Topics in Complex & Hypercomplex Geometry

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1 Introduction

Like many of the topics in modern geometry, complex manifold theory can be thought to have begun with Riemann. Indeed, Riemann's work on what are now known as *Riemann surfaces*—i.e., one-dimensional complex manifolds—dates back to the mid 19th century and indubitably marks the beginning of what is now an active, blossoming field which has found many applications in various areas including mathematics, physics, and other physical sciences.

The purpose of this paper is to give a somewhat self-contained overview of a handful of the generalizations and applications of complex geometry. Length considerations require that the exposition err on the side of breadth rather than depth, and so the goal will be to cover a variety of related topics deeply enough to yield insight but (regrettably) not comprehensively. Ideally, the sources cited throughout will provide the diligent reader a clear enough road map to accompany further study.

It will be assumed, initially, that the reader is familiar with some of the basics of complex geometry, namely of complex manifolds, their properties, some of their characterizing results, etc. Everything needed to catch up with such knowledge (and probably a bit more) can be found in the appendices.

2 Almost-Hypercomplex & Hypercomplex Geometry

In appendix 1.3, results stemming from the existence of a single (almost-)complex structure are discussed. A natural progression, then, is to examine the possibility of having multiple (almost-)complex structures on the tangent space TM of a single manifold M; this question was also hinted at in appendix 1.3. Of course, it's possible for many such structures to exist *without* the inclusion of any additional algebraic structure on TM; inarguably, however, the imposing of an interconnectedness on the (almost-)complex structures by way additional algebraic structure makes for a more interesting mathematical situation. This section examines (some of) that.

2.1 Preliminaries and Basic Results

First, a definition.

Definition 2.1. Let M be a real-differentiable manifold of (real) dimension $4n$, $n \in \mathbb{N}$.

1. An *almost-hypercomplex structure* on M is a triple $(I, J, K = IJ)$ of automorphisms on TM which satisfy the algebraic relations of the imaginary units of the Hamilton quaternions $\mathbb{H}$:
$$I^2 = J^2 = K^2 = -\operatorname{id}_{TM}, \quad IJ = K = -IJ, \quad IK = -J = -KI, \quad JK = I = -KJ.$$
2. A *hypercomplex structure* on M is an almost-complex structure (I, J, K) on M for which each of I, J, and K is integrable in the sense of definition A1.4 and/or theorem A1.8.

Remark. As mentioned in appendix 1.3, the Nijenhuis Tensor condition (see definition A1.7 and the last condition of theorem A1.8) is often the easiest-used test for integrability of an almost-complex structure.

Due to the enormous overlap of the study of (almost-)hypercomplex manifolds to other areas including algebraic geometry, differential geometry, geometric topology, etc., there are a number of alternative ways to express definition 2.1. For example, the authors of [24] note the existence of the three integrable almost-hypercomplex structures I, J, K on a hypercomplex manifold M^{4n} only after defining such a manifold in terms of the existence of a G-structure, $G = \mathrm{GL}(n, \mathbb{H})$ with $n = (\dim M)/4$, admitting a unique, torsion-free Obata connection and while these definitions are the same in theory, they're often drastically different in terms of the background material necessary. Diligent readers interested in the machinery utilized in [24] are encouraged to consider, e.g., appendix 2.1.

Due to the fact that $\dim M = 4n$ for any hypercomplex manifold M, the simplest and most readily-understood cases of hypercomplex manifolds come in the case $n = 1$. To that end, one has a somewhat-classical result due to Boyer [21]:

Theorem 2.2. Every compact hypercomplex 4-manifold is conformally equivalent to either

(a) a (flat) torus,

(b) a K3 surface with a hyperkähler Yau metric, or

(c) a coordinate quaternionic Hopf surface with its standard conformally flat metric.

In particular, given a compact hypercomplex 4-manifold (M^{4n}, I_M, J_M, K_M), there exists a conformal transformation mapping M to one of the three spaces above, call it N, and mapping (I_M, J_M, K_M) to the (I_N, J_N, K_N) structure on N.

The proof of theorem 2.2—which is both lengthy and dependent upon considerable background material—is in [21]; some particular details can be found in the appendix, particularly in appendix 2.3. In addition to theorem 2.2, [24] also points out that every compact hypercomplex 4-manifold is locally conformally hyperkähler, has Kodaira dimension $-\infty$ or 0, and has second Betti number either 0 or 22; moreover, [24] also points out that of the manifolds listed in theorem 2.2, only the K3 surface is simply-connected. It's probably not a surprise, then, that the existence of a hypercomplex structure on a compact dimension-four manifold is considered to be quite restrictive, both geometrically and topologically [24].

As it turns out, very little progress has been made towards such results for non-compact manifolds in dimension 4 *or* in dimensions $4n$, $n > 1$ [70]. The results which have been discovered thus far are—at best—few and far between, and many have come about because of or in relation to advances in modern physics. In fact, not only is there very little progress towards an overall classification [70], but even providing examples of compact, irreducible hypercomplex manifolds in dimension 8 and higher has been a challenge [27]. The goal of the following sections will be to present (albeit briefly) some of the various results and to attempt a somewhat comprehensive presentation of sources thereon.

2.2 Hyperkähler & Locally Conformally Hyperkähler Manifolds

One natural extension of a hypercomplex manifold consists of the addition of a metric to its tangent space. Assuming certain parameters are satisfied by the aforementioned metric, one may form what's called a hyperkähler manifold, the following definition of which is borrowed from [39]. Recall that a manifold is said to be Kähler assuming it has three structures,—a complex structure, a Riemannian metric, and a symplectic structure—all of which are compatible[1].

Definition 2.3. A *hyperkähler manifold* is a Riemannian manifold (M, g) with three covariant constant orthogonal automorphisms I, J, K of the tangent bundle which satisfy the quaternionic identities $I^2 = J^2 = K^2 = IJK = -1$.

Informally, the definition 2.1 can be summarized as essentially definition 2.3 without the existence of the Riemannian metric g [70]. It's worth noting that Hitchin's definition 2.3 actually says a lot more than it appears to say; in particular, the covariant constancy of the automorphisms I, J, and K imply that the three induced Kähler two-forms (see [39])

$$\omega_1(X, Y) = g(IX, Y) \qquad \omega_2(X, Y) = g(JX, Y) \qquad \omega_3(X, Y) = g(KX, Y)$$

are closed [3] Although hyperkähler manifolds are interesting and well-documented areas of study in their own right (see [39], [42], [70], for example), they also provide some additional machinery which aids in understanding classification results for hypercomplex manifolds of dimension greater than or equal to 8. For example, the (now somewhat-outdated) result from [17],[24]:

Proposition 2.4. A compact 4-manifold is hyperkähler if and only if it is a K3 surface (see theorem 2.2 above). Moreover, a compact hyperkähler manifold M^{4n}, $n > 1$, is deformation equivalent to either

(a) the Hilbert scheme of points on a K3 surface, or

(b) a generalized Kummer variety.

Notice that both of these examples are simply-connected.

The "outdated" note preceding proposition 2.4 is a reference to the 1997 paper [42] which demonstrates that any Kähler manifold admitting an everywhere non-degenerate two-form ω which is birationally equivalent[2]. Worth noting, too, is that the hyperkähler manifolds

[1] One way to summarize the Kählerian compatibility of structures on a Riemannian manifold (M, g) with almost-complex structure I is to say that for each $x \in M$, the induced scalar product g_x on T_xM satisfies $g_x(u, v) = g_x(I(u), I(v))$ for all $u, v \in T_xM$ *and* that the so-called "fundamental $(1, 1)$-form" $\omega(*, *) = g(I(*), *)$ is closed with respect to exterior differentiation d [41]. Note that ω is non-degenerate by definition and that $g(*, *) = \omega(*, I(*))$, implying compatibility.

[2] According to [53], smooth compact manifolds M and N are *birationally equivalent* if there exists a sequence of smooth compact manifolds $M = M_0, M_1, \ldots, M_n = N$ such that, for all $j = 1, 2, \ldots, n$, either M_{j-1} is the result of blowup of M_j along a proper submanifold of M_j or M_j is the result of a blowup of M_{j-1} along a proper submanifold of M_{j-1}. to a hyperkähler is itself hyperkähler.

are considered (see [24]) to be among the simplest examples of hypercomplex manifolds in dimensions greater than 4.

Authors of [24] go on to say that excluding manifolds which are hyperkähler, the simples examples of hypercomplex manifolds are those which are locally conformally hyperkähler. Moving forward, the term *hyperhermitian* will be used for a hypercomplex manifold (M^{4n}, I, J, K) equipped with a Riemannian metric g which is Hermitian with respect to each of I, J, and K[3]. Now, a definition from [71].

Definition 2.5. A hyperhermitian manifold (M, I, J, K, g) is *locally conformally hyperkähler* provided that M admits a closed form θ and that (M, g, ∇, θ) is a closed Weyl manifold. Here, ∇ is the Obata connection on M (see section 2.1).

And, with this particular definition established and in-hand, the following result from [24] may now be stated.

Proposition 2.6. All hypercomplex manifolds which are locally conformally hyperkähler (but not hyperkähler but) are generalized Hopf manifolds admitting a natural one-dimensional foliation $\mathcal{F}$. Moreover, when the leaves of $\mathcal{F}$ are compact, $\mathcal{F}$ has a compact 3-Sasakian orbifold as its space of leaves.

Remark.

1. Information on Weyl manifolds and the Obata connection can be found in appendix 2.1, definitions B2.3 and B2.5, respectively. The terminology used in definitions B2.3 and 2.5 is largely classical; as such, one will find that both the collection of literature and the variation of notation is expansive. For other perspectives, the reader is encouraged to consult sources cited in [71], as well as [35], [72], and the sources cited therein.

2. Like definition 2.5, the statement of proposition 2.6 relies on a large amount of outside material, some classical, some not. Information regarding generalized Hopf manifolds can be found in appendix 2.1, specifically definition B2.4. A casual, (hopefully-)inviting discussion about foliation-theory and its terminology lives in appendix 2.2, as does a very brief interlude on orbifolds. Finally, note that the topic of 3-Sasakian structures (and hence, the definition thereof) is the focus of section 2.3 below.

In the realm of locally conformally hyperkähler spaces[4], much work has been done in recent years. Indeed, many authors (see, e.g., [57], [25], [29]) have been investigating the possibility of hypercomplex structures on such manifolds, doing so in what's essentially two completely separate cases. To that end, consider the following definition, borrowed here (and adapted slightly) from [73].

[3]In particular, then, every hyperkähler manifold is hyperhermitian but that the converse fails for structures I, J, K which *aren't* covariant constant.

[4]From this point forward, a space discussed with the unmodified descriptor "locally conformally hyperkähler" will be assumed to mean locally conformally hyperkähler *and not hyperkähler.*

Definition 2.7. A hypercomplex manifold M is called *homogeneous* if, given any two points of M, there exists an analytic homeomorphism of M carrying one point to the other. A space which is not homogeneous is called *inhomogeneous*.

Of the above-mentioned sources, [57] is responsible for a complete classification of homogeneous locally conformally hyperkähler manifolds by way of proposition 2.6 combined with a complete classification of homogeneous 3-Sasakian manifolds (see section 2.3). Contrarily, while no complete classification of inhomogeneous locally conformally hyperkähler manifolds exists presently, further applications of 3-Sasakian techniques by [25], [29] has exhibited a large family of examples thereof.

At the core of the paragraph above is the overwhelming indication that the current level of developed machinery has been exhausted. As such, it's time to transition into the next section, where details of and results pertaining to 3-Sasakian structures are discussed.

2.3 3-Sasakian Spaces

The purpose of this section, ultimately, is to continue the above program and to present some of the classification results for hypercomplex manifolds in (real-)dimension greater than four. As the conclusion of the previous section indicates, doing so without at least a brief foray into 3-Sasakian geometry is an impossibility, and as it turns out, discussions of Sasakian structures are essentially rooted in the area of *contact geometry*.

The theory of contact geometry has become extraordinarily vast over the past fifty years, and so an even partially self-contained discussion thereof far exceeds the scope of this paper. For this particular section, only the definitions upon which fundamental material hinges will be mentioned; in later sections (appendix 2.2, in particular), the topic will be discussed a bit more satisfactorily, particularly in comparison and contrast with notions in foliation theory.

To begin, let (M, g) be a Riemannian manifold and define the so-called *Riemannian cone* $\mathcal{C}(M)$ to be the product $M \times \mathbb{R}^+$ of M with the upper half-line $\mathbb{R}^+$. This Riemannian cone has a naturally induced *cone metric* $t^2 g + dt^2$, $t \in \mathbb{R}^+$. Suppose further that M has defined on it a 1-form θ. In this scenario, the data (M, g, θ) is said to be a *contact manifold* if the induced 2-form

$$t^2 d\theta + 2tdt \cdot \theta \tag{2.3.1}$$

defined on the Riemannian cone $\mathcal{C}(M)$ of M is symplectic (i.e., is skew-symmetric, totally isotropic, and non-degenerate). With these definitions in hand, one can now define the following.

Definition 2.8. A contact Riemannian manifold (M, g, θ) is *Sasakian* if $\mathcal{C}(M)$ with the cone metric is Kähler with Kähler form (2.3.1). In this case, the metric cone $\mathcal{C}(M)$ is sometimes called the *Kähler cone*.

As the section title indicates, specific attention will be given to manifolds (and orbifolds) which are defined to be *3-Sasakian*. What exactly does that mean? Before answering that question directly, it's worthwhile to express the Sasakian property à la definition 2.8 in terms

of an actual *Sasakian structure*, whereby the notion 3-Sasakian will refer to a space (manifold or orbifold) admitting three such structures. Consider the following definitions, all borrowed from [26].

Definition 2.9. Let (M, g) be a Riemannian manifold and let ∇ denote the Levi-Civita connection of g. Then (M, g) has a Sasakian structure if there exists a Killing vector field ξ of unit length on M so that the tensor field $\Phi = \nabla\xi$ of type $(1, 1)$ satisfies the condition

$$(\nabla_X \Phi)(Y) = \eta(Y)X - g(X, Y)\xi$$

for η satisfying (i) $g(Y, \xi) = \eta(Y)$ and (ii) $(\nabla_X \eta)(Y) = g(Y, \Phi X)$ and for any pair of vector fields $X, Y \in \mathfrak{X}(M)$. The triple (Φ, ξ, η) is called the *Sasakian structure* on (M, g).

Definition 2.10. Let (M, g) be a Riemannian manifold that admits three distinct Sasakian structures $\{\Phi^\alpha, \xi^\alpha, \eta^\alpha\}_{\alpha=1,2,3}$ such that $g(\xi^\alpha, \xi^\beta) = \delta_{\alpha\beta}$ and $[\xi^\alpha, \xi^\beta] = 2\varepsilon_{\alpha\beta\gamma}\xi^\gamma$ for $\alpha, \beta, \gamma = 1, 2, 3$. Then (M, g) is called a *3-Sasakian Manifold with Sasakian 3-structure* (M, g, ξ^α).

Remarks.

1. In [62], it's noted that Sasakian geometry is the "odd-dimensional cousin of Kähler geometry" because (i) the spaces (manifolds or orbifolds) in question are always odd--dimensional—real-dimension $2\dim_{\mathbb{C}} \mathcal{C}(M) - 1$ for general Sasakian spaces M [62] and real-dimension $4k + 3$, $k \in \mathbb{N}$, for 3-Sasakian spaces [26]—and (ii) Sasakian geometry is the natural intersection of CR, contact, and Riemannian geometries, whereas Kähler geometry is the natural intersection of complex, symplectic, and Riemannian geometries.

2. The world of Sasakian geometry and topology is now a vast one. In addition to [62] and [26], the interested reader is referred to [22], [23], and the sources cited therein.

Upon having entered the realm of Sasakian geometry, one of the partial classifications hinted at in section 2.2 can be stated. For completeness, recall proposition 2.6, originally from [24] and mentioned in section 2.2.

Proposition 2.6. All hypercomplex manifolds which are locally conformally hyperkähler (but not hyperkähler) are generalized Höpf manifolds admitting a natural one-dimensional foliation $\mathcal{F}$. Moreover, when the leaves of $\mathcal{F}$ are compact, $\mathcal{F}$ has a compact 3-Sasakian orbifold as its space of leaves.

Also, before proceeding, consider the following classification result from [26] regarding homogeneous 3-Sasakian structures.

Theorem 2.11. Let (M, g, ξ^α) be a 3-Sasakian manifold with a transitive action of the group of automorphisms of the Sasakian 3-structure, i.e. let M be homogeneous and 3-Sasakian. Then M is precisely one of the following homogeneous spaces:

$$\begin{gathered}\frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)} \simeq S^{4n-1}, \quad \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)\times\mathbb{Z}_2} \simeq \mathbb{RP}^{4n-1}, \quad \frac{\mathrm{SU}(m)}{\mathrm{S}(\mathrm{U}(m-2)\times\mathrm{U}(1))}, \\ \frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4)\times\mathrm{Sp}(1)}, \quad \frac{\mathrm{G}_2}{\mathrm{Sp}(1)}, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)}, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)}, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)}, \quad \frac{\mathrm{E}_8}{\mathrm{E}_7}.\end{gathered} \tag{2.3.2}$$

In the above notation, $n \geq 1$, $\mathrm{Sp}(0) = \{1\}$, $m \geq 3$, and $k \geq 7$.

