lemmas 3.12 and 3.16 can be applied, thereby yielding a separate (sutured manifold) decomposition of (M, γ) satisfying the properties claimed by (O.IV): (i) No decomposing surface S_i is a torus unless $\partial M = \varnothing$, $H_2(M)$ is generated by tori, and $i = 1$; and (ii) if V is a component of $R(\gamma_{i-1})$, then $V \cap S_i$ consists of k (≥ 0) parallel oriented nonseparating simple closed curves.Finally, by applying the constructions detailed above (for outline item (O.II)) to the decomposition produced by lemmas 3.12 and 3.16, one arrives at foliations on (M, γ) which satisfy *all* of the properties claimed in the statement of theorem 3.13. Hence, the result. $\square$

3.5 Implications and Corollaries

One can completely understate the importance of the results of [Gab83] by saying that the work itself is "significant;" indeed, the true level of significance is hardly fathomable. Here are some things that follow from having proven theorem 3.13.

Theorem 3.17. Let M be a compact connected irreducible oriented 3-manifold whose boundary ∂M is a (possibly empty) union of tori. Let S be a norm minimizing surface representing a nontrivial class $z \in H_2(M, \partial M)$. Then there exist transversely oriented foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of M such that:

(1) For $i = 0, 1$, $\mathcal{F}_i \pitchfork \partial M$ and $\mathcal{F}_i | \partial M$ has no Reeb components.
(2) Every leaf of $\mathcal{F}_0$ and $\mathcal{F}_1$ nontrivially intersects a transverse closed curve.
(3) $\mathcal{F}_0$ is of finite depth.
(4) $\mathcal{F}_1$ is C^∞ except possibly along toral components of S.
(5) S is a compact leaf of both $\mathcal{F}_0$ and $\mathcal{F}_1$.

Proof. Cut along S to yield a decomposition

$$(M, \partial M) \xrightarrow{S} (M_1, \gamma_1),$$

noting that *a priori* this decomposition isn't a sutured manifold decomposition. Due to the norm-minimality of S, however, it follows that (M_1, γ_1) is taut and that $H_2(M_1, \gamma_1) \neq 0$. In particular, by lemma 3.12, this decomposition can be upgraded to a sutured manifold decomposition, whereby theorem 3.13 can be applied to obtain foliations $\mathcal{F}'_\alpha$ on (M_1, γ_1), $\alpha \in \{1, 2\}$. Identifying the sets S^+ and S^- (as subsets of S), one can glue to "reconstruct" M and can foliate the resulting manifold to yield the foliations $\mathcal{F}_\alpha$ as defined. □

Remark. Despite the proof of theorem 3.13 being the self-described crux of this exposition, the result of 3.17 is perhaps the most powerful single result contained herein. In particular, it yields *pay dirt* (i.e., Reebless foliations) on an extremely large, general class of 3-manifolds which are characterized by a surprisingly loose set of conditions.

The construction from theorem 3.13 and its use to prove theorem 3.17 yield a slew of interesting corollaries related to areas such as knot theory. Some of those are stated below without proof.

Corollary 1. Let L be an oriented nonsplit link[20] in S^3. Then S is a surface of minimal genus for L[21] if and only if there exists a C^∞ transversely-oriented foliation $\mathcal{F}$ of $S^3 - \mathring{N}(L)$ such that:

(1) $\mathcal{F} \pitchfork \partial N(L)$ and $\mathcal{F}|\partial N(L)$ has no Reeb components.
(2) $\mathcal{F}$ has no Reeb components.
(3) S is a compact leaf of $\mathcal{F}$.

Corollary 2. A nontrivial link L in S^3 is nonsplit if and only if L is the set of cores of Reeb components of some foliation $\mathcal{F}$ of S^3 where here, one defines the *core* of a Reeb compnent $V = D^2 \times S^1$ to be a smooth simple closed curve δ in V isotopic to $t \times S^1$ for some $t \in D^2$.

As noted in [Gab83], corollary 2 answers the "Reeb placement problem" of Laudenbach and Roussarie ([Rou70]) by showing which links in S^3 are cores of Reeb components and can be considered a special case of a more general result: A link L in a 3-manifold M has an irreducible ∂-incompressible complement if and only if L is the set of cores of Reeb components of some foliation $\mathcal{F}$ on M. The proof is as in Corollary 6.5 and can be found in [Gab83, p.478].