Remark. Though strictly beyond the scope of this paper, it's worth noting that some additional conclusions to theorem 2.11 are examined by [26]. Earlier in [26], the authors show that such an M is a principal G-bundle over a so-called *Wolf space* $\mathcal{W}$, $G \in \{\mathrm{Sp}(1), \mathrm{SO}(3)\}$, and as part of theorem 2.11, they conclude that the fiber F is $\mathrm{Sp}(1)$ if and only if $M = S^{4n-1}$ and that $F = \mathrm{SO}(3)$ for all other listed M.

The purpose of presenting theorem 2.11 regarding classifications of 3-Sasakian homogeneous manifolds is to motivate two of the classification results regarding higher-dimensional hypercomplex structures as presented in [57]. Note that (the here-repeated) proposition 2.6 is one of the key steps in going from theorem 2.11 to propositions 2.12, 2.13 below. In particular:

Proposition 2.12. The class of complex locally conformal hyperkähler homogeneous manifolds coincides with that of flat principal S^1-bundles over one of the 3-Sasakian homogeneous manifolds in theorem 2.11, equation (2.3.2).

Proposition 2.13. Let M be a compact locally conformal hyperkähler homogeneous manifold. Then M is one of the following:

(i) The Möbius band, i.e. the unique nontrivial princpal S^1-bundle over $\mathbb{RP}^{4n-1}$.

(ii) A product $(G/H) \times S^1$, where G/H is one of the 3-Sasakian homogeneous manifolds in theorem 2.11, equation (2.3.2).

Example 2.14. The result of proposition 2.13 above gives an explicit list of all compact locally conformal hyperkähler homogeneous 8-manifolds:

(i) $S^7 \times S^1$

(ii) $\mathbb{RP}^7 \times S^1$

(iii) $\{\mathrm{SU}(3)/S(\mathrm{U}(1) \times \mathrm{U}(1))\} \times S^1$

(iv) The Möbius band over $\mathbb{RP}^7$.

What's more, note that the first exceptional example appears in real-dimension 12 in the form of the bundle $\{\mathrm{G}^2 / \mathrm{Sp}(1)\} \times S^1$. The base of this bundle is (though not obviously) diffeomorphic to the Stiefel manifold $V_2(\mathbb{R}^7)$, a better understanding of which will (hopefully) lie at the end of section 2.4 below.

With the presentation of propositions 2.12 and 2.13, the subclass of homogeneous locally conformal hyperkähler manifolds—examination of which began in section 2.2 above—has been fully classified. This is a major facet of the classification of all hypercomplex manifolds inasmuch as this classification exists currently. Of course, the elephant in the room at this point is that nothing has been said about locally conformal hyperkähler manifolds which

are *inhomogeneous*, nor about hypercomplex manifolds which are neither hyperkähler nor locally conformally hyperkähler. This, of course, raises a number of questions: Do such things exist? Have they been classified? What's known, and how can that be elaborated using elementary techniques?

Part of the goal moving forward will be to answer—or at least, to address—these questions. The upshot is that experts in the field have spent considerable time over the last two decades working in various areas of geometry, constructing and discovering new links to hypercomplex structures and their properties. Despite this, however, the literature is largely sparse and incomplete on these other two categories. The authors of [24] have done considerable work constructing "large families of inhomogeneous hypercomplex manifolds" constructed using various techniques (including but not limited to 3-Sasakian techniques), though no far-sweeping classification has been derived. Moreover, the case when M is not locally conformally hyperkähler—despite being "the most intriguing" [24]—is no less incomplete than the inhomogeneous case.

Suffice it to say that a considerable amount of additional machinery will be needed to gain any worthwhile understanding of the remaining results. As a decent *next logical step*, consider the Stiefel manifold.

2.4 Stiefel Manifolds

Before winding too far ahead, consider first the definition.

Definition 2.15. The *Stiefel manifold* $V_{n,k} = V_k(\mathbb{K}^n)$ is the manifold whose points are all orthonormal k-tuples of vectors in $\mathbb{K}^n$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Such k-tuples are called *orthonormal k-frames*.

For preciseness, note that definition 2.15 is often called the *compact* Stiefel manifold, while it's sometimes helpful to consider the *non-compact* Stiefel manifold consisting of all *linearly independent* k-frames in $\mathbb{K}^n$. The difference is relatively trivial due to the fact that Gram-Schmidt spells out a well-defined sequential algorithm for deformation retracting the non-compact version to its compact counterpart; historically the notation $V^*_{n,k}$ is sometimes used for the non-compact Stiefel manifold when necessary.

It's not immediately obvious that the above definition yields a manifold. However, rewriting k-tuples as column vectors allows an arbitrary Stiefel manifold element $\boldsymbol{y}$ to be considered as a matrix Y in the collection $M_{n\times k}(\mathbb{K})$ of $(n\times k)$-matrices over $\mathbb{K}$ [1]. From this perspective, one can rewrite $V_{n,k}(\mathbb{K})$ as[5]

$$V_{n,k}(\mathbb{K}) = \{X \in M_{n\times k}(\mathbb{K}) \,:\, X^\dagger X = \mathrm{id}_k\}, \tag{2.4.1}$$

whereby the collection $V_{n,k}(\mathbb{K})$ has a manifold structure given by the subspace topology

[5] In (2.4.1), $X^\dagger$ denotes the conjugate transpose of a matrix X and id_k denotes the $k \times k$ identity matrix.

inherited from $M_{n\times k}(\mathbb{K})$. One can show ([1], [67]) that

$$\begin{aligned} \dim V_{n,k}(\mathbb{R}) &= nk - \frac{1}{2}k(k+1) \\ \dim V_{n,k}(\mathbb{C}) &= 2nk - k^2 \\ \dim V_{n,k}(\mathbb{H}) &= 4nk - k(2k-1) \end{aligned} .$$

Even less obvious is the fact that such an object would be the source for interesting geometry. This issue is more pressing to the intent of the paper, so before transitioning into hypercomplex results, a bit of background on the Stiefel manifold will be given for motivation.

2.4.1 Background, Preliminaries, and Some Geometry

Viewing $\mathbb{K}$ as $\mathbb{R}^{2m}$, $m = \frac{1}{2}, 1, 2$, allows one to realize the Stiefel manifold $V_{n,k}(\mathbb{R}^{2m})$ as a subset of k copies of S^{2m-1} where here, S^{2m-1} is the boundary of the unit ball $B^{2m} \subset \mathbb{R}^{2m}$ of vectors $x \in \mathbb{R}^{2m}$ having Euclidean norm $\|x\|$ less than or equal to 1. From this perspective, the topology on $V_{n,k}$ is the subspace topology of the k-fold product of S^{2m-1} and in particular, elements of $V_{n,k}$ correspond to norm-preserving linear transformations from $\mathbb{K}^k \cong \mathbb{R}^{2mk}$ into $\mathbb{K}^n \cong \mathbb{R}^{2mn}$ [43].

Unsurprisingly, this perspective yields a strong geometric framework upon which to study Stiefel manifolds which in turn has obvious connections to Lie groups and Lie theory. As pointed out in [38], the case of $\mathbb{K} = \mathbb{R}$ yields a natural projection $\pi : \mathrm{O}(n) \twoheadrightarrow V_{n,k}$ sending each orthogonal $n \times n$ matrix A to the k-frame consisting of the last k columns of A (that is, to images under A of the last k standard basis vectors in $\mathbb{R}^n$). Because π is surjective and because the preimages of points $\boldsymbol{y} \in V_{n,k}$ are the cosets[6] $A \cdot \mathrm{O}(n-k)$, $V_{n,k}(\mathbb{R})$ can be viewed as the quotient $\mathrm{O}(n)/\mathrm{O}(n-k)$ which thus inherits the quotient topology[7]. Taking this construction one step farther, one can prove that $k \lneq$ yields surjective projection $\mathrm{SO}(n) \twoheadrightarrow V_{k,n}$ and thus that $V_{k,n} \cong \mathrm{SO}(n)/\mathrm{SO}(n-k)$ [38]; moreover, taking the above construction farther still yields surjective projections $G \twoheadrightarrow V_{n,k}(\mathbb{K})$ with respect to various Lie groups G where here,

$$\begin{array}{lll} k \leq n, & \mathbb{K} = \mathbb{C} & \Longrightarrow \quad G = \mathrm{U}(n) \\ k \lneqq n, & \mathbb{K} = \mathbb{C} & \Longrightarrow \quad G = \mathrm{SU}(n) \\ k \leq n, & \mathbb{K} = \mathbb{H} & \Longrightarrow \quad G = \mathrm{Sp}(n) = \mathrm{U}(n, \mathbb{H}) \\ k \lneqq n, & \mathbb{K} = \mathbb{H} & \Longrightarrow \quad G = \mathrm{SU}(n, \mathbb{H}) \end{array} . \tag{2.4.2}$$

The appropriate quotients can then be formed[8].

The author of [38] further points out that one can also reach the geometry of the Stiefel manifolds by way of fiber bundles over Grassmannians. Indeed, one can verify that the projection maps $V_{n,k}(\mathbb{K}) \twoheadrightarrow G_k(\mathbb{K}^n)$ are actually fiber bundles by using Gram-Schmidt to

[6] Here, $\mathrm{O}(n-k)$ is embedded in $\mathrm{O}(n)$ as the orthogonal transformations of the first $n-k$ coordinates of $\mathbb{R}^n$.

[7] One can check that the topology here is the same as the previously-defined topology.

[8] One must, of course, pay special attention to left- versus right-cosets in the case of $\mathbb{K} = \mathbb{H}$.

obtain continuously varying orthonormal bases for all n-planes P' in a neighborhood of some fixed n-plane $P \in G_k(\mathbb{K}^n)$ and by identifying k-frames in these n-planes with k-frames in $\mathbb{K}^n$ [38]. Upon completing (and making rigorous) this process, one thereby derives a collection for all $k = \mathbb{N} \cup \{\infty\}$ of sequences a lá (2.4.2):

$$\begin{array}{ccccc} \mathrm{O}(n) & \longrightarrow & V_{n,k}(\mathbb{R}) & \longrightarrow & G_k(\mathbb{R}^n) \\ \mathrm{U}(n) & \longrightarrow & V_{n,k}(\mathbb{C}) & \longrightarrow & G_k(\mathbb{C}^n) \\ \mathrm{Sp}(n) & \longrightarrow & V_{n,k}(\mathbb{H}) & \longrightarrow & G_k(\mathbb{H}^n) \end{array} . \tag{2.4.3}$$

The case $k = \infty$ is merely a technicality, though its truth is obtained by realizing $V_{n,\infty}(\mathbb{K})$, respectively $G_\infty(\mathbb{K}^n)$, as unions

$$V_{n,\infty}(\mathbb{K}) = \bigcup_k V_{n,k}(\mathbb{K}), \text{ respectively } G_\infty(\mathbb{K}^n) = \bigcup_k G_k(\mathbb{K}^n).$$

Further geometrical and topological aspects of Stiefel manifolds can be found in various sources, most notably [38], [43], and sources cited therein. Most of those details will do little to advance the reader's understanding of results central to the current paper, so most will be omitted. The lone exceptions are the following examples—taken from [38]—which have far-reaching consequences broad enough to justify their inclusion. In the sections that follow, a brief detour will be taken for the sake of mentioning a novel relationship between Stiefel manifolds and Clifford algebras before moving on to the most pertinent results.

Examples 2.16.

1. For $j < k \leq n$, there are fiber bundles

$$V_{k-j}(\mathbb{K}^{n-j}) \longrightarrow V_k(\mathbb{K}^n) \xrightarrow{p} V_j(\mathbb{K}^n) \tag{2.4.4}$$

   where the projection p sends a k-frame onto the j-frame formed by its first j vectors. In particular, the fiber consists of $(k-j)$-frames in the $(n-j)$-plane orthogonal to a given j-frame. Details for constructing the associated local trivializations can be found in [38], as well as special cases of (2.4.4) for specific values of j.

2. The bundle (2.4.4) yields well known bundles when $j = 1$ and $k = n$. Basic substitution yield for $j = 1$ a bundle

$$V_{k-1}(\mathbb{K}^{n-1}) \longrightarrow V_k(\mathbb{K}^n) \xrightarrow{p} V_1(\mathbb{K}^n), \tag{2.4.5}$$

   where $V_1(\mathbb{K}^n)$ can be shown isomorphic to S^{n-1}. Moreover, $k = n$ transforms (2.4.5) into the following bundles corresponding to the three options for $\mathbb{K}$:

$$\begin{array}{lccccc} \mathbb{K} = \mathbb{R}: & \mathrm{O}(n-1) & \longrightarrow & \mathrm{O}(n) & \xrightarrow{p} & S^{n-1} \\ \mathbb{K} = \mathbb{C}: & \mathrm{U}(n-1) & \longrightarrow & \mathrm{U}(n) & \xrightarrow{p} & S^{2n-1} \\ \mathbb{K} = \mathbb{H}: & \mathrm{Sp}(n-1) & \longrightarrow & \mathrm{Sp}(n) & \xrightarrow{p} & S^{4n-1} \end{array} . \tag{2.4.6}$$

   In all cases of (2.4.6), the map p is evaluation of a transformation—an orthogonal, unitary, or symplectic transformation, respectively—on a fixed unit vector. Studying the homotopy groups associated to the groups in the bundle sequences of (2.4.6) yields the so-called Bott Periodicity Theorem, details of which can be found in, e.g., [38].

2.4.2 A Brief Detour for the Sake of Clifford

As shown in equation (2.4.5), $V_{n,k}$ naturally fibers over S^{n-1} by taking one vector—the last, say—from each k-frame [43]. Because of this, it's natural to define some sort of inverse relation.

Definition 2.17. A *cross-section* $f : S^{n-1} \to V_{n,k}$ is a map which associates with each point $v \in S^{n-1}$ an orthonormal k-frame $(v_1, \ldots, v_{k-1}, v)$. Here, $(v_1, \ldots, v_{k-1}) \stackrel{\text{def}}{=} g(v)$ is an orthonormal $(k-1)$-frame, and the map $g : S^{n-1} \to V_{n,k-1}$ is called the *projection* of f.x

One can regard $g(v)$ as a $(k-1)$-frame of tangents to S^{n-1} at the point v, and so cross-sections of $V_{n,k}$ over S^{n-1} is equivalent to a so-called $(k-1)$-field, that is, a field of orthonormal tangent $(k-1)$-frames. In particular, one can show that any such $(k-1)$-field spans a field of tangent $(k-1)$-planes, though as [43] points out, not every field of tangent $(k-1)$-planes can be spanned by a $(k-1)$-field. The natural question one can ask, then, is: For what values of n and k does $V_{n,k}$ admit a cross-section over S^{n-1}? Surprisingly, the answer lies in the study of (Euclidean) Clifford algebras.

For $n = 0, 1, 2, \ldots$, let $\mathcal{C}\ell_{0,n}$ denote the real Euclidean Clifford algebra of dimension 2^n generating by a collection of anticommuting basis elements $e_1, e_2, \ldots, e_n$ which all square to $e_i^2 = -1$, $i = 1, 2, \ldots, n$. Following this convention, the first few such Clifford algebras are

$$\mathbb{R}, \quad \mathbb{C}, \quad \mathbb{H}, \quad \mathbb{H} \oplus \mathbb{H}, \quad \mathbb{H}(2), \quad \mathbb{C}(4), \quad \mathbb{R}(8), \quad \mathbb{R}(8) \oplus \mathbb{R}(8), \ldots, \tag{2.4.7}$$

where here, given an algebra A and a positive integer $q \in \mathbb{Z}^+$, $A(q)$ denotes the collection of $q \times q$ matrices with elements in A. Moreover, the classification of real Clifford algebras along with Bott periodicity shows that $\mathcal{C}\ell_{0,n} \cong \mathcal{C}\ell_{0,q} \otimes_{\mathbb{R}} \mathbb{R}(16p)$ where $n = 8p + q$, $q = 0, 1, 2, \ldots, 7$; one can also confirm that $\mathcal{C}\ell_{0,n}(16) \cong \mathcal{C}\ell_{0,n+8}$, whereby it follows that all Euclidean Clifford algebras can be expressed in terms of matrix algebras over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

For a given value k, the goal will be to construct cross-sections of $V_{n,k}$ for certain values n. To that end, let $\sigma(k)$ denote the number of integers s in the range $0 < s < k$ satisfying $s \equiv c \mod 8$, $c \in \{0, 1, 2, 4\}$. As shown in [43], $\mathbb{R}^n$ can be represented as a $\mathcal{C}\ell_{0,k-1}$-module whenever $n \equiv 0 \mod a_k$ where $a_k = 2^{\sigma(k)}$. Given such a representation, one can orthogonalize so that the generators $e_1, \ldots, e_{k-1}$ of $\mathcal{C}\ell_{0,k-1}$ correspond to orthogonal transformations; in this case, a cross-section $f : S^{n-1} \to V_{n,k}$ can be given by

$$f : v \mapsto (e_1 \cdot v, \ldots, e_{k-1} \cdot v, v) \text{ for } v \in S^{n-1}.$$

Cross-sections of this type are known as *Clifford cross-sections* and have played significant roles in various results Eckmann, Hurwitz, and Radon [43].

In addition to the above-stated results, Clifford cross-sections are a fundamental tool in Adams' proof that there exist no more than $\rho(n) - 1$ linearly independent vector fields on S^{n-1} where here, $\rho(n) = 2^c + 8d$ and $n = (2a+1)2^{c+4d}$, $0 \leq c \leq 3$ [2]. For the time being, however, this result of Adams is phrased in terms of Stiefel manifolds as follows and is succeeded by a somewhat illustrative example.

Theorem 2.18. The Stiefel manifold $V_{n,k}$ admits a cross-section over S^{n-1} if and only if $n \equiv 0 \mod a_k$ for a_k as above.

The aforementioned example, given below, comes from [43]. Indeed, notice that if

$$(n, k) = (16, 9),$$

then $a^k = 2^{\sigma(k)} = 2^4 = 16$ and so $n \equiv 0 \mod a_k$. Assuming the result of theorem 2.18 is correct, one would thereby expect that $V_{16,9}$ admits a cross-section over S^{15}, and not only is this true, but the example itself can be written down explicitly.

$\boldsymbol{v_1}$	$\boldsymbol{v_2}$	$\boldsymbol{v_3}$	$\boldsymbol{v_4}$	$\boldsymbol{v_5}$	$\boldsymbol{v_6}$	$\boldsymbol{v_7}$	$\boldsymbol{v_8}$	$\boldsymbol{v_9}$
x_8	$-x_7$	$-x_6$	$-x_5$	$-x_4$	$-x_3$	$-x_2$	$-x_1$	x_0
$-x_9$	x_6	$-x_7$	$-x_4$	x_5	$-x_2$	x_3	x_0	x_1
$-x_{10}$	$-x_5$	$-x_4$	x_7	x_6	x_1	x_0	$-x_3$	x_2
$-x_{11}$	$-x_4$	x_5	$-x_6$	x_7	x_0	$-x_1$	x_2	x_3
$-x_{12}$	x_3	x_2	x_1	x_0	$-x_7$	$-x_6$	$-x_5$	x_4
$-x_{13}$	x_2	$-x_3$	x_0	$-x_1$	x_6	$-x_7$	x_4	x_5
$-x_{14}$	$-x_1$	x_0	x_3	$-x_2$	$-x_5$	x_4	x_7	x_6
$-x_{15}$	x_0	x_1	$-x_2$	$-x_3$	x_4	x_5	$-x_6$	x_7
$-x_0$	$-x_{15}$	$-x_{14}$	$-x_{13}$	$-x_{12}$	$-x_{11}$	$-x_{10}$	$-x_9$	x_8
x_1	$-x_{14}$	x_{15}	x_{12}	$-x_{13}$	x_{10}	$-x_{11}$	x_8	x_9
x_2	x_{13}	x_{12}	$-x_{15}$	$-x_{14}$	$-x_9$	x_8	x_{11}	x_{10}
x_3	x_{12}	$-x_{13}$	x_{14}	$-x_{15}$	x_8	x_9	$-x_{10}$	x_{11}
x_4	$-x_{11}$	$-x_{10}$	$-x_9$	x_8	x_{15}	x_{14}	x_{13}	x_{12}
x_5	$-x_{10}$	x_{11}	x_8	x_9	$-x_{14}$	x_{15}	$-x_{12}$	x_{13}
x_6	x_9	x_8	$-x_{11}$	x_{10}	x_{13}	$-x_{12}$	$-x_{15}$	x_{14}
x_7	x_8	$-x_9$	x_{10}	x_{11}	$-x_{12}$	$-x_{13}$	x_{14}	x_{15}

Figure 1

The first eight column vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_8$ are tangent to S^{15} at the point given by the ninth column vector $\boldsymbol{v_9}$

2.4.3 Hypercomplex Structures on Stiefel Manifolds

Many of the results addressed in the beginning of this section will come from [27], and so of particular interest here will be the Stiefel manifolds $V_{n,k}(\mathbb{C}) = V_k(\mathbb{C}^n)$. Unless otherwise noted, then, $V_{n,k}$ will be used in this section to denote $V_{n,k}(\mathbb{C})$ and throughout, the detail-oriented reader is encouraged to notice the prevalence of the $k = 2$ case especially. In the latter parts of the section, results from [24] will be presented, including an adapted quotient procedure given in [46], adapted from [40] and [36]. These results are particularly notation-heavy and machinery-dependent, whereby it follows that a number of definitions and intermediate results will be necessary. Here, the objective will be to err on the side of clarity rather than brevity.

Recall from section 2.4.1 that $V_{n,2} = V_{n,2}(\mathbb{C})$ can be realized both as the quotient $\mathrm{U}(n)/\,\mathrm{U}(n-2)$ (see also [27]) and as an embedded submanifold

$$V_{n,2} \subset S^{4n-1} \subset \mathbb{H}^n$$

as well. As it turns out, however, the most interesting results on $V_{n,2}$ from the standpoint of hypercomplex geometry stem from a third representation—one that likens this particular Stiefel manifold to the 3-Sasakian structures mentioned (albeit briefly) in section 2.3. This is the avenue that will be pursued first, and the construction henceforth largely mimics that in [27].

First, consider an arbitrary 3-Sasakian manifold $(\mathcal{S}, g_{\mathcal{S}}, \xi^\alpha)$, $\alpha = 1, 2, 3$, with nontrivial group $I_0(\mathcal{S}, g_{\mathcal{S}})$ of 3-Sasakian isometries, i.e. isometries of $\mathcal{S}$ which leave invariant the associated tensor structures $\{\Phi^\alpha, \xi^\alpha, \eta^\alpha\}$. As shown in section 2.3, $\mathcal{S}$ being Sasakian implies that the Riemannian cone $M \stackrel{\text{def}}{=} \mathcal{S} \times \mathbb{R}^+$ is necessarily hyperkähler with respect to the cone metric (see also [26]). As a result, [40] shows that any (Lie) subgroup $G \subset I_0(\mathcal{S}, g_{\mathcal{S}})$ gives rise to a so-called *hyperkähler moment map* $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ associated to G. When necessary, denote by $\mu_{\mathcal{S}}$ the restriction $\mu|_{\mathcal{S}}$ and denote by $\mu^\alpha_{\mathcal{S}}$, $\alpha = 1, 2, 3$, the components of $\mu_{\mathcal{S}}$ with respect to the standard basis of $\mathbb{R}^3$ (identified with the purely imaginary quaternions).

At this point, it would appear that we've been dredging through abstraction for drudgery's sake. As it turns out, this isn't exactly the case. The next result links the idea of Stiefel manifolds to the seemingly anachronistic 3-Sasakian terminology above, but in order for it to make sense, define $\mathbb{K}^{\mathrm{op}}$ to be $\mathbb{H}$, $\mathbb{C}$, or $\mathbb{R}$ whenever $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively, and let $k = [\mathbb{K} : \mathbb{R}]$.

Theorem 2.19. Under a suitable rescaling, the zero set $N(\mathbb{K}) \stackrel{\text{def}}{=} \mu_{\mathcal{S}}^{-1}(0)$ is precisely the Stiefel manifold $V_{n,k}(\mathbb{K}^{\mathrm{op}})$ where here,

$$V_{n,k}(\mathbb{K}^{\mathrm{op}}) = \{A \in M_{n\times k}(\mathbb{K}^{\mathrm{op}}) \,:\, A^\dagger A = I_k\}$$

a lá (2.4.1) above. In this case, $\iota : N(\mathbb{K}) \hookrightarrow S^{4n-1}$ is a smooth compact submanifold of dimension $4n + 2 - 3k$.

One reason theorem 2.19 is helpful is because it helps connect the Stiefel manifold to the previous notions from 3-Sasakian geometry; another is that it allows for the introduction of 3-Sasakian moment maps. The importance of these maps is far-reaching, and as such, their appearance in the current section is hardly finished.

Moving forward, define for $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$ arbitrary[9] the manifold $\mathcal{N}(\boldsymbol{p})$ to be

$$\mathcal{N}(\boldsymbol{p}) = \mu(\boldsymbol{p})^{-1} \cap S^{4n-1} \subset \mathbb{H}^n \setminus \{0\}$$

where here, μ denotes the specific hyperkähler moment map

$$\mu(\boldsymbol{p})(\boldsymbol{u}^*, \boldsymbol{u}) = \langle \boldsymbol{u}^*, i\boldsymbol{p}\boldsymbol{u} \rangle$$

induced from the standard moment map $\mu(\boldsymbol{p})$ on $\mathbb{H}^n$, $\langle \cdot, \cdot \rangle$ the flat Hermitian inner product on $\mathbb{H}^n$, $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{H}^n$ with (quaternionic) adjoint $\boldsymbol{u}^*$. Next, let

$$\Xi : \mathfrak{t}_n \to \Gamma(T(\mathbb{H}^n \setminus \{0\}))$$

be the map associating to each element of $\mathfrak{t}_n$ the corresponding vector field on $\mathbb{H}^n \setminus \{0\}$ and define the subset $\widetilde{\mathfrak{t}}_n$ of $\mathfrak{t}_n$ (see footnote [9] above) such that

$$\widetilde{\mathfrak{t}}_n = \{\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathfrak{t}_n \,:\, p_i \neq 0 \text{ for all } i = 1, \ldots, n\}.$$

By making further impositions on the vector fields $\Xi(\boldsymbol{p})$, a main result can be stated. First, consider a definition.

Definition 2.20. An almost-hypercomplex structure $\{I^\alpha\}_{\alpha=1}^3$ on a smooth manifold M is said to be Sp(1)-*compatible* if there are a smooth action of Sp(1) and a vector field Ξ on M such that

(i) Ξ is an infinitesimal automorphism of I^α, $\alpha = 1, 2, 3$,

(ii) The vector space V spanned by $\{I^\alpha\}_{\alpha=1}^3$ is the adjoint representation of Sp(1), and

(iii) For all $\alpha, \beta, \gamma = 1, 2, 3$,

$$I^\alpha \xi^\alpha = -\epsilon_{\alpha\beta\gamma}\xi^\gamma + \delta_{\alpha\beta}\Xi$$

where ξ^α are the infinitesimal generators for the Sp(1)-action. Here, δ denotes Kronecker's delta and ϵ denotes the Levi-Civita permutation symbol.

An Sp(1)-compatible hypercomplex structure is denoted by the triple $(I^\alpha, \xi^\alpha, \Xi)$.

Note that a vector field Ξ which is nowhere vanishing on M combines with ξ^1, ξ^2, ξ^3 to span a trivial subbundle $\mathcal{V}_4$ of TM and that these vector fields give rise to a nested series of foliations $\mathcal{F}_1 \subset \mathcal{F}_2^\alpha \subset \mathcal{F}_4$ generated by subbundles $\mathcal{V}_1 = \text{span}\{\Xi\}$, $\mathcal{V}_2^\alpha = \text{span}\{\Xi, \xi^\alpha\}$, and $\mathcal{V}_4$, respectively. In addition, note that each $\boldsymbol{p}$ in $\widetilde{\mathfrak{t}}_n$ generates associated vector fields

[9] Here, $\boldsymbol{p}$ is identified as an element $\boldsymbol{p} \sim \text{diag}(p_1, \ldots, p_n)$ of the Lie algebra $\mathfrak{t}_n$ associated to the maximal torus T^n lying a U(n) subgroup of Sp(n) where Sp(n) is the maximal compact subgroup of GL($n, \mathbb{H}$).

$\Xi(\boldsymbol{p}), \xi^\alpha(\boldsymbol{p})$, $\alpha = 1,2,3$, thereby generating associated subbundles $\mathcal{V}_i(\boldsymbol{p})$, $i = 1,2,4$, and thence a sequence of nested multifoliate structures $\mathcal{F}_1(\boldsymbol{p}) \subset \mathcal{F}_2^\alpha(\boldsymbol{p}) \subset \mathcal{F}_4(\boldsymbol{p})$.

Now that this added machinery is in-place, note that the vector fields $\Xi(\boldsymbol{p})$ associated to elements $\boldsymbol{p}$ from $\widetilde{\mathfrak{t}}_n$ generate a G-action on $\mathcal{N}(\boldsymbol{p})$, $G \in \{S^1, \mathbb{R}\}$, denoted $G(\boldsymbol{p}) \in \{S^1(\boldsymbol{p}), \mathbb{R}(\boldsymbol{p})\}$, respectively. In each case, there is a corresponding chain of subgroups $G_1(\boldsymbol{p}) < G_2^\alpha(\boldsymbol{p}) < G_4(\boldsymbol{p})$ given, respectively, by:

$$\begin{aligned} &G_1(\boldsymbol{p}) = S^1(\boldsymbol{p}), \quad G_2^\alpha(\boldsymbol{p}) = S^1(\boldsymbol{p}) \times S^1_\alpha, \quad G_4(\boldsymbol{p}) = S^1(\boldsymbol{p}) \times \mathrm{SU}(2) \simeq \mathrm{U}(2)(\boldsymbol{p}), \text{ and} \\ &G_1(\boldsymbol{p}) = \mathbb{R}(\boldsymbol{p}), \quad G_2^\alpha(\boldsymbol{p}) = \mathbb{R}(\boldsymbol{p}) \times S^1_\alpha, \quad G_4(\boldsymbol{p}) = \mathbb{R}(\boldsymbol{p}) \times \mathrm{SU}(2)(\boldsymbol{p}). \end{aligned} \tag{2.4.8}$$

In (2.4.8), the superscript α denotes the structure with respect to the vector field ξ^α, $\alpha = 1,2,3$. Note, too, that these nested subgroups $G_i(\boldsymbol{p})$ yield associated foliations $\mathcal{F}_1(\boldsymbol{p}) \subset \mathcal{F}_2^\alpha(\boldsymbol{p}) \subset \mathcal{F}_4(\boldsymbol{p})$ on $\mathcal{N}(\boldsymbol{p})$, and to these one can associate splittings

$$\mathcal{H}(\boldsymbol{p}) \supset \mathcal{H}_2^\alpha(\boldsymbol{p}) \supset \mathcal{H}_4(\boldsymbol{p}) \tag{2.4.9}$$

of $T\mathcal{N}(\boldsymbol{p})$, thought of has horizontal foliations thereof.

As mentioned prior to definition 2.20 above, the goal of introducing such a large collection of machinery was to state some major results which further the classification of hypercomplex structures. Currently, it may not seem like any progress has been made, but there's actually more to the present exposition than meets the eye.

Theorem 2.21. Let $\boldsymbol{p} \in \widetilde{\mathfrak{t}}_n$ and let $\mathcal{I}^\alpha(\boldsymbol{p})$ denote the sections of $\operatorname{End}\mathcal{V}_4(\boldsymbol{p}) \oplus \operatorname{End}\mathcal{H}_4(\boldsymbol{p})$ defined by

(a) On $\mathcal{V}_4(\boldsymbol{p})$, $\mathcal{I}^\alpha(\boldsymbol{p})\xi^\alpha = -\epsilon_{\alpha\beta\gamma}\xi^\gamma + \delta_{\alpha\beta}\Xi$ and $\mathcal{I}^\alpha(\boldsymbol{p})\Xi(\boldsymbol{p}) = -\xi^\alpha$, and

(b) On $\mathcal{H}_4(\boldsymbol{p})$, $\mathcal{I}^\alpha(\boldsymbol{p}) = I^\alpha_+$ where here, I^α_+ is the positive component of the two (equivalent) flat hypercomplex structures on $\mathbb{H}^n$ given with respect to the standard quaternionic coordinates $(u_1, \ldots, u_n)$ by

$$I^\alpha_\pm = \sum \left(\frac{\partial}{\partial u^\alpha_j} \otimes du^0_j - \frac{\partial}{\partial u^0_j} \otimes du^\alpha_j \pm \epsilon_{\alpha\beta\gamma} \frac{\partial}{\partial u^\beta_j} \otimes du^\gamma_j \right)$$

for all $\alpha, \beta, \gamma = 1,2,3$.

These endomorphisms $\mathcal{I}^\alpha(\boldsymbol{p})$ define an integrable Sp(1)-compatible almost-hypercomplex structure on $\mathcal{N}(\boldsymbol{p})$.

And now, the kicker:

Theorem 2.22. For all $\boldsymbol{p} = (p_1, \ldots, p_n) \in \widetilde{\mathfrak{t}}_n$, the subspace $\mathcal{N}(\boldsymbol{p})$ is a manifold of real dimension $4n-4$ which is diffeomorphic to the complex Stiefel manifold $V_{n,2}(\mathbb{C})$ and which *is not* locally conformally hyperkähler.

It follows, then, that the authors of [27] have actually proven the following:

Theorem 2.23. Let $n > 2$ and $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$ be an n-tuple of non-zero real numbers. For each such $\boldsymbol{p}$, there is a compact hypercomplex manifold $(\mathcal{N}(\boldsymbol{p}), \mathcal{I}^\alpha(\boldsymbol{p}))$ where $\mathcal{N}(\boldsymbol{p})$ is diffeomorphic to $V_{n,2}(\mathbb{C})$ and so, in particular, there exist uncountably many distinct hypercomplex structures on Stiefel manifolds of complex 2-planes in complex n-space.

Remarks.

1. The authors of [27] note that of the uncountably many hypercomplex structures constructed on $V_{n,2}(\mathbb{C})$, all are inhomogeneous *except* when the components p_i of $\boldsymbol{p}$ are all identical. Therefore, despite the fact that no general classification exists in the inhomogeneous case for dimension greater than 4 (see comments in section 2.3 above), theorem 2.23 shows that such structures are certainly not small in number. More details on these examples and examples related thereto can be found in, e.g., [28].