Corollary 3. Let S_i be a Seifert surface[22] for the oriented link $L_i \subset S^3$ for $i = 1, 2$, and S be any Murasugi sum or generalized plumbing[23] of S_1 and S_2 with $L = \partial S$. Then S is a minimal genus surface for the oriented link L if and only if each S_i is a minimal genus surface for the oriented link L_i.

[20] Recall that L is *nonsplit* in S^3 if there exists no embedded $S^2 \subset S^3$ such that $S^2 \cap L = \varnothing$ but each component of $S^3 - S^2$ intersects L nontrivially. This is equivalent to saying that $\pi_2(S^3 - L) = 0$.

[21] A *surface of minimal genus* for an oriented link L in S^3 is an oriented embedded surface S in S^3 containing no closed components, whose oriented boundary is L and $\chi(S) \geq \chi(T)$ for any other surface T satisfying the above conditions. Here, χ denotes the Euler characteristic.

[22] A surface S is a *Seifert surface* if its boundary is a given knot or link.

[23] A *Murasugi sum*, also known as a generalized plumping or a start product by Murasugi, is an operation which combines two Seifert surfaces along a disk in each.

As noted in [Gab83], corollary 3 generalizes a classical fact of Seifert which says that the connected sum of minimal genus surfaces is minimal genus.

Corollary 4. Let M be a compact irreducible connected 3-manifold such that its boundary ∂M is a (possibly empty) union of tori, and $H_2(M, \partial M)$ is not generated by tori and annuli (i.e., $x(z) \neq 0$ for some $z \in H_2(M, \partial M)$). Then there exists a C^∞ transversely oriented foliation $\mathcal{F}$ of M such that $\mathcal{F} \pitchfork \partial M$, $\mathcal{F}|\partial M$ has no 2-dimensional Reeb components, and no leaf of $\mathcal{F}$ is compact.

The proof of corollary 4 is two-and-a-half pages of difficult mathematics in [Gab83, pp.480–482], though the heaviest of the lifting comes from theorem 3.13 above.

Corollary 5. Let M either be (i) a compact 3-manifold with boundary ∂M whose interior has a complete hyperbolic metric and $H_2(M, \partial M) \neq 0$, or (ii) $S^3 - \mathring{N}(L)$ where L is a nonsplit link in S^3. Then there exists a C^∞ transversely oriented foliation $\mathcal{F}$ of M such that $\mathcal{F}$ has no compact leaves, $\mathcal{F} \pitchfork \partial M$, and $\mathcal{F}|\partial M$ has no Reeb components.

Corollary 6. Suppose M is a compact irreducible 3-manifold, ∂M is a (possibly empty) union of tori, and $H_2(M, \partial M)$ is not generated by tori and annuli. Then there exists a Riemannian metric and a foliation $\mathcal{F}$ on M such that $\mathcal{F} \pitchfork \partial M$, and every leaf is minimal (i.e., has mean curvature zero).

As noted in [Gab83], corollary 6 was conjectured by Thurston in [Thu86].

Corollary 7. Let M be compact and orientable. Let $p : \widetilde{M} \to M$ be an n-fold covering map, and let $z \in H_2(M) = H^1(M, \partial M)$ or $z \in H_2(M, \partial M) = H^1(M)$. Then $n(x(z)) = x(p^*(z))$.

Corollary 8. Let M be a compact oriented 3-manifold. Then on $H_2(M)$ or $H_2(M, \partial M)$, $x_s = x = g/2$ where x denotes the Thurston norm, x_s is the norm based on singular surfaces, and g is the Gromov norm[24].

The equality of the singular and Thurston norms was conjectured by Thurston in [Thu86]. Meanwhile, corollary 9 below is a generalization of Dehn's lemma and the loop and sphere theorems to higher genus surfaces. How such theorems generalize was an open question before Gabai's work [Gab83] dating back to the works of Papakyriakopoulos in the mid 1950s. In particular he asked about the relationship between the immersed genus and the genus of a knot, a question also answered by 10 below.