2. [24] points out that while the diffeomorphism in theorem 2.22 fundamentally hinges on the definition of $\widetilde{\mathfrak{t}}_n$ (and, in particular, the fact that $p_i \neq 0$ for all components of $\boldsymbol{p} = (p_1, \ldots, p_n)$, one can still produce a natural hypercomplex structure on $\mathcal{N}(\boldsymbol{p})$ despite the fact that $\mu(\boldsymbol{p})^{-1}$ is a "singular stratified space."

3. Authors of [24] note that Joyce's construction (see [45]) yields *homogeneous* hypercomplex structures on $V_{2,n}(\mathbb{C})$. Results from [45] will be discussed in further detail in section **??** below.

It's worthwhile to note that the usefulness of Stiefel manifolds in the understanding and creation of hypercomplex structures in *unclassified cases* is far from exhausted by the above results. Indeed, in [24], the authors show that the manifold $V_{2,n}(\mathbb{C})$ is fundamental to the classification of so-called *circle V-bundles* over 3-Sasakian orbifolds which admit hypercomplex structures. In that same paper, Joyce's method of hypercomplex quotients (see [46]) is also adapted to $\mathcal{N}(\boldsymbol{p})$ and yields myriad new hypercomplex structures, both when $\mathcal{N}(\boldsymbol{p})$ is a manifold and otherwise (see remark 2 above). Some of these methods will be detailed hereafter (albeit briefly) to close out this section.

2.4.4 Hypercomplex Quotients and Structures on Circle Bundles

For the remainder of the section, let $\mathcal{S}$ be a complete orbifold[10] with a 3-Sasakian structure, that is, with an almost contact 3-structure which is a 3-Sasakian structure with respect to a metric g which itself is invariant under all the local uniformizing groups of the orbifold. The first goal will be to examine a canonical almost hypercomplex structure on all circle V-bundles $H(\mathcal{S})$ which, with only somewhat more rigid assumptions, can be proved integrable. In order for such a conversation to be worthwhile, a series of definitions is no doubt in order.

[10]Roughly, an orbifold is a topological space which is locally the quotient of Euclidean space by the linear action of a finite group. Contrast this with a manifold, which is instead locally Euclidean. For more details on orbifolds, see section 2.2.

Definitions 2.24.

1. An orbifold $\mathcal{S}$ has an *almost contact 3-structure* if there are vector fields ξ^α, one-forms η^α, and $(1,1)$ tensor fields Φ^α, $\alpha = 1,2,3$, that are invariant under the action of all the local uniformizing groups of the orbifold and which satisfy the conditions (i) $\eta^\alpha(\xi^\beta) = \delta_{\alpha\beta}$, (ii) $\Phi^\alpha\xi^\beta = -\epsilon_{\alpha\beta\gamma}\xi^\gamma$, and (iii) $\Phi^\alpha \circ \Phi^\beta - \xi^\alpha \times \eta^\beta = -\epsilon_{\alpha\beta\gamma}\Phi^\gamma - \delta_{\alpha\beta}\,\mathrm{id}$.

2. A *circle V-bundle* $H(\mathcal{S})$ over a 3-Sasakian orbifold $\mathcal{S}$ with local uniformizing systems $\{U, \Gamma, \phi\}$ is described by[11] locally trivial bundles $U \times S^1$ over the local uniformizing neighborhoods together with a map $\gamma \mapsto h_U$ defined by

$$h_U(\gamma)(x,u) = (\gamma^{-1}x, \eta_U(x)\cdot u)$$

where $\gamma \mapsto \eta_U(\gamma)(x)$ is a group homomorphism from Γ into the group of the bundle S^1.

3. An integrable almost hypercomplex structure on $H(\mathcal{S})$ is said to be a *compatible hypercomplex structure* or is said to be *compatible with the 3-Sasakian structure on* $\mathcal{S}$.

4. Given a circle V-bundle $H(\mathcal{S})$ over $\mathcal{S}$, let $\widehat{g}$ be a Riemannian metric on $H(\mathcal{S})$ so that $\pi : (H(\mathcal{S}), \widehat{g}) \to (\mathcal{S}, g)$ is a Riemannian submersion. There exists a nowhere vanishing section Ξ of $\mathcal{V}_1$—the vertical subbundle of the tangent bundle $TH(\mathcal{S})$ to $H(\mathcal{S})$—for which rescaling along the fibers of π by a factor of $\widehat{g}(\Xi,\Xi)^{-1}$ yields a hyperhermitian metric. The pair $(H(\mathcal{S}), \Xi)$ is said to be a *framed circle bundle on* $\mathcal{S}$.

Remarks.

1. It's shown in [24] that the total space of a circle V-bundle $H(\mathcal{S})$ over an orbifold $\mathcal{S}$ is a smooth manifold if and only if the homomorphism η_U is a monomorphism everywhere on $\mathcal{S}$. This is worthwhile knowledge, though it's not entirely pressing for the results studied here.

2. Generally, the term *bundle* is used for a circle V-bundle unless otherwise necessary.

Before proceeding, the construction of the aforementioned canonical hypercomplex structure on $H(\mathcal{S})$ will be given, mimicked almost directly from [24]. Throughout, let $\widehat{X}$ denote the lift of a vector field X from $\mathcal{S}$ to $H(\mathcal{S})$[12].

Let $\pi : H(\mathcal{S}) \to \mathcal{S}$ be the natural projection and let $\widehat{g}$, $TH(\mathcal{S})$, Ξ, and $\mathcal{V}_1$ be as in definition (2.24.4) so that Ξ generates the S^1 action on $H(\mathcal{S})$. The structural properties of $\mathcal{S}$ ensures that $\widehat{g}$ splits $TH(\mathcal{S})$ as

$$TH(\mathcal{S}) \simeq \widehat{\mathcal{H}} \oplus \mathcal{V}_1, \tag{2.4.10}$$

[11] This definition is given with more rigor (though framed differently) in [11], [12], [48], etc.

[12] In particular, $\widehat{X}$ can be thought of as a *basic vector field*, i.e. as the unique such lifted vector field which is π-related to X.

and it follows that π_* induces an isometry between the horizontal vector space $\widehat{\mathcal{H}}_p$ at a point $p \in H(\mathcal{S})$ and the tangent space $T_{\pi(p)}\mathcal{S}$. Moreover, the vector fields $\widehat{\xi}^\alpha$ generate a subbundle $\widehat{\mathcal{V}}_3$ of $\widehat{\mathcal{H}}$ that is isometric at every point to the bundle $\mathcal{V}_3$ on $\mathcal{S}$. Denoting $\widetilde{\mathcal{H}}$ the orthogonal complement to $\widehat{\mathcal{V}}_3$ in $\widehat{\mathcal{H}}$ allows (2.4.10) to be re-expressed as

$$TH(\mathcal{S}) \simeq \widetilde{\mathcal{H}} \oplus \widehat{\mathcal{V}}_3 \oplus \mathcal{V}_1, \tag{2.4.11}$$

and observing the fact that the tensor fields Φ^α are sections of $\operatorname{End}\mathcal{H} \oplus \operatorname{End}\mathcal{V}_3$ on S, they can be lifted to sections $\widehat{\Phi}^\alpha$ of $\operatorname{End}\widetilde{\mathcal{H}} \oplus \operatorname{End}\widehat{\mathcal{V}}_3$ on $H(\mathcal{S})$ by defining

$$\widehat{\Phi}^\alpha \widehat{X} = \widehat{\Phi^\alpha X}$$

and extending to arbitrary sections of $\operatorname{End}\widetilde{\mathcal{H}} \oplus \operatorname{End}\widehat{\mathcal{V}}_3$ by linearity (see footnote [12]). By further imposing the hypotheses that $\widehat{\xi}^\alpha$, $\widehat{\Phi}^\alpha$ are invariant under the local uniformizing groups of $H(\mathcal{S})$, one can define endomorphisms $\mathcal{I}^\alpha$ on $TH(\mathcal{S})$ by

$$\mathcal{I}^\alpha X = -\widehat{\Phi}^\alpha X + \pi^*\eta^\alpha(X)\Xi \qquad \text{and} \qquad \mathcal{I}^\alpha \Xi = -\widehat{\xi}^\alpha$$

for sections X of $\widehat{\mathcal{H}}$ (that is, so-called *horizontal vector fields* on $H(\mathcal{S})$). At this point, one can easily verify that $\mathcal{I}^\alpha$, $\alpha = 1, 2, 3$, defines a hypercomplex structure on $H(\mathcal{S})$. The diligent reader is encouraged to notice the similarity with the structures $\mathcal{I}^\alpha(\boldsymbol{p})$ from theorem 2.21 above.

As pointed out in [24], the bundle $H(\mathcal{S})$ possesses a number of interesting, useful, and otherwise-desirable properties. Some of these are immediately desirable for the sake of the results in this section; others are either tangentially so or are worthwhile because of discussions contained in other sections, or in sources cited throughout. Unsurprisingly, an exhaustive treatment of these results is far beyond the scope of the present, and for that reason, the following treatment must be sufficient. Where appropriate, remarks will be made to help bridge the gap.

Proposition 2.25. Throughout, let $\mathcal{S}$ be a 3-Sasakian orbifold and let $H(\mathcal{S})$ be a compatible hypercomplex circle bundle over $\mathcal{S}$.

1. The fibers of $H(\mathcal{S})$ are totally geodesic.

2. The subbundles $\mathcal{V}_1$, $\widehat{\mathcal{V}}_3$ from the splitting in equation (2.4.11) are integrable, whereby it follows that the subbundle $\mathcal{V}_4 = \mathcal{V}_1 \oplus \widehat{\mathcal{V}}_3$ is also integrable and thus defines a four-dimensional foliation $\mathcal{F}_4$ on $H(\mathcal{S})$ whose leaves are of the form $S^1 \times S^3/\Gamma$ where $\Gamma < \mathrm{SU}(2)$ is a finite subgroup.. Moreover, $\mathcal{F}_4$ also splits: $\mathcal{F}_4 = \mathcal{F}_1 \oplus \mathcal{F}_3$.

3. The leaves of the foliation $\mathcal{F}_4$ are totally geodesic.

4. The vector field $Z^\alpha = \Xi + i\widehat{\xi}^\alpha$ is nowhere vanishing and holomorphic with respect to the complex structure $\mathcal{I}^\alpha$.

5. Z^α generates a holomorphic foliation, say $\mathcal{F}_2$, on $H(\mathcal{S})$.

6. If $\mathcal{S}$ is complete, the complex structures in the two-sphere of complex structures on $H(\mathcal{S})$ are all equivalent, whereby it follows that the hypercomplex structure on $H(\mathcal{S})$ defines a unique complex structure.

7. For a given bundle $H(\mathcal{S})$, changing the framing from Ξ to $\lambda\Xi$, $\lambda \in \mathbb{R}^+$, does not alter the bundle but *does* alter the associated hypercomplex structure. In particular, each circle bundle $H(\mathcal{S})$ over S has a real one-parameter family of inequivalent hypercomplex structures and each such structure determines an inequivalent hypercomplex structure on $H(\mathcal{S})$.

8. $H(\mathcal{S})$ has no symplectic structure and—in particular—admits no Kähler metric.

9. In the event that $H(\mathcal{S})$ is smooth and that the orbifold bundle $\pi : H(\mathcal{S}) \to \mathcal{S}$ is flat, then $H(\mathcal{S})$ is locally conformally hyperkähler.

10. In the event that $H(\mathcal{S})$ is hypercomplex homogeneous (that is, that the group of hypercomplex symmetries on $H(\mathcal{S})$ acts transitively), all the leaves of the U(2) action are diffeomorphic.

In addition to the multitude of properties outlined in proposition 2.25, there's one fundamental result linking the above-described hypercomplex geometry results more directly to the Stiefel manifold $V_{2,n}(\mathbb{C})$. In particular:

Theorem 2.26. Suppose that $\mathcal{S}$ is as in the above proposition and suppose that $H(\mathcal{S})$ is hypercomplex homogeneous. Then $H(\mathcal{S})$ is one of the following:

(i) $H(\mathcal{S}) = V_{2,n}(\mathbb{C})$;

(ii) $H(\mathcal{S}) = V_{2,n}(\mathbb{C})/\mathbb{Z}_k$ with $2 \leq k \in \mathbb{Z}^+$; or

(iii) $H(\mathcal{S})$ is locally conformally hyperkähler and is one of the spaces: $(G/H)\times S^1$ with G/H equal to S^{4n-1}, $\mathbb{RP}^{n-1}$, $\mathrm{SU}(m)/S(\mathrm{U}(m-2)\times\mathrm{U}(1))$ for $m \geq 1$, $\mathrm{SO}(k)/\,\mathrm{SO}(k-4)\times\mathrm{Sp}(1)$ for $k \geq 7$, $\mathrm{G}_2\,/\,\mathrm{Sp}(1)$, $\mathrm{F}_4\,/\,\mathrm{Sp}(3)$, $\mathrm{E}_6\,/\,\mathrm{SU}(6)$, $\mathrm{E}_7\,/\,\mathrm{Spin}(12)$, $\mathrm{E}_8\,/\,\mathrm{E}_7$; or the unique nontrivial principal S^1-bundle over $\mathbb{RP}^{4n-1}$. All these bundles are flat.

In each of these cases, there is a real one parameter family of hypercomplex structures.

Remark. The list in item (iii) of theorem 2.26 is the same as the list in equation (2.3.2) and is nearly identical to the list found in [57] of all complex, locally conformally hyperkähler homogeneous manifolds. [24] points out this correspondence, stating that the list in item (iii) above therefore corresponds to the complete classification of all such manifolds. [57] goes on to prove that all *compact* locally conformally hyperkähler homogeneous manifolds either have the form $(G/H) \times S^1$ for G/H from list item (iii), or are the unique nontrivial principal S^1-bundle over $\mathbb{RP}^{4n-1}$ (i.e., the Möbius band).

In [24], the authors use exposition similar to the above as a logical segue into a somewhat-thorough construction of "new" hypercomplex structures, combining the results listed above with the so-called *hypercomplex quotient* procedure from [46]. This hypercomplex quotient procedure is an immediate extension of already-known quotient algorithms for hyperkähler (see [40]) and quaternionic Kähler manifolds (see [36]) [46]. Summarizing this procedure, including relevant results obtained therefrom, will serve as the conclusion of this particular section.

[46] summarizes the idea behind the algorithm which is a two-stage process. First, a moment-map μ is defined mapping a manifold M into a vector space or vector bundle satisfying certain properties. In this regard, one can show that the quotient of the zero set of μ by a group F of Lie group isometries will inherit some of the structure of the original manifold. It's noted by [46] that, while the existence and uniqueness of the moment map can be shown under "reasonable conditions," these conditions can't be proven in general and so there exist cases in which one cannot define the reduction of a hypercomplex manifold by a "respectable group" because no moment map exists. It's also noted in [46] that in other cases, there may be an exceptionally-large number of distinct reductions of a single manifold by a fixed group. The results presented will largely stem from [24].

Throughout, let $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{H}^n$, define S^{4n-1} to be the subset of HQ^n consisting of all $\boldsymbol{u}$ for which $\sum_{\alpha=1}^n \overline{u_\alpha} u_\alpha = 1$, and denote by g the flat metric on S^{4n-1} induced by the inclusion $S^{4n-1} \hookrightarrow \mathbb{H}^n$. Note that S^{4n-1} has two natural 3-Sasakian structures with respect to g depending on whether $\mathbb{H}^n$ is acknowledged to be a left- or right-quaternionic vector space, and for the sake of matching the literature (see [24]), the convention of a left-module structure on $\mathbb{H}^n$ is adopted. Because of this, the convention will be to let $\xi^\beta = \xi_r^\beta$ be (right) 3-Sasakian vector fields, whereby one notes that the subgroup $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) < \mathrm{O}(n)$ of the isometry group $\mathrm{O}(n)$ of (S^{4n-1}, g) normalizes this particular 3-Sasakian structure. Here,

$$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) = (\mathrm{Sp}(n) \times \mathrm{Sp}(1)) / \mathbb{Z}_2,$$

and $\mathrm{Sp}(1)$ is the group generated by the vector fields ξ^β.

As noted in the results preceding this exposition, there is a 3-Sasakian moment map $\mu_G : S^{4n-1} \to \mathfrak{g}^* \otimes \mathbb{R}^3$ associated to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of any subgroup $G < \mathrm{Sp}(n)$. Moving forward, note that the reduction on which this focuses is a so-called *toral reduction*; this means, in particular, that special attention will be paid to a maximal torus $T^k \subset \mathrm{Sp}(n)$ (for various values k) whose action on $\mathbb{H}^n$ is of the form $u_\alpha \mapsto \tau_\alpha u_\alpha$ and can be expressed as a $(k \times n)$ diagonal matrix $\mathcal{T}$ of the form

$$\mathcal{T} = \mathrm{diag}\left(\prod_{j=1}^k \tau_j^{a_i^j}\right),$$

$i = 1, \ldots, n$, $a_j^i \in \mathbb{Z}$. This gives rise to a so-called *weight matrix* Ω, a $(k \times n)$ integral matrix satisfying

$$(\Omega_{ji}) = a_j^i, \quad j = 1, \ldots, k, \quad i = 1, \ldots, n. \tag{2.4.12}$$

As above, let $\mathfrak{t}_k$ denote the Lie algebra associated to the k-torus T^k for any k.

Given the expression (2.4.12) and the definition of $\mathfrak{t}_k$, it's easy to see that Ω defines an element of $\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{t}_n, \mathfrak{t}_k) \simeq \mathfrak{t}_k \otimes \mathfrak{t}_n^*$, a module which parameterizes the representations of T^k in $\mathbb{H}^n$ and in turn gives rise to a moment map $\mu_\Omega = \sum_j \mu_\Omega^j e_j : S^{4n-1} \to \mathfrak{t}_k^* \otimes \mathbb{R}^3$ of the form

$$\mu_\Omega^j(\boldsymbol{u}) = \sum_\alpha \overline{u}_\alpha i a_\alpha^j u_\alpha, \tag{2.4.13}$$

$\{e_j\}_{j=1}^k$ denoting the standard basis of $\mathbb{R}^k \simeq \mathfrak{t}_k^*$ and a_α^j denoting the (re-indexed) weights from matrix (2.4.12). Ideally, one would like to show that the quotient[13] $\mu_\Omega^{-1}(0)/T^k(\Omega)$ is somehow "well-behaved" and to discuss potential hypercomplex structures existing thereon. Results quantifying what "well-behaved" means and what conditions must be imposed on Ω to achieve the desired result are summarized here, where the notation $\mathcal{S}(\Omega)$ is shorthand for the quotient $\mathcal{S}(\Omega) = \mu_\Omega^{-1}(0)/T^k(\Omega)$.

Theorem 2.27. If all the $k \times k$ minor determinants[14] $\Delta_{\alpha_1,\ldots,\alpha_k}$ of Ω are non-vanishing, then $\mathcal{S}(\Omega)$ is a 3-Sasakian orbifold. If, in addition, d_k denotes the so-called *kth determinantal divisor* of Ω—that is, d_k is the greatest common divisor of all $k \times k$ minor determinants of Ω—and if

$$\gcd\left(\Delta_{\alpha_2,\ldots,\alpha_{k+1}}, \ldots, \Delta_{\alpha_1,\ldots,\widehat{\alpha}_s,\ldots,\alpha_{k+1}}, \ldots, \Delta_{\alpha_1,\ldots,\alpha_k}\right) = d_k$$

for all sequences $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n$, then $\mathcal{S}(\Omega)$ is a smooth manifold.

Remark. An Ω for which all $k \times k$ minor determinants are non-vanishing is said to be *non-degenerate*, and if the gcd condition of theorem 2.27 also holds, Ω is called *admissible*. The assumption that Ω be non-degenerate will be made from hereon unless otherwise stated.

To relate the quotients $\mathcal{S}(\Omega)$ to the above-exposited results on circle V-bundles, the goal moving forward will be to examine the circle V-bundles $H(\mathcal{S}) = H(\mathcal{S}(\Omega))$ over $\mathcal{S}(\Omega)$ and to construct hypercomplex structures on the total space of these bundles. This will be done using the hypercomplex quotient method of [46].

To begin, choose a subgroup $T^{k-1} \subset T^k$ which naturally yields an exact sequence of $\mathbb{Z}$-modules of the form

$$0 \longrightarrow \mathfrak{t}_{k-1} \longrightarrow \mathfrak{t}_k \longrightarrow \mathfrak{t}_1 \longrightarrow 0, \tag{2.4.14}$$

and tensoring (2.4.14) with the free $\mathbb{Z}$-module $\mathfrak{t}_n^*$ yields an exact sequence

$$0 \longrightarrow \mathfrak{t}_{k-1} \otimes \mathfrak{t}_n^* \longrightarrow \mathfrak{t}_k \otimes \mathfrak{t}_n^* \longrightarrow \underbrace{\mathfrak{t}_1 \otimes \mathfrak{t}_n^*}_{\simeq \mathfrak{t}_n^*} \longrightarrow 0. \tag{2.4.15}$$

Now, recall that the $k \times n$ matrix Ω is of the form (2.4.12) and corresponds to an element from $\mathfrak{t}_k \otimes \mathfrak{t}_n^*$, whereby it follows that any $(k-1) \times n$ submatrix Ω_1 of Ω thus corresponds to

[13] Here, the notation $T^k(\Omega)$ is used to designate the k-torus whose $\mathbb{H}^n$-action gives rise to the weight matrix Ω. This is used in [24] due to here-unstated results regarding equivalences of weight matrices Ω and Ω', where the tori $T^k(\Omega)$, $T^k(\Omega')$ are different and hence should be studied as such.

[14] Here, $1 \leq \alpha_1 < \cdots < a_k \leq n$ label the columns of Ω.

an element of $\mathfrak{t}_{k-1} \otimes \mathfrak{t}_n^*$. By exactness, one can consider splitting the sequence in (2.4.15) to yield

$$\mathfrak{t}_k \otimes \mathfrak{t}_n^* \cong (\mathfrak{t}_{k-1} \otimes \mathfrak{t}_n^*) \oplus \mathfrak{t}_n^*,$$

which in turn allows one to write Ω as $\Omega = \begin{pmatrix} \boldsymbol{p} & \Omega_1 \end{pmatrix}^T$ for some integral point $\boldsymbol{p}$ in $\mathfrak{t}_n^*$. At this point, it makes sense to talk about the moment map $\mu_{\boldsymbol{p}} : S^{4n-1} \to \mathbb{R}^3$ associated to this particular integral point $\boldsymbol{p}$ and to conclude the construction of the hypercomplex structure by pursuing avenues similar to those which led to theorems 2.21, 2.22, and 2.23. As above, let $\mathcal{N}(\boldsymbol{p})$ denote $\mu_{\boldsymbol{p}}^{-1}(0)$ with hypercomplex structure a lá theorem 2.21 and, without loss of generality, suppose that $\boldsymbol{p}$ corresponds to the first row of Ω.

As shown above, the torus $T^{k-1}(\Omega_1)$ can be viewed as a subgroup of the group $\operatorname{Aut}\mathcal{N}(\boldsymbol{p})$ of hypercomplex automorphisms of $\mathcal{N}(\boldsymbol{p})$. Let $\nu_{\Omega_1} : \mathcal{N}(\boldsymbol{p}) \to \mathfrak{t}_{k-1} \otimes \mathbb{R}^3$ denote the restriction to $\mathcal{N}(\boldsymbol{p})$ of the projection of μ_Ω (see equation (2.4.13) and the discussion immediately preceding it) onto the last $k-1$ coordinates of $\mathfrak{t}_k$. One can show that the action of $T^{k-1}(\Omega_1)$ on $\nu_{\Omega_1}^{-1}(0) = \mu_\Omega^{-1}(0)$ is locally free, whence it follows that the quotient space

$$H(\boldsymbol{p}, \Omega_1) \stackrel{\text{def}}{=} \nu_{\Omega_1}^{-1}(0)/T^{k-1}(\Omega_1) \tag{2.4.16}$$

is also an orbifold. Moreover, because the circle group $S^1(\boldsymbol{p})$ acts on $H(\boldsymbol{p}, \Omega_1)$ locally freely as well, one can say a bit more.

Theorem 2.28. The orbifold $H(\boldsymbol{p}, \Omega_1)$ is a circle V-bundle over the 3-Sasakian orbifold $\mathcal{S}(\Omega)$ and has a naturally induced hypercomplex structure which is compatible with the 3-Sasakian structure on $S(\Omega)$ (see item (3) of definition 2.24). Furthermore, if the gcd condition of theorem 2.27 is satisfied, then $H(\boldsymbol{p}, \Omega_1)$ is a hypercomplex manifold.

Proof Sketch. The idea is to apply the hypercomplex reduction procedure of [46] to the moment map $\nu_{\Omega_1} : \mathcal{N}(\boldsymbol{p}) \to \mathfrak{t}_{k-1} \otimes \mathbb{R}^3$ obtained from $\mu_{\boldsymbol{p}}$ as described above[15]. Recall that this reduction hinges on finding a Lie group G (with associated Lie algebra $\mathfrak{g}$ and Lie algebra dual $\mathfrak{g}^*$) acting locally freely on a hypercomplex manifold (M, I^α) and then looking for a moment map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ satisfying two conditions, namely that (i) $I^i\, d\mu^i = I^j\, d\mu^j$ for $i, j = 1, 2, 3$, and that (ii) for every $\xi \in \mathfrak{g}$, $I^\alpha\, d\mu^\alpha_\xi(\Xi) \neq 0$ for all $\alpha = 1, 2, 3$ where Ξ is the vector field on M corresponding to $\xi \in \mathfrak{g}$.

First, notice $d\nu^\alpha$, $\alpha = 1, 2, 3$, is a section of $\mathfrak{t}_{k-1} \otimes T^*\mathcal{N}(\boldsymbol{p})$ stemming from a restriction of a quadratic 1-form written in flat coordinates on $\mathbb{H}^n$. Using results stated elsewhere (see, e.g., [27]), one can recognize $\mathcal{N}(\boldsymbol{p})$ as the total space of a U(2) principal V-bundle over a quaternionic Kähler orbifold, an observation which produces an exact sequence

$$0 \longrightarrow \mathcal{V}^*(\boldsymbol{p}) \longrightarrow T^*\mathcal{N}(\boldsymbol{p}) \longrightarrow Q^*(\boldsymbol{p}) \longrightarrow 0 \tag{2.4.17}$$

of vector bundles on $\mathcal{N}(\boldsymbol{p})$ where here, $\mathcal{V}^*(\boldsymbol{p})$ is spanned by the connection 1-form. Exposition in [27] shows that the hypercomplex structure $\{\mathcal{I}^\alpha(\boldsymbol{p})\}_{\alpha=1,2,3}$ on $\mathcal{N}(\boldsymbol{p})$ coincides on $Q^*(\boldsymbol{p})$

[15] To be technical, one must differentiate the case that all components p_i of $\boldsymbol{p} = (p_1, \ldots, p_n)$ are non-zero from the case that some p_i vanish (see [24]), though very little will be said about that in this sketch.

with the restriction of the flat hypercomplex structure I_+^α on $\mathbb{H}^n$ (see theorem 2.21 to recall the definitions of $I_\pm$) associated with right quaternion multiplication.

Next, define ν_j to be the jth component of ν_{Ω_1} with respect to the basis of $\mathfrak{t}_{k-1}$ determined by the $k-1$ rows of Ω_1 and let $\Xi_j(\Omega)$ be the corresponding vector field on $\mathcal{N}(\boldsymbol{p})$ for $j = 2, 3, \ldots, k$. Also, let $\eta_j^0(\Omega)$ be the 1-form dual to the vector field $\Xi_j(\Omega)$ with respect to the restriction g of the flat metric in $\mathbb{H}^n$ to $\mathcal{N}(\boldsymbol{p}) \cap \nu_{\Omega_1}^{-1}(0)$. These definitions allow proof that

$$\mathcal{I}^\alpha(\boldsymbol{p})\, d\nu_j^\alpha = -\eta_j^0(\Omega) \tag{2.4.18}$$

and that

$$\mathcal{I}^\alpha(\boldsymbol{p})\, d\nu_j^\alpha(\Xi_j(\Omega)) = -g(\Xi_j(\Omega, \Xi_j(\Omega))). \tag{2.4.19}$$

As shown in [24], (2.4.18) proves the first necessary condition while (2.4.19) is a direct precursor to the proof of the second. Therefore, $H(\boldsymbol{p}, \Omega_1)$ is a hypercomplex orbifold[16] which becomes a manifold in the presence of the gcd condition from theorem 2.27. □

2.5 Lie Groups & Lie Algebras

Unsurprisingly, many notions from hypercomplex geometry arise naturally in contexts which are of interest to modern physicists. For example, utilization of hypercomplex geometry has led to a considerable amount of headway being made in the area of supersymmetry (see, e.g., [40] for evidence). Sometimes this relationship has also worked in reverse, which is precisely what happened in 1988 when a group of physicists studying supersymmetric σ-models on group manifolds unearthed a variety of hypercomplex geometric ideas lurking, then-undiscovered, among the study of Lie theory [65]. Since then, these ideas have been rediscovered and furthered by a variety of authors, so much so that some very precise classification-style results exist concerning Lie groups and algebras which possess hypercomplex structures. The purpose of this section is to elaborate on some of these results, drawing mainly from [13] and [45].

2.5.1 Some Results on Lie Groups

A significant classification result for Lie groups G admitting hypercomplex structures can be found in [45], though the results themselves may seem rather arbitrary without the exposition given therein. Therefore, before jumping into the results themselves, consider the following *example* as a bit of motivation.

Given a hypercomplex manifold M with twistor space[17] Z, G a Lie group, and P a principal G-bundle over M, let $\widetilde{P}$ denote the lift of P to Z and let $\widetilde{P}^c$ denote the complexification of $\widetilde{P}$ with fibers the complexified group G^c induced by G. Let Φ denote the natural bundle

[16] For the sake of completeness, note that this sketch has only addressed that $p_i \neq 0$ for all i. When some p_i vanish, one can essentially consider the smooth locus $\mathcal{N}_0(\boldsymbol{p})$ of $\mathcal{N}(\boldsymbol{p})$ (ignoring its singular locus) and use the sketch given to construct a hypercomplex structure on $\mathcal{N}_0$, though the resulting space is but a singular stratified space and hence loses the orbifold structure.

[17] Twistor spaces are discussed briefly in appendix 2.3 below.

action of G on P, noting that Φ is transitive on the fibers, and let $\Psi : G \to \mathrm{Aut}(M)$ denote the natural action of G on M. The lift to P of the map Ψ (call it Ψ by abuse of notation) preserves the principal bundle structure and commutes with Φ, facts which can be used to prove two fundamental results[18]: (i) The manifold $N = P/\Psi(G)$ has a natural hypercomplex structure provided that $\Psi(G)$ acts freely on P, and (ii) For the map $\Delta : G \to \mathrm{Aut}(P)$ mapping $g \mapsto \Phi(g)\Psi(g)$, the manifold $N = P/\Delta(G)$ also has a hypercomplex structure provided that $\Delta(G)$ acts freely on P.

Why is this exposition relevant? In the event that $M = \mathbb{CP}^2$ with $G = \mathrm{U}(1)$ and P a principal $\mathrm{U}(1)$-bundle over $\mathbb{CP}^2$, one can prove that $N \cong \mathrm{SU}(3)$! In particular, then, there's at least one well-known Lie group having a hypercomplex structure. What's more, the extended versions of the two results mentioned in the above paragraph indicate that $\mathrm{SU}(3)$ actually has possesses a family of homogeneous hypercomplex structures, indicating an even deeper degree of geometry on $\mathrm{SU}(3)$. This raises the question: Is this example unique or are there others? That, in part, is the question that the latter half of [45] sets out to answer.

In the 1950s, [61] shows that every compact Lie group G of even dimension has a complex structure (see appendices 1.1 and 1.3) with respect to which left translation defines a holomorphic map. Samelson's proof hinges on splitting the complexified Lie algebra $\widetilde{\mathfrak{g}}$ corresponding to the Lie algebra $\mathfrak{g}$ of G into so-called *root subspaces*

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{h}} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha, \tag{2.5.1}$$

$\mathfrak{h}$ the Lie algebra associated to a maximal torus H of G, Δ a finite subset of nonzero elements of $\widetilde{\mathfrak{h}}^*$ called *roots*, and $\mathfrak{g}_\alpha$ the one-dimensional subspace of $\mathfrak{g}$ defined by

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,:\, [h, x] = \alpha(h)x \,\forall h \in \mathfrak{h}\}.$$

After putting this machinery in place, the proof concludes by isolating a *positive root system* $P \subseteq \Delta$ satisfying $P \cap (-P) = \varnothing$ and $P \cup (-P) = \Delta$, and using the set W of $(1,0)$-forms in $\widetilde{\mathfrak{h}}$ corresponding to a complex structure I' on $\mathfrak{h}$ to define a collection $\mathfrak{m}$,

$$\mathfrak{m} = W + \sum_{\alpha \in P} \mathfrak{g}_\alpha,$$

which turns out to be precisely the Lie algebra associated to a group M which makes $G \cong \widetilde{G}/M$ a complex manifold. The complex structure on $\mathfrak{g}$ can be written explicitly since $\widetilde{\mathfrak{g}} = \mathfrak{g} + \mathfrak{m}$ (as real vector spaces), thus implying that $\mathfrak{g} = \widetilde{\mathfrak{g}}/\mathfrak{m}$ as a quotient of complex vector spaces upon which $\mathfrak{m}$ is exactly the $(1,0)$-forms. The idea moving forward will be to mimic these details in the hypercomplex case.

To begin, consider the following result analogous to the splitting in (2.5.1) above.

[18]These are proven for the statements which replace "hypercomplex" with "quaternionic," though this result can be proven also.

Lemma 2.29. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{b} + \sum_{k=1}^{n} \mathfrak{d}_k + \sum_{k=1}^{n} \mathfrak{f}_k, \tag{2.5.2}$$

where $\mathfrak{b}$ is Abelian, $\mathfrak{d}_k$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$, $\mathfrak{b} + \sum_k \mathfrak{d}_k$ contains the Lie algebra of a maximal torus of G, and $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$ are (possibly empty) vector subspaces of $\mathfrak{g}$ such that, for each $k = 1, 2, \ldots, n$, $\mathfrak{f}_k$ satisfies (i) $[\mathfrak{d}_\ell, \mathfrak{f}_k] = \{0\}$ whenever $\ell < k$, and (ii) $\mathfrak{f}_k$ is closed under the Lie bracket with $\mathfrak{d}_k$, and the Lie bracket action of $\mathfrak{d}_k$ on $\mathfrak{f}_k$ is isomorphic to the sum of m copies of the action of $\mathfrak{su}(2)$ on $\mathbb{C}^2$ by left multiplication for some integer m.