[24]Here,

$$x_s(z) = \inf\{x(T)/n \,:\, f : T \to M \text{ and } f_*[T] = n \text{ where } f \text{ is a proper map of a compact oriented surface}\}$$

and

$$g(z) = \inf\{\sum |a_i| \left[\sum a_i \sigma_i\right] = z \text{ where } \sum a_i \sigma_i \text{ is a singular chain}\}.$$

Corollary 9 (Higher Genus Dehn's Lemma)**.** Let M be a compact oriented 3-manifold, S a compact oriented connected surface with connected boundary, $f : S \to M$ a map such that $f|\partial S$ is an embedding and $f^{-1}(f(\partial S)) = \partial S$. Then there exists a compact embedded oriented surface T in M such that $\partial T = \partial S$ and $\text{genus}(T) \leq \text{genus}(S)$.

As a special case of Corollary 9:

Corollary 10. If K is a knot in S^3, then the immersed genus[25] equals the embedded genus[26] and, more generally, if K is nontrivial and $f : T \to S^3 - \mathring{N}(K)$ is a proper map of an oriented surface no component of which is closed then $x(T) \geq (2g-1)|n|$ where $f_*[T] = [n] \in H_2(S^3 - \mathring{N}(K), \partial N(K)) = \mathbb{Z}$.

The corollaries stated above are but a small subset of the results stemming from theorems 3.13 and 3.17 above. Indeed, the entirety of [Gab87a] and [Gab87b] further Gabai's theory of sutured manifolds and produces a further collection of outstanding results. The diligent reader is encouraged to consult all three of Gabai's volumes [Gab83], [Gab87a], and [Gab87b], both for the sake of completing the omissions in this particular exposition and for the benefit of being exposed to a corpus of truly amazing mathematics.

[25]The *immersed genus* of a knot K in S^3 is the smallest g such that K bounds a pucntured immersed surface of genus g which is nonsingular along the boundary, i.e. $f : S \to S^3$ and $f^{-1}(K) = \partial S$.

[26]The *embedded genus* of a knot K in S^3 is the smallest g such that k bounds a punctured embedded surface S_g of genus g.

4 Possible Directions for Future Work

A number of interesting future projects have emerged from the research conducted up to this point, some of which are summarized (briefly) in the subsections that follow.

4.1 Sutured Manifolds

Unsurprisingly, the theory of sutured manifolds is a vast one. Works by [Juh10] and [Sch90] are but a small subset of this theory and yet are hugely illustrative in terms of what exists for the theory. Though the main focus of my work moving forward will likely center on foliation theory—and despite the fact that the foliation theory associated to sutured manifold theory has slowed down slightly—a number of recent results in the area have shown that there is still foliation theory to be done. Moreover, I think it would be interesting to delve deeper into sutured manifold concepts (e.g., sutured Floer homology) and to attempt to (i) catch up on the work that exists, and (ii) develop the existing body of work even further.I can easily say that this is the area with which I'm least familiar, but even so, I look forward to the challenge of digging deeper therein.

4.2 Universal Circles of Finite-Depth Foliations

There's an enormous body of work that links foliation theory to hyperbolic geometry. Despite going unaddressed in this particular essay, a fair portion of my work moving forward will be to exploit these relationships.

In [Can93], a strong connection is made, not between hyperbolic geometry and foliation theory *directly*, but between the geometry and the theory of laminations. Roughly speaking, a *(surface) lamination* in a Riemannian 3-manifold $M = (M^3, g)$ is a foliation $\mathcal{F}_\Omega$ of a closed subset Ω of M by 2-dimensional leaves which are complete (as metric spaces). The gap between this result and foliation theory is strengthened by a theorem of [Con89] which states that there exists a leafwise hyperbolic metric on a properly-foliated[27] manifold $(M, \mathcal{F})$ if and only if (i) each component of ∂M is a leaf of $\mathcal{F}$, and (ii) no leaf of $\mathcal{F}$ is a torus or sphere.. Succinctly stated: The collection of foliated manifolds upon whose leaves one can construct a hyperbolic metric is extremely large. The connection between this metric and the topology of the foliation is also very enticing.