The proof of lemma 2.29 relies on a great deal of machinery regarding the structure of Lie algebras, and while the result itself is of fundamental importance moving forward, the proof is neither necessary for the paper as a whole nor particularly enlightening. For that reason, it's worthwhile to forgo proof for the sake of moving forward. The upshot, however, is that virtually no other machinery will be needed to prove the main result, captured in the theorem that follows. Note that the proof of this theorem *is* given, particularly because it allows for the hypercomplex structures in question to be explicitly stated.

Theorem 2.30. Let G be a compact Lie group. Then there exists an integer k with $0 \leq k \leq \max(3, \operatorname{rank} G)$ such that $\mathrm{U}(1)^k \times G$ admits a homogeneous hypercomplex structure.

Proof. Let G be a compact Lie group and have associated Lie algebra $\mathfrak{g}$ By lemma 2.29, $\mathfrak{g}$ admits a splitting of the form (2.5.1) whose constituent pieces satisfy certain conditions; next, define k and m as follows:

- If $\dim \mathfrak{b} \leq n$, n from (2.5.1), define $k = n - \dim \mathfrak{b}$ and let $m = 0$.
- If $\dim \mathfrak{b} > n$, choose $k \in \{0, 1, 2, 3\}$ such that $\dim \mathfrak{b} + k = n + 4m$ for some positive integer $m \in \mathbb{Z}^+$.

Now, noting that the Lie algebra of $\mathrm{U}(1)^k \times G$ is $k\mathfrak{u}(1) + \mathfrak{g}$, it suffices to define a hypercomplex structure on this Lie algebra; this step will yield an almost-hypercomplex structure on the group (by left translation), whereby the characterization in [61] can be used to prove integrability on the structure of the group.

Identify $k\mathfrak{u}(1) + \mathfrak{g} \longleftrightarrow \mathbb{H}^m + \mathbb{R}^n$ as real vector spaces (henceforth writing "=" instead of "$\longleftrightarrow$"), noting that such an identification yields a total of $(n + 4m)^2$ "free parameters[19]." Let $\{e_i\}_{i=1}^n$ denote the standard basis for $\mathbb{R}^n$, and for each k, choose an isomorphism[20] $\phi_k : \mathfrak{su}(2) \to \mathfrak{d}_k$. Note that $\mathfrak{su}(2)$ can be written as $\langle i_1, i_2, i_3 \rangle$ where the i_k satisfy

$$[i_1, i_2] = 2i_3 \qquad [i_2, i_3] = 2i_1 \qquad [i_3, i_1] = 2i_2,$$

[19] The result of this fact is that producing one hypercomplex structure on $k\mathfrak{u}(1) + \mathfrak{g}$ automatically yields infinitely many (nonisomorphic) such structures on $\mathrm{U}^k(1) \times G$.

[20] According to Samelson [61]: "There are $3n$ parameters of freedom in doing this, but the different ways will lead to hypercomplex structures isomorphic up to conjugacy."

and using these relations, complex structures I_k, $k = 1, 2, 3$, can be defined component-wise on $\mathfrak{g}$ as follows:

(a) Let I_1, I_2, and I_3 act as "imaginary unit quaternion multiplication" on $\mathbb{H}^m$ as per usual.

(b) Let the actions of I_1, I_2, and I_3 on $\mathbb{R}^n + \sum_j \mathfrak{d}_j$ be given by

$$I_\alpha(e_j) = \phi_j(i_\alpha), \quad I_\alpha(\phi_j(i_\alpha)) = -e_j, \quad I_\alpha(\phi_j(i_\beta)) = \phi_j(i_\gamma), \quad I_\alpha(\phi_j(i_\gamma)) = -\phi_j(i_\beta)$$

whenever $(\alpha\ \beta\ \gamma)$ is an even permutation of $(1\ 2\ 3)$.

(c) Let the actions of I_1, I_2, and I_3 on $\mathfrak{f}_j$ be given by $I_\alpha(v) = [v, \phi_j(i_\alpha)]$ for each $v \in \mathfrak{f}_j$.

It suffices to prove I_1, I_2, and I_3 are complex structures on $k\mathfrak{u}(1) + \mathfrak{g}$, that $I_3 = I_1 I_2$, and that the almost complex structures induced on $\mathrm{U}(1)^k \times G$ generated by left translation are integrable.

For the remainder of the proof, let $A = \{1, 2, 3\}$. Note that the collection $\{I_\alpha\}_{\alpha \in A}$ consists of complex structures satisfying $I_1 I_2 = I_3$ on each of $\mathbb{H}^m$, $\mathbb{R}^n + \sum_j \mathfrak{d}_j$ by way of parts (a) and (b), respectively. Moreover, the second condition given in lemma 2.29 allows one to view the action of the I_α on $\mathfrak{f}_j$ as being equivalent to the isomorphism resulting from the action of $\mathfrak{d}_j$ on $\mathfrak{f}_j$ by conjugation. Because this action is isomorphic to the natural action of the purely imaginary quaternions on $\mathbb{H}^\ell$ for some ℓ, the collection $\{I_\alpha\}_{\alpha \in A}$ consists of complex structures satisfying $I_1 I_2 = I_3$ on the component of (c) as well, whereby it follows that $\{I_\alpha\}_{\alpha \in A}$ is a hypercomplex structure on $k\mathfrak{u}(1) + \mathfrak{g}$. The result will be concluded provided that I_1, I_2, and I_3 generate homogeneous integrable complex structures on $\mathrm{U}(1)^k \times G$ by left translation (proven using the results from [61]).

For $\alpha \in A$, define $\mathfrak{t}$ by

$$\mathfrak{t} = \mathbb{H}^m + \mathbb{R}^n + \langle \phi_1(i_a), \ldots, \phi_n(i_\alpha) \rangle. \tag{2.5.3}$$

In particular, $\mathfrak{t}$ is the Lie algebra associated to a maximal torus $T \subset \mathrm{U}(1) \times G$. Define now a subset $V \subset k\widetilde{\mathfrak{u}}(1) + \widetilde{g}$ consisting of all $(1,0)$-forms of I_α in $k\widetilde{\mathfrak{u}}(1) + \widetilde{g}$ where $\widetilde{\cdot}$ denotes the complexification. The remainder of the proof uses methods similar to those in Samelson's proof in [61] involving a positive system of roots for $k\widetilde{\mathfrak{u}}(1) + \widetilde{g}$ relative to $\mathfrak{t}$, whereby his, Samelson's, result will show that I_α is an integrable complex structure on $\mathrm{U}(1)^k \times G$ for all $\alpha \in A$. This will complete the proof.

To begin, note that the subset V can be clearly described by examining the $(1,0)$-forms on each component of (a), (b), and (c) from above[21]. For (a), note that the $(1,0)$-forms are the usual such forms on $\mathbb{H}^m$; for (b), the $(1,0)$-forms of $\mathbb{R}^n + \sum_j \mathfrak{d}_j$ are

$$\langle e_1 + i\phi_1(\alpha), \ldots, e_n + i\phi_n(i_\alpha), \phi_1(i_\beta) + i\phi_1(i_\gamma), \ldots, \phi_n(i_\beta) + i\phi_n(i_\gamma) \rangle$$

[21] Verification of parts (b) and (c) of the claim that follows relies on structure theory from [61] and is briefly touched on in [45].

where again, $(\alpha\ \beta\ \gamma)$ is any even permutation of $(1\ 2\ 3)$; for (c), the $(1,0)$-forms of $\widetilde{\mathfrak{f}}_j$ are given by the subset

$$V \cap \widetilde{\mathfrak{f}}_j = \sum_{\substack{B \in \Delta_j : B \neq A_j \\ B(i\phi_j(i_\alpha)) > 0}}$$

of all root subspaces of $\mathfrak{b}_{j-1}$ corresponding to roots B other than A_j which satisfy $B(i\phi_j(i_\alpha)) > 0$.

At this point, the machinery needed for the remainder of the argument is completely in place: The roots of $\widetilde{\mathfrak{b}}_j$ are the roots of $\mathfrak{g}$ which give zero when evaluated by $\phi_1(i_\alpha), \ldots, \phi_j(i_\alpha)$ as these are the roots which centralize $\mathfrak{d}_1, \ldots, \mathfrak{d}_j$. In particular, one can define a subset $P \subset \Delta$, Δ the set of all roots, to be

$$\begin{aligned} P = \{A \in \Delta\ : & A(\phi_1(i_\alpha)) = \cdots = A(\phi_{j-1}(i_\alpha)) = 0, \\ & A(i\phi_j(i_\alpha)) > 0 \text{ for some } j \in \{1, 2, \ldots, n\}\}. \end{aligned}$$

One can show that this P is a positive root system, and can verify that

$$V = V \cap \widetilde{\mathfrak{t}} + \sum_{A \in P} \mathfrak{g}_A,$$

i.e. that V (i.e. the collection of $(1,0)$-forms of I_α) is the sum of $(1,0)$-forms of some complex structure on $\mathfrak{t}$ together with a positive system of roots. In the language of [61], this proves that the left translation of I_α is a homogeneous complex structure on $\mathrm{U}(1)^k \times G$ which is necessarily integrable. Hence, the result. □

One of the most intriguing parts of Joyce's paper [45] is that, in addition to theorem 2.30, a number of other considerable results involving hypercomplex (and later, quaternionic) structures on Lie groups and Lie algebras are also given. One instance of this can be found near the end of the paper, where yet another analogue of a complex geometry result is proven. For the sake of completeness, that result is stated below sans proof, after which some examples showing the power of these theoretical ideas are used to round out the section. First, consider the following definitions, where unless otherwise stated, K denotes a simply-connected compact semisimple Lie group, G a compact Lie group with associated Lie algebra $\mathfrak{g}$, H a maximal torus in G with associated Lie algebra $\mathfrak{h}$, and where one can find a subalgebra of the complexification $\widetilde{\mathfrak{g}}$ isomorphic to $\mathfrak{su}(2)$ generated by $\mathfrak{g}_{\pm A}$ for some *highest root* A.

Definitions 2.31.

1. A *C-subgroup* of K is a closed and connected subgroup whose semisimple part coincides with the semisimple part of the centralizer of a toral subgroup of K.

2. A *D-subgroup* of G is the centralizer in G of any $\mathfrak{su}(2)$ embedded in $\mathfrak{g}$ that comes from a highest root A.

3. An *E-subgroup* of G is any subgroup $E < G$ for which there exists a chain of subgroup inclusions $G = G_0 \supset G_1 \supset \cdots \supset G_j = E$ such that G_{i+1} is a D-subgroup of G_i. Here, j is called the *length* of E and can be shown to be well-defined.

Next, consider the following result of Wang, given here as a proposition.

Proposition 2.32. Let X be a C-subgroup of a simply connected compact semisimple Lie group K. If K/X is even-dimensional, then K/X has a homogeneous complex structure.

And finally, Joyce's adaptation to the realm of hypercomplex geometry:

Theorem 2.33. Let G be a compact Lie group, let E be an E-subgroup of G of length j, let F be the semisimple part of E, and let X be any closed subgroup of G for which $F \subseteq X \subseteq E$. Then there exists an integer k with $0 \leq k \leq \max(3, j)$ such that $\mathrm{U}(1)^k \times G/X$ admits a homogeneous hypercomplex structure and thus one that is preserved by left translations in $\mathrm{U}(1)^k \times G$.

For the last major result of this part, consider the following examples of theorems 2.30 and 2.33, respectively.

Examples 2.34.

1. First, it will be shown that $U(1) \times \mathrm{SO}(6)$ admits a hypercomplex structure a lá theorem 2.30. To make this structure align with what's needed to use the theorem, let $G = \mathrm{SO}(6)$, let $H \subset \mathrm{U}(3) \subset \mathrm{SO}(6)$ be the collection of all diagonal matrices[22], and note that the Lie algebra $\mathfrak{h}$ consists of all matrices in $\mathfrak{u}(3) \subset \mathfrak{so}(6)$ of the form $\mathrm{diag}(i\lambda_1, i\lambda_2, i\lambda_3)$, $\lambda_\alpha \in \mathbb{R}$ for $\alpha = 1, 2, 3$.

   Next, define a coordinate system (x_1, x_2, x_3) on $\widetilde{\mathfrak{h}}^*$—the vector space dual of the complexification $\widetilde{\mathfrak{h}}$ of $\mathfrak{h}$—such that

   $$(x_1, x_2, x_3) : \mathrm{diag}(i\lambda_1, i\lambda_2, i\lambda_3) \mapsto 2(x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3).$$

   Given these coordinates, one can verify that the twelve roots of $\mathrm{SO}(6)$ are given by $(\pm i, \pm i, 0)$, $(0, \pm i, \pm i)$, and $(\pm i, 0, \pm i)$; also, one can confirm that all these roots are equivalent under automorphisms of G preserving H, whereby it follows that every root is a *highest root*.

   After some computation (needed to perform the Lie algebra decomposition used in the proof of the theorem), one can choose an arbitrary highest root to generate $\mathfrak{d}_1$ which in turn allows isolation of expressions for $\mathfrak{d}_2$, $\mathfrak{b}_k$, and $\mathfrak{f}_k$ for $k = 1, 2$. Some finagling shows that the values $(n, \dim \mathfrak{b}_2, k, 1) = (2, 1, 1, 0)$ "work" for theorem 2.30, thereby showing that $\mathrm{U}(1) \times \mathrm{SO}(6)$ has a hypercomplex structure whose choice has four real parameters of freedom.

2. Here, the machinery laid out in the first example is combined with theorem 2.33 to show that $\mathrm{SO}(6)/\mathrm{SU}(2)$ is a homogeneous hypercomplex manifold. In particular, use the decomposition of $\mathfrak{so}(6)$, isolate the semisimple part

   $$\widetilde{\mathfrak{f}} = \langle \mathrm{diag}(i, -i, 0) \rangle + \mathfrak{g}_{(i,-i,0)} + \mathfrak{g}_{(-i,i,0)},$$

[22] One can show that the collection of all such matrices form a maximal torus in G.

and choose X such that $F \subseteq X \subseteq E$. For this example, suppose $X = F$. In particular, then, X is generated by block diagonal matrices and satisfies

$$X = \{\mathrm{diag}(A, 1) \,:\, A \in \mathrm{SU}(2)\} \subset \mathrm{U}(3) \subset \mathrm{SO}(6).$$

After some work, one shows that the Lie algebra $\mathfrak{e}$ splits into $\mathfrak{e} = \mathfrak{x} + \langle \mathrm{diag}(0, 0, i) \rangle$ where the latter component is the Lie algebra of a torus, and so for $k = 0$, one gets a homogeneous hypercomplex structure on

$$\mathrm{U}(1)^0 \times \mathrm{SO}(6)/\,\mathrm{SU}(2) \cong \mathrm{SO}(6)/\,\mathrm{SU}(2).$$

In addition, the freedom in making the complex structure is that of one real parameter.

2.5.2 Some Results on Lie Algebras

Due to the extensiveness of the literature concerning Lie algebras with hypercomplex structures (see section 2.5.3 below), an even somewhat-thorough analysis is beyond the scope of this paper. For that reason, this section will be dedicated to presenting the results presented in [13] which serve to be both enlightening and considerable simultaneously. The goal will be to maintain relative precision and depth while not drifting too far away from the core ideas behind this paper.

Worth noting that the work of Barberis in [13] can, in some ways, be thought of as an extension of [45]. In particular, the self-proclaimed goal of [13] is to parameterize the equivalence classes of invariant hypercomplex structures on 4-dimensional simply connected real Lie groups, whereby the logical progression is to define a meaningful notion of *equivalence* and to consider the machinery laid down by [45] as a first step towards completing this goal. As is often the case, it's most logical to begin at the beginning.

To that end, let $\mathfrak{g}$ denote a real Lie algebra and consider a hypercomplex structure on $\mathfrak{g}$ as a pair J_1, J_2 of endomorphisms of $\mathfrak{g}$ satisfying both the algebraic relations of the purely imaginary quaternions and a suitable integrability condition (see, e.g., definition A1.7 in appendix 1.1 regarding the Nijenhuis tensor). In the material that follows, let $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ denote the so-called *derived Lie algebra* of $\mathfrak{g}$ and let $\mathfrak{z}$ be the center of $\mathfrak{g}$.

Definition 2.35. Two hypercomplex structures $\{J_\alpha\}_{\alpha=1,2}$, $\{J'_\alpha\}_{\alpha=1,2}$ on $\mathfrak{g}$ are said to be equivalent if there exists an automorphism $\phi \in \mathrm{Aut}\,\mathfrak{g}$ such that $\phi J_\alpha = J'_\alpha \phi$ for $\alpha = 1, 2$.

This definition serves as the foundation for the results that come afterwards. Between the definition and the main components of the classifications, a number of preliminary lemmas are stated with various levels of proof. Ideally, some of the main results from [13] will be stated here with at least sketches of proofs, and because of the heavy dependence on these lemmas, it would be unsatisfactory to omit them entirely. As a compromise, consider the following three-part lemma—a massive condensation of the author's presentation in [13] but a presentation suitable for the purposes of this paper.

Lemma 2.36.

1. If $\{J_1, J_2\}$ is a pair of endomorphisms on $\mathbb{R}^4$ which satisfy the algebraic properties of the purely imaginary quaternions and if $W \subset \mathbb{R}^4$ is an arbitrary two-dimensional subspace, then there exists $x = (x_1, x_2, x_3) \in S^2$ such that $J_x = \sum_{\alpha=1}^{3} x_\alpha J_\alpha$ preserves W. Here, $J_3 = J_1 J_2$.