One question asked by Bill Thurston was whether the *circles of infinity* $S^1_\infty(\lambda)$ of the leaves λ of a foliated 3-manifold $(M, \mathcal{F})$ could be "glued together" in a way which is "coherent." This question was the center of the unfinished manuscript [Thu98] and gave way to what is now known as the *universal circle*. The universal circle is defined formally by Calegari and Dunfield in [CD03] and is addressed (in various cases and forms) in a number of

[27]A foliated manifold $(M, \mathcal{F})$ is said to be *proper* if every leaf L of $\mathcal{F}$ is proper, i.e. if any of the equivalent conditions are satisfied for each leaf L: (i) The relative topology of L in M coincides with the manifold topology of L; (ii) each point $x \in L$ lies in a foliation chart $\mathcal{U} \subset M$ such that $L \cap \mathcal{U}$ is a single plaque; (iii) L is not asymptotic to itself.

other works including [Cal00], [Cal07], and [Fen02] and is closely linked to topics in foliation theory including tautness. The definition is withheld for brevity.

The punchline of one a theorem from the work of Calegari and Dunfield on the subject [CD03, Theorem 6.2] is that universal circles exist for any taut foliation of an orientable 3-manifold with hyperbolic leaves. Among the many questions posed by this result is how the existence of this universal circle (a largely geometric object) pertains to the topology of the host manifold. That route is one I would like to investigate further.

One project of particular interest is to restrict attention to the case of finite-depth foliations (see section 3) and to methodically construct the corresponding universal circle in hopes that the details of the construction will help to learn more about the geometry and topology of the underlying manifold. In general, finite-depth foliations can still be very exotic and for that reason, this project will begin with the modest goal of understanding the construction for depth-one foliations before attempting to delve to farther depths. Some work in this direction has already been begun.

4.3 Slitherings

One type of foliation not defined in the body of this exposition is that induced by a *slithering*. Defined first in [Thu97], a manifold M is said to *slither* around a manifold N when $\widetilde{M}$ fibers over N so that the deck transformations of $\widetilde{M}$ are bundle automorphisms. As stated by Thurston: 3-manifolds that slither around S^1 are like a hybrid between 3-manifolds that fiber over S^1 and certain kinds of Seifert-fibered[28] 3-manifolds.

[Thu97] mentions a number of results for such objects. For example, it's shown that M slithers around the circle if and only there exists a *uniform foliation*[29] $\mathcal{F}$ on M. What's more, these induced uniform foliations are understood to be hybrids of fibrations over the circle with foliated circle bundles over surfaces. There are many examples of slithered structures and a number of well-studied examples of foliations exist as foliations induced by slitherings; as such, the geometry and topology birthed from this type of structure are very rich.

Unfortunately, all that exists of Thurston's original exposition on this is the single preprint [Thu97]. In that paper, the author predicts penning two additional manuscripts in which a number of properties were to be discussed; unfortunately, the author's untimely passing occurred before any subsequent volumes were penned. Other resources such as [Cal07] delve a bit deeper into the topic, though objectively, a number of topics related thereto still remain uninvestigated. This topic will serve as the foundation for one future project, with the initial goal of plucking the low-hanging fruits related to these structures. Time permitting, this project could grow into something more expansive.

[28] A *Seifert-fibered manifold* is a closed 3-manifold together with a decomposition into a disjoint union of circles (called *fibers*) such that each fiber has a tubular neighborhood which forms a standard fibered torus.

[29] A foliation $\mathcal{F}$ on a 3-manifold M is said to be *uniform* if any two leaves in the lifted foliation $\widetilde{\mathcal{F}}$ of the universal cover $\widetilde{M}$ are a bounded distance apart. The bound in general depends on the pair of leaves.

References

[Cal00] Danny Calegari, *The Geometry of* $\mathbb{R}$*-covered Foliations*, Geom. Topol. **4** (2000).

[Cal07] Danny Calegari, *Foliations and the geometry of 3-manifolds*, Oxford Mathematical Monographs, Oxford Univ. Press, UK, Oxford, 2007.

[Can93] Alberto Candel, *Uniformization of surface laminations*, Annales scientifiques de l'cole Normale Suprieure **26** (1993), no. 4, 489–516.

[CC81] John Cantwell and Lawrence Conlon, *Poincar-Bendixson Theory for Leaves of Codimension One*, Transactions of the American Mathematical Society **265** (1981), no. 1, 181–209.