2. If J_α is an endomorphism of $\mathfrak{g}$ such that $J_\alpha^2 = -1$ and if $\{X_1, J_\alpha X_1, \ldots, X_n, J_\alpha X_n\}$ is a basis of $\mathfrak{g}$, then the associated Nijenhuis tensor N_α of J_α is identically zero if and only if $N_\alpha(X_i, X_j) = 0$ for all $i < j$.

3. If $\{J_\alpha\}_{\alpha=1,2}$ is a family of endomorphisms of a real vector space V which satisfy the algebraic properties of the purely imaginary quaternions, then V admits an inner product such that J_1 and J_2 are orthogonal. Moreover, when $\dim V = 4$, this inner product is unique up to a constant multiple.

Now, with the statements of these results in-place, the heart of the matter—i.e., the classification and parametrization of all equivalence classes of hypercomplex structures on 4-dimensional real Lie algebras—can finally be addressed. The results are broken down based on whether $\mathfrak{g}$ is solvable or not.

Theorem 2.37. If $\mathfrak{g}$ is not solvable, then $\mathfrak{g}$ admits a hypercomplex structure if and only if $\mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{so}(3)$ and this particular hypercomplex structure is unique up to equivalence.

Proof. ($\Longrightarrow$) To show that $\mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{so}(3)$ implies a hypercomplex structure on $\mathfrak{g}$, it suffices to exhibit such a structure on $\mathbb{R} \oplus \mathfrak{so}(3)$. To that end, let $\{Z, X, Y, W\}$ be a basis for $\mathbb{R} \oplus \mathfrak{so}(3)$ such that $Z \in \mathbb{R}$ and

$$[X, Y] = W, \qquad [Y, W] = X, \qquad [W, X] = Y. \tag{2.5.4}$$

Now, define endomorphisms $J_1, J_2 \in \mathrm{End}(\mathbb{R} \oplus \mathfrak{so}(3))$ as follows:

$$\begin{aligned} J_1 Z &= X, \quad J_1 Y = W, \quad J_1^2 = -I \\ J_2 Z &= Y, \quad J_2 W = X, \quad J_2^2 = -I. \end{aligned} \tag{2.5.5}$$

These endomorphisms commute, and by simple Nijenhuis computations:

$$\begin{aligned} N_1(Z, Y) &= [J_1 Z, J_1 Y] - J_1[Z, J_1 Y] - J_1[J_1 Z, Y] - [Z, Y] \\ &= [X, W] - J_1[Z, W] - J_1[X, Y] - [Z, Y] \text{ by (2.5.5)} \\ &= -Y - J_1[Z, W] - J_1 W - [Z, Y] \text{ by (2.5.4)} \\ &= -Y - (-Y) \text{ by (2.5.4)+(2.5.5)} \\ &= 0. \end{aligned}$$

Similarly for $N_2(Z, W)$,

$$\begin{aligned} N_2(Z, W) &= [J_2 Z, J_2 W] - J_2[Z, J_2 W] - J_2[J_2 Z, W] - [Z, W] \\ &= [Y, X] - J_2[Y, W] \\ &= -W - (-W) = 0. \end{aligned}$$

Therefore, by item (ii) in lemma 2.36, J_1 and J_2 are integrable, thus defining a hypercomplex structure $\mathcal{H} = \{J_1, J_2\}$ on $\mathbb{R} \oplus \mathfrak{so}(3)$.

($\Longleftarrow$) Conversely, suppose that $\mathfrak{g}$ admits a hypercomplex structure $\{J_1, J_2\}$. Using well-known results about Lie algebras, it follows that either $\mathfrak{g} \cong \mathfrak{z} \oplus \mathfrak{so}(3)$ or $\mathfrak{g} \cong \mathfrak{z} \oplus \mathfrak{sl}(2, \mathbb{R})$ where here, the direct sum represents a direct sum of ideals. Choose a nonzero element Z of $\mathfrak{z}$, and define

$$X = J_1 Z, \qquad Y = J_2 Z, \qquad W = J_1 J_2 Z. \tag{2.5.6}$$

From here, one notes that $\{Z, X, Y, W\}$ is necessarily a basis for $\mathfrak{g}$, and by writing $[X, Y] = aZ + bX + cY + dW$ and by noting that $N_1(Z, Y) = 0 = N_2(Z, X)$ as follows by equation (2.5.6), one can confirm that

$$J_1[X, Y] = [X, W] \qquad \text{and} \qquad J_2[X, Y] = [Y, W]. \tag{2.5.7}$$

Using the expression for $[X, Y]$ in terms of a, b, c, and d along with equations (2.5.6) and (2.5.7) yields expressions for $[W, X]$ and $[Y, W]$, namely

$$[W, X] = bZ - aX + dY - cW \qquad \text{and} \qquad [Y, W] = -cZ + dX + aY - bW.$$

Finally, expanding $\mathrm{Jac}(X, Y, W) \stackrel{\text{def}}{=} [[X, Y], W] + [[Y, W], X] + [[W, X], Y]$ and applying the above equations confirms that the coefficient of Z in $\mathrm{Jac}(X, Y, W)$ is precisely $a^2 + b^2 + c^2$. Clearly, $\mathrm{Jac}(X, Y, W)$ should be an expression free of Z terms, whereby it follows that $a^2 + b^2 + c^2 = 0$. In particular, $a = b = c = 0$, so rewriting above expressions yields that

$$[X, Y] = dW, \qquad [Y, W] = dX, \qquad [W, X] = dY.$$

Now, $\mathfrak{g}$ being unsolvable implies that $\mathfrak{g}$ is necessarily non-Abelian; in particular, $d \neq 0$, whereby it follows that $\{X, Y, W\}$ generates a three-dimensional Lie algebra which, by equations (2.5.6) and (2.5.7), is isomorphic to $\mathfrak{so}(3)$. Therefore, $\mathfrak{g} \cong \mathfrak{z} \oplus \mathfrak{so}(3) \cong \mathbb{R} \oplus \mathfrak{so}(3)$.

The theorem will be proven if the uniqueness result can be shown. Indeed, showing that any two such hypercomplex structures are equivalent can be done using elementary considerations, the argument for which can be found in its entirety in [13]. Hence, the result. $\square$

Remarks.

1. The simply connected Lie group with Lie algebra $\mathbb{R} \oplus \mathfrak{so}(3)$ is the multiplicative group $\mathbb{H}^*$ of nonzero quaternions.

2. In the proof of theorem 2.37, a stronger statement is proven: If $\mathfrak{z} \neq \{0\}$, then $J_1\mathfrak{z} \oplus J_2\mathfrak{z} \oplus J_3\mathfrak{z}$ is either isomorphic to $\mathfrak{so}(3)$ or is Abelian. Hence, a 4-dimensional Lie algebra $\mathfrak{g}$ with nontrivial center admits a hypercomplex structure if and only if $\mathfrak{g}$ is Abelian or $\mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{so}(3)$.

3. The Lie algebra $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})$ mentioned in the proof of theorem 2.37 obviously doesn't admit a hypercomplex structure. Outside results have shown that $\mathfrak{g}_0$ *does* admit an invariant complex structure.

4. An immediate result of the last theorem is that the Lie groups $\mathbb{R} \oplus \mathrm{SO}(3)$, $S^1 \times S^3$, and $S^1 \times \mathrm{SO}(3)$ also admit hypercomplex structures. The last two are Hopf surfaces, characterized by Boyer's classification (see theorem 2.2), and because of the diffeomorphism $S^1 \times S^3 \cong \mathrm{U}(2)$, $\mathrm{U}(2)$ must also admit a hypercomplex structure.

To characterize hypercomplex structures on $\mathfrak{g}$ which *is* solvable, it's necessary to consider cases on the dimension of the derived subalgebra $\mathfrak{g}'$. In particular, there are results for $\dim \mathfrak{g}' = 0, 1, 2, 3$.

Theorem 2.38. If $\dim \mathfrak{g}' = 0$, there is a one-to-one correspondence between hypercomplex structures on $\mathfrak{g}$ and points in the space $\mathrm{GL}(4n, \mathbb{R}) / \mathrm{GL}(n, \mathbb{H})$. All such hypercomplex structures are equivalent.

Proof. If $\dim \mathfrak{g}' = 0$, $\mathfrak{g}$ is Abelian. As a result, the integrability condition of a hypercomplex structure on $\mathfrak{g}$—a structure which can be formed simply by choosing two endomorphisms satisfying the imaginary unit quaternion arithmetic conditions. To determine the correspondence suggested, fix a hypercomplex structure $\{J^0_\alpha\}_{\alpha=1,2}$ and consider the map $T \mapsto \{TJ^0_\alpha T^{-1}\}_{\alpha=1,2}$ for all $T \in \mathrm{GL}(4n, \mathbb{R})$. The fact that the conjugated endomorphism $TJ^0_\alpha T^{-1}$, $\alpha = 1, 2$, is in the quotient $\mathrm{GL}(4n, \mathbb{R}) / \mathrm{GL}(n, \mathbb{H})$ is obvious, as is the fact that any such endomorphism is equivalent to J^0_α, $\alpha = 1, 2$ (see definition 2.35 above). □

Something of a diametrically opposite result is true when $\dim \mathfrak{g}' = 1$:

Theorem 2.39. If $\dim \mathfrak{g}' = 1$, then $\mathfrak{g}$ does not admit any hypercomplex structure.

Proof. Because of the second remark above, one may assume that if $\mathfrak{g}$ admits a hypercomplex structure, then $\mathfrak{z} = \{0\}$. Now, let X be a nonzero element of $\mathfrak{g}'$, noting necessarily then that there exists a $Y \in \mathfrak{g}$ for which $[Y, X] = X$. Using machinery from Lie algebra theory, this fact allows the splitting of $\mathfrak{g}$ into

$$\mathfrak{g} = \ker(\mathrm{ad}_X) \cap \ker(\mathrm{ad}_Y) \oplus \mathbb{R}X \oplus \mathbb{R}Y.^{23}$$

Choosing arbitrary elements $U, V \in \ker(\mathrm{ad}_X) \cap \ker(\mathrm{ad}_Y)$ and applying the Jacobi identity to U, V, Y, one can show that $[U, V] = 0$, whereby it follows that $\mathfrak{z} = \ker(\mathrm{ad}_X) \cap \ker(\mathrm{ad}_Y)$. This obviously contradicts the assumption that $\mathfrak{z} = \{0\}$, whereby the result follows. □

[23] For a given vector field $W \in \mathfrak{g}$, $\mathrm{ad}_W \in \mathrm{GL}(\mathfrak{g})$ denotes the adjoint action in $\mathfrak{g}$ induced by the action $\mathrm{Ad}_g : G \to G$ defined by $\mathrm{Ad}_g : h \mapsto ghg^{-1}$ for all $g, h \in G$ [33].

The remaining cases are when $\dim \mathfrak{g}' \in \{2, 3\}$, and the proofs for those are a bit more involved. Moving forward, let $\mathrm{Aff}(\mathbb{K})$ denote the affine motion group on $\mathbb{K}$, that is, the group of all invertible affine transformations from $\mathbb{K}$ to itself under the group operation of composition, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$[24]. One well-known fact is that

$$\begin{aligned} \mathrm{Aff}(\mathbb{K}) &\cong \mathbb{K} \rtimes \mathrm{GL}(\mathbb{K}) \\ &\cong \mathbb{K} \rtimes \mathbb{K}^* \end{aligned}$$

where $\mathbb{K}^*$ denotes the dual of $\mathbb{K}$. Of particular interest moving forward will be the case of $\mathbb{K} = \mathbb{C}$, in which case the associated Lie algebra $\mathfrak{aff}(\mathbb{C})$ decomposes as the direct sum of an Abelian ideal and an Abelian subalgebra, the first structure corresponding to the normal subgroup of translations while the second corresponds to multiplication by a scalar in $\mathbb{C}^*$. In this case, there are bases X, Y of the ideal and bases Z, W of the subalgebra so that

$$[X, Z] = X, \qquad [Y, Z] = Y, \qquad [X, W] = Y, \qquad [Y, W] = -X. \tag{2.5.8}$$

These facts, though mentioned briefly in passing, will play somewhat crucial roles in the results that follow.

Theorem 2.40. If $\dim \mathfrak{g}' = 2$, then (i) $\mathfrak{g}$ admits a hypercomplex structure if and only if $\mathfrak{g} \cong \mathfrak{aff}(\mathbb{C})$, and (ii) the equivalence classes of hypercomplex structures on $\mathfrak{g}$ are parameterized by the space $\mathbb{RP}^2$.

Proof. The key to proving the second claim is to exhibit a one-to-one correspondence between hypercomplex structures on $\mathfrak{g}$ and the space $\mathrm{O}(2) \setminus \mathrm{SO}(3) \cong \mathbb{RP}^2$, the details of which are spelled out in [13]. The most significant part of the proof is in the proof of the first claim which is detailed here.

First, construct a hypercomplex structure on $\mathfrak{aff}(\mathbb{C})$ as follows: Let $\mathcal{H} = \{J_\alpha\}_{\alpha=1,2}$ be the family of endomorphisms of $\mathfrak{aff}(\mathbb{C})$ defined by

$$\begin{array}{lll} J_1 X = -W, & J_1 Y = Z, & J_1^2 = -I, \\ J_2 X = Y, & J_2 Z = -W, & J_2^2 = -I. \end{array} \tag{2.5.9}$$

Using part (2) of lemma 2.36, one can check the integrability of J_1 and J_2, while the anti-commutativity follows immediately from the definition. Hence, $\mathfrak{aff}(\mathbb{C})$ has a hypercomplex structure.

Conversely, suppose $\{J_1, J_2\}$ defines a hypercomplex structure on $\mathfrak{g}$. Given that a hypercomplex structure yields a 2-sphere S^2 of complex structure[25] and using part (1) of lemma 2.36, it can be assumed without loss of generality that $J_2 : \mathfrak{g}' \to \mathfrak{g}'$ so that $\mathfrak{g}$ splits as

$$\mathfrak{g} = \mathfrak{g}' \oplus J_1 \mathfrak{g}'.$$

[24]The notion of affine groups is much more general than the three cases given here. These are given in particular because of the well-known fact that $\mathrm{Aff}(\mathbb{K})$ is a Lie group whenever $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

[25]Given a hypercomplex structure $\{J_1, J_2\}$ and a 3-tuple $(x, y, z) \in S^2$ satisfying $x^2 + y^2 + z^2 = 1$, one can easily show that

$$J_\alpha = xJ_1 + yJ_2 + zJ_1J_2$$

is yet another hypercomplex structure.

Let $\{X', Y'\}$ be a basis of $\mathfrak{g}'$ such that $Y' = J_2X'$, whereby one easily confirms that $\{X', Y', J_1X', J_1Y'\}$ is a basis of $\mathfrak{g}$. Next, note the existence of two skew-symmetric bilinear forms α, β on $\mathfrak{g}$ so that

$$[V, W] = \alpha(V, W)X' + \beta(V, W)Y' \text{ for all } V,\, W \in \mathfrak{g}.$$

Now, $\mathfrak{g}$ solvable implies that $\mathfrak{g}'$ is Abelian; this, along with the integrability condition $N_1(X', Y') = 0$, yields that $[J_1X', J_1Y'] = 0$ and that $[X', J_1Y'] = [Y', J_1X']$. Moreover, $N_2(X', J_1X') = 0$ implies that

$$[X', J_1X'] = -[Y', J_1Y'].$$

Applying the Jacobi identity yields that $\alpha(X', J_1X') = \beta(X', J_1Y')$ and that $\alpha(X', J_1Y') = -\beta(X', J_1X')$, which in particular gives a parametrization of the bracket in $\mathfrak{g}$ in terms of $c \overset{\text{def}}{=} \alpha(X', J_1X')$ and $d \overset{\text{def}}{=} \alpha(X', J_1Y')$ as follows:

$$\begin{aligned} [X', J_1X'] &= cX' - dY', & [Y', J_1X'] &= dX' + cY', \\ [X', J_1Y'] &= dX' + cY', & [Y', J_1Y'] &= -cX' + dY'. \end{aligned} \tag{2.5.10}$$

By dimensional considerations, if $c = 0 = d$, then one would reach a contradiction of the fact that $\dim \mathfrak{g}' = 2$; therefore, either $c \neq 0$ or $d \neq 0$. In particular, this allows one to define a new collection of elements in $\mathfrak{g}$ as follows:

$$\begin{aligned} X &= (c^2 + d^2)^{-1}(dX' + cY'), & Y &= (c^2 + d^2)^{-1}(-cX' + dY'), \\ Z &= (c^2 + d^2)^{-1}(cJ_1X' + dJ_1Y'), & W &= (c^2 + d^2)^{-1}(-dJ_1X' + cJ_1Y'). \end{aligned} \tag{2.5.11}$$

At this point, the proof is complete: Simple computation verifies that the elements

$$\{X, Y, Z, W\} \subset \mathfrak{g}$$

satisfy precisely the relations in (2.5.8) above, whereby it follows that $\mathfrak{g} \cong \mathfrak{aff}(\mathbb{C})$. □

Remark. Though not pressing, [13] takes the added step of combing equations (2.5.9) and (2.5.11) to present the hypercomplex structure given by J_1 and J_2 relative to the basis $\{X, Y, Z, W\}$:

$$\begin{aligned} J_1X &= aZ - bW, & J_1Y &= bZ + aW, & J_1^2 &= -I, \\ J_2X &= Y, & J_2Z &= -W, & J_2^2 &= -I, \end{aligned}$$

where $a = 2cd(c^2 + d^2)^{-1}$ and $b = (d^2 - c^2)(c^2 + d^2)^{-1}$ satisfy $a^2 + b^2 = 1$.