[CC00] Alberto Candel and Lawrence Conlon, *Foliations*, Graduate Studies in Mathematics, American Mathematical Society, 2000.

[CC03] ———, *Foliations II*, Graduate Studies in Mathematics, American Mathematical Society, 2003.

[CD03] Danny Calegari and Nathan Dunfield, *Laminations and groups of homeomorphisms of the circle*, Invent. Math. **152** (2003).

[CN85] César Camacho and Alcides Lins Neto, *Geometric Theory of Foliations*, Birkhäuser Boston, Boston, MA, 1985.

[Con89] John Cantwell; Lawrence Conlon, *Leafwise hyperbolicity of proper foliations*, Commentarii Mathematici Helvetici **64** (1989).

[Dur72] Alan H. Durfee, *Foliations of Odd-Dimensional Spheres*, Annals of Mathematics **96** (1972), no. 2, pp. 407–411.

[Fen02] S. R. Fenley, *Foliations, topology and geometry of 3-manifolds: R-covered foliations and transverse pseudo-Anosov flows*, Commentarii Mathematici Helvetici **77** (2002), no. 3, 415–490.

[Gab83] David Gabai, *Foliations and the topology of 3-manifolds*, Journal of Differential Geometry **18** (1983), no. 3, 445–503.

[Gab87a] ———, *Foliations and the topology of* 3*-manifolds. II*, J. Differential Geom. **26** (1987), no. 3, 461–478. MR 910017 (89a:57014a)

[Gab87b] ———, *Foliations and the topology of* 3*-manifolds. III*, J. Differential Geom. **26** (1987), no. 3, 479–536. MR 910018 (89a:57014b)

[Juh10] Andras Juhasz, *Problems in Sutured Floer Homology*, 2010.

[Law71] H. Blaine Lawson, *Codimension-one foliations of spheres*, Bulletin of the American Mathematical Society **77** (1971), no. 3, 437–438.

[Lic62] W. B. R. Lickorish, *A Representation of Orientable Combinatorial 3-Manifolds*, Annals of Mathematics **76** (1962), no. 3, pp. 531–540.

[Lic65] ______, *A Foliation for 3-Manifolds*, Annals of Mathematics **82** (1965), no. 3, pp. 414–420.

[Nov65] S. P. Novikov, *Topology of Foliations*, Trans. Moscow Math. Soc. (1965), pp. 268–304.

[Pal78] Carlos Frederico Borges Palmeira, *Open Manifolds Foliated by Planes*, Annals of Mathematics **107** (1978), no. 1, pp. 109–131.

[Ros68] Harold Rosenberg, *Foliations by planes*, Topology **7** (1968), no. 2, 131–138.

[Rou70] Franois Laudenbach; Robert Roussarie, *Un exemple de feuilletage sur* S^3, Topology **9** (1970).

[RRS03] J. Shareshian R. Roberts and M. Stein, *Infinitely Many Hyperbolic 3-Manifolds Which Contain no Reebless Foliation*, Journal of the American Mathematical Society **16** (2003).

[Sch90] Martin Scharlemann, *Lectures on the theory of sutured manifolds*, Proceedings of Kaist Mathematics Workshop (S.H. Bae and G.T. Jin, eds.), Korea Advanced Institute of Science and Technology, Mathematics Research Center, 1990, pp. 25–46.

[Tam72] Itiro Tamura, *Every Odd Dimensional Homotopy Sphere has a Foliation of Codimension One*, Commentarii mathematici Helvetici **47** (1972), 164–170.

[Thu76] W. P. Thurston, *Existence of Codimension-One Foliations*, Annals of Mathematics **104** (1976), no. 2, pp. 249–268.

[Thu86] William P. Thurston, *A norm for the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–vi and 99–130. MR 823443 (88h:57014)

[Thu97] William P. Thurston, *Three-manifolds, Foliations and Circles, I*, 1997.

[Thu98] William P. Thurston, *Three-manifolds, Foliations and Circles, II*, Unfinished Manuscript, 1998.

[Wal60] A.H. Wallace, *Modifications and Cobounding Manifolds*, Can. J. Math. **12** (1960), pp. 503–528.

[Woo69] John W. Wood, *Foliations on 3-Manifolds*, Annals of Mathematics **89** (1969), no. 2, pp. 336–358.