Barberis' classification in [13] is completed by the result for $\dim \mathfrak{g}' = 3$. Though interesting, the details of the proof require more outside machinery than any of the results presented thus far. For that reason, this final case is stated without proof, though the interested reader is encouraged to consult [13], theorem 3.4 for the full details.

Theorem 2.41. If $\dim \mathfrak{g}' = 3$, the one of the following holds:

(a) $\mathfrak{g}'$ is Abelian, in which case $\mathfrak{g}$ admits a hypercomplex structure if and only if $\mathfrak{g}$ corresponds to the space $\mathbb{R}H^4$, the 4-dimensional real hyperbolic space. Moreover, $\mathfrak{g}$ has a unique hypercomplex structure up to equivalence.

(b) $\mathfrak{g}'$ is a Heisenberg algebra[26], in which case $\mathfrak{g}$ admits a hypercomplex structure if and only if $\mathfrak{g}$ corresponds to the complex hyperbolic space $\mathbb{C}H^2$. Moreover, the equivalence classes of hypercomplex structures on $\mathfrak{g}$ are parametrized by the space $\mathbb{RP}^2$.

This is the end of the road for this section, and for the current investigation of the work in [13]. Worth noting is that this particular paper actually has one other section which serves to classify hyperhermitian metrics on 4-dimensional Lie groups. Due to the lack of attention paid in this work to hyperhermitian metrics, these results are omitted. It would be unfortunate not to mention that the author of [13] *does* have a number of other publications which *are* more applicable to the current exposition but which themselves are omitted for the sake of brevity (see section 2.5.3 below). For example, the author of [13] is also a contributing author on [6], a paper classifying all Abelian complex structures on 6-Dimensional Lie algebras; naturally this work is more fitting in the context of this discussion, but because of the array of other topics even *more* fitting, it, too, has been omitted. The diligent reader is encouraged to seek out this author's entire body of work[27] as it's nothing short of a treasure trove of clever insight and perspective.

2.5.3 Lie Theory: Some Closing Remarks

Nearly all of the results given above come from [13], [45], and sources therein. This is done in part because of the need to remain brief (hence, selective) and in part because of the preferences of the author. Worth noting, though, is that this seemingly narrow viewpoint should in no way be interpreted as an indication of how much literature addresses the intersection of hypercomplex geometry and Lie theory or of how many results have been proven in these areas. The purpose of this *addendum* is to both address some of these results and to hopefully represent the vastness of what exists beyond the scope of this paper with a bit more accuracy.

[26]The three-dimensional Heisenberg algebra $\mathfrak{h}_3$ consists of square matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$. This can be extended to $2n+1$ dimensional analogues $\mathfrak{h}_{2n+1}$ by replacing $a, b \in \mathbb{R}$ by $\boldsymbol{a} \in \mathbb{R}^{1\times n}$, $\boldsymbol{b} \in \mathbb{R}^{n\times 1}$, and by replacing the central "1" with the $n \times n$ identity matrix I_n.

[27]A somewhat complete list of the publications of Barberis can be found at

`http://www.famaf.unc.edu.ar/~barberis/pub.htm`.

By all accounts, Joyce's paper [45] was one of the foremost of its kind when it was first authored. Nearly any paper on hypercomplex geometry which also discusses Lie group theory cites either [45] or one of the foundational results upon which Joyce elaborated. Later, [13] was published to comparably high reception, serving as one of the most fundamental results for classifying 4-dimensional hypercomplex Lie groups based on structural properties of their corresponding Lie algebras. Since that time, the literature has grown significantly. For example, a great deal of work has been done with hypercomplex structures with respect to 4-dimensional Lie groups. In [54] and [55], for example, this topic is considered from the perspective of differential geometry including analysis of various topics such as scalar curvature, connections, and so-called *Randers metrics*. An even more vast library of progress can be found by analyzing the sources cited therein, etc.

Like the study of Lie groups, a considerable amount of literature has emerged regarding hypercomplex structures on $4n$-dimensional Lie algebras as well, many coming also from a variety of perspectives. For example, [14] looks at hypercomplex structures on nilpotent and solvable Lie groups while other papers such as [15] and [19] investigate the interactions of Lie algebras which satisfy certain properties with both (hyper)complex structures and their generalizations. This branch of study is particularly fruitful because, unsurprisingly, the constant evolution of perspective—starting at complex geometry, going to hypercomplex and quaternionic geometries, and now advancing to topics such as Clifford geometry—has led to an always-expanding collection of literature. There really is no way to convey how much knowledge is out there.

In addition to its shortcomings depth-wise, an observant reader can probably deduce that this paper's firm separation of Lie groups from Lie algebras is, at best, artificial. For example, both [45] and [13] use a great deal of Lie algebra mechanics to produce their results, and despite [13] being included in the section on Lie *algebras*, her paper is actually rooted in a Lie *group* classification. Simply put, the interplay between these two structures is indivisible, as is illustrated by the above exposition, and that any classification of Lie groups will essentially require extensive examination of the closely-related properties of its associated Lie algebra.

The fact of the matter is that an entire manuscript could be devoted to the need for a more thorough and well-designed survey of hypercomplex structures in Lie theory; the iceberg is so tall that even its tip is barely coverable. In short, there's no purely worthwhile way to cover the enormity of Lie theory—even when viewed as its intersection with hypercomplex geometry—within such a short treatise. Here, the author errs on the side of brevity, and with the exception of a forthcoming discussion related to [19] in section 3 below, the focus of the remainder of the paper will shift almost completely away from Lie theory altogether.

2.6 Conclusion and *General Closing Remarks*

The current section on almost-hypercomplex and hypercomplex structures is, by far, the most significant part of this exposition. It was stated early on that the purpose of this section was to give an overall survey of the current status of the classification of all hypercomplex structures on manifolds (and associated structures), and while a somewhat vast chunk of the landscape has been presented, the amount that hasn't been touched on is indubitably more

vast still.

For example, Pedersen [59] accomplishes a classification result similar to Joyce [45] [46], listing explicit homogeneous hypercomplex structures on a variety of structures, and takes the bar one level higher by constructing hypercomplex structures on several other spaces such as $(S^3 \times S^1)^n$, the associated bundle $\mathcal{U}(M)$ on quaternionic manifolds M, and the so-called *Swann bundle* $\mathcal{V}(M) = \mathcal{U}(M)/\mathbb{Z}$. He then goes on to consider hypercomplex structures on more advanced structures derived from instantons, the most notable of which is the so-called *Twisted Swann Bundle* $\mathcal{V}_P(M) = P \times S^1\mathcal{V}(M)$ associated to an S^1-instanton P on M. This work is extended even farther in [60].

Then, of course, there's the indescribable amount of literature produced by authors Boyer, Galicki, and Mann. Some of their work is touched upon here (see [21], [22], [24], [25], [26], [27], [28], [27], [36], etc.), but nothing said here even begins to describe the depth or breadth of the results they've derived. Moreover, a glimpse at the publication list of any of those three authors will uncover no fewer than two dozen other papers whose results couldn't even be mentioned here for fear of this project getting even more out of hand than it's already gotten. To say that their work in the field is second-to-none is hardly an understatement.

So, the point of this section is to re-emphasize, on a more global scale, the sentiments in section 2.5.3 above: The stuff that exists *out there* is so vast and deep and amazing that this paper hardly does any of it justice. The interested reader is urged, strongly, to use this exposition as nothing more than a vessel by which to get a flavor of what's being done. Hoping for anything more than that is expecting far too much.

3 Some Stuff on Cliffordian Structures

Overall, the consideration of so-called *Cliffordian Structures* on manifolds is a relatively new perspective. In particular, the literature seems sparse and not entirely consistent notationally. For that reason, any exposition on this topic will require a considerable amount of foundation. That's where the exposition begins.

3.1 Preliminaries, Definitions, and Notation

Before jumping into any worthwhile conversation on Cliffordian geometry, it's necessary to start with some fundamentals. Perhaps the most elementary fundamental that should be addressed is the complete lack of consistency in studying structures referred to as *Cliffordian structures*. As noted in [56], several approaches to the concept of Clifford (or Cliffordian; the terms will be used interchangeably throughout) structures on manifolds can be found throughout the literature and, indeed, the same terminology is often used to describe entirely different circumstances. Some such notions have been:

- Several (or, according to [56], *most*) authors use the phrase *Clifford structures* to indicate a collection of global almost complex structures which satisfy the algebraic relations of the basis generators of $\mathcal{C}\ell_{0,n}$. These are sometimes referred to as *flat Clifford structures*.

- The phrase *Clifford structures* has been used in much of the literature related to the so-called *Osserman Conjecture*; in this context, the term refers to a structure on a Riemannian manifold (M, g) similar to the above-mentioned flat Cliffordian structures with additional assumptions placed on the Riemannian curvature tensor.

- In [30], the author makes use of the phrase *Clifford-Kähler manifold* to describe something related to a collection of *local* almost complex structures obtained from a particular Clifford algebra bundle and from local orthonormal frames of the Clifford bundle. This notion is related to what's sometimes called an *even Cliffordian structure*.

- Finally, in [56], the term *Clifford structure* refers to a structure on a Riemannian manifold (M, g) which has even Cliffordian structures as a special case. Similar to the description immediately above, the idea here is to study a special Euclidean vector bundle (E, h) over M called the *Clifford bundle* along with a representation of the so-called *Clifford algebra bundle* $\mathcal{C}\ell(E, h)$ (see above) on the tangent bundle TM. In this context, the notion of *even Cliffordian structure* corresponds to a subbundle $\mathcal{C}\ell^0(E, h)$ of the bundle mentioned above.

As if this weren't confusing enough, there's also the fact that some authors use terms *other than* "Cliffordian" to refer to structures related to $\mathcal{C}\ell_{p,q}$ for specific values of p and q. For example, as mentioned above, [19] uses the term "para-hypercomplex" to refer to geometric structures associated to $\mathcal{C}\ell_{1,1}$, while these same structures are referred to as "split-quaternionic" by the authors of [8], [7], etc. Suffice it to say, there's absolutely no uniformity abound whatsoever, so the best possible methodology to present any sort of snapshot of the current state of the subject at large is to do so in a somewhat piecemeal fashion with a variety of stand-alone expositions minus the typically sought-after cohesiveness. That'll be the strategy adapted below.

3.2 "Flat" Cliffordian Structures and Related Topics

As mentioned above, there are a number of various notions of Cliffordian structures. Inarguably, however, the so-called *flat* Clifford structure appears to be the most immediate generalization of, e.g., hypercomplex geometry. For that reason, this particular avenue seems like it could be the most readily-understood place to start. The majority of the information found herein will come first from [47] and then from [16]. The section begins with generalities.

To begin, consider a manifold M^n with an associated G-structure (see section 2.1), G a Lie subgroup of $\mathrm{GL}(n)$. By a *geometric structure* on M, one means a G-structure on M which satisfies some (possibly trivial) partial differential equation called an *integrability condition* involving only the G-structure. Examples include the Riemannian metric ($G = \mathrm{O}(n)$ with trivial integrability condition), orientation ($G = \mathrm{GL}_+(n)$ with trivial integrability condition) and complex and hypercomplex structures ($G = \mathrm{GL}(n, \mathbb{C})$ and $G = \mathrm{GL}(n, \mathbb{H})$, respectively, satisfying, e.g., the vanishing Nijenhuis tensor condition). These ideas can be combined in

order to define a structure which is closely related to the Clifford algebra $\mathcal{C}\ell_{0,n}$. It's easily verified that this construction yields the sequence given in equation (2.4.7) above.

Definition 3.1. Let B be a subset of

$$\{j \in \mathrm{GL}(2n) \,:\, j^2 = -1\} \subset \mathrm{GL}(2n)$$

and let G be the subgroup of $\mathrm{GL}(2n)$ of the form

$$G = \{x \in \mathrm{GL}(2n) \,:\, xj = jx \text{ for all } j \in B\}.$$

Here, a G-structure on a manifold M^{2n} induces an almost complex structure J on M for every element j of B, and so one can define the *geometric structure associated to* B to be the G-structure along with the integrability condition that every almost complex structure induced by an element of B be integrable.

The connection between definition 3.1 and the usual Clifford algebra $\mathcal{C}\ell_{0,n}$ seems immediate enough. This connection is written about more succinctly in [47] given some machinery related to algebras and modules which is summarized briefly as follows: Given $B \subset \mathrm{GL}(2n)$ as above, define A to be the subalgebra of $M(2n, \mathbb{R})$ generated over $\mathbb{R}$ by B so that $B \subset \{a \in A \,:\, a^2 = -1\}$ and so that $\mathbb{R}^{2n}$ is a natural A-module; moreover, one can conversely construct from any A-module a geometric structure associated to $B \subset \{a \in A \,:\, a^2 = -1\}$ for a unital algebra A. This gives an explicit connection between unital algebras and sets B inducing geometric structures; it also motivates a couple very immediate examples.

Examples 3.2.

1. When $A = \mathbb{H}$ and $B = \{i, j, k\}$, the A-module $\mathbb{H}^n$ gives the hypercomplex structure in dimension $4n$.

2. (Flat) Geometric structures related to Clifford algebras $\mathcal{C}\ell_{0,n}$ come about in this way as well. Briefly, let $V = \mathbb{R}^n$ with the usual metric $|\cdot|$ and let $T^n = \bigoplus_{i=0}^{\infty} \otimes^i V$ where $\otimes^0 V = \mathbb{R}$ and where multiplication is by tensor products in the obvious way. Defining I_n to be the two-sided ideal of T_n generated by elements of the form $x \otimes x + |x|^2 \cdot 1$, one can construct $\mathcal{C}\ell_{0,n}$ as the quotient $\mathcal{C}\ell_{0,n} = T_n / I_n$.

   Now, given an orthonormal basis $(j_1, \ldots, j_n)$ of V, it follows that the elements j_k are elements of $\mathcal{C}\ell_{0,n}$ which satisfy $j_k^2 = -1$ and $j_k j_\ell = -j_\ell j_k$ for $k \neq \ell = 1, 2, \ldots, n$. Thus, setting $A = \mathcal{C}\ell_{0,n}$ and $B = \{j_1, \ldots\}$ shows that any A-module gives rise to a geometric structure consisting of n anticommuting complex structures; conversely, any such geometric structure comes from an A-module as well. Finally, note that exposition in [9] classifying all modules over Clifford algebras in turn classifies all such geometric structures composed of these anticommuting complex structures. Integrability comes from the vanishing Nijenhuis tensor condition.

A number of algebraic properties of Clifford algebras $\mathcal{C}\ell_{0,n}$ are given in [47], [9], and also in [16]. Because those results are largely outside the scope of the current line of study, they'll be mostly omitted except when they're pertinent herein. Now, focus shifts towards the results in [16], an article which focuses on the above-detailed Cliffordian structures as they exist on families of manifolds and Lie algebras. The geometry utilized therein is built from the machinery discussed above. For the sake of clarity, the terminology is explicitly stated before continuing.

Definition 3.3. A *Clifford structure* on a connected differentiable manifold M is a family $\{J_\alpha\}_{\alpha\in I}$ of anticommuting complex structures where I is some index set.

In particular, any manifold M possessing a Clifford structure necessarily has the structure of a Clifford $\mathcal{C}\ell_{0,n}$-module on T_pM for each $p \in M$. Whenever this action is faithful, M is said to have a *Clifford structure of order* n or a $\mathcal{C}\ell_{0,n}$*-structure*, and such an M is sometimes said to be a $\mathcal{C}\ell_{0,n}$-manifold. The quintessential examples of Clifford structures are examples discussed elsewhere in this paper: A Clifford structure of order 1 is a complex structure and a Clifford structure of order 2 is a hypercomplex structure. Also, in the same vein:

Definition 3.4. An *automorphism* of the Clifford structures on a manifold M is a diffeomorphism of M which is holomorphic with respect to J_α, $1 \leq \alpha \leq n$.

Barberis [16] goes on to construct several nontrivial examples, some of which will be discussed below.

3.2.1 Clifford Structures on Some Lie Groups

Related to definition 3.3 above is the notion of a Clifford structure on a Lie group. In this scenario, the following definitions are important.

Definitions 3.5.

1. A Clifford structure on a connected Lie group G is said to be *invariant* if, for each $x \in G$, left translation by x is holomorphic with respect to J_α, $\alpha = 1, \ldots, n$.

2. A *Clifford structure of order* n or a *Clifford* $\mathcal{C}\ell_{0,n}$*-structure* on a real Lie algebra $\mathfrak{g}$ is a family $\{J_\alpha\}_{\alpha\in I}$ of endomorphisms of $\mathfrak{g}$ satisfying for all $1 \leq \alpha, \beta \leq n$:

   (a) $J_\alpha^2 = -1, J_\alpha J_\beta + J_\beta J_\alpha = 0$ for $\alpha \neq \beta$.

   (b) $N_{J_\alpha}(X, Y) = 0$ for all $X, Y \in \mathfrak{g}$ where here, N_{J_α} is the Nijenhuis tensor for J_α:

   $$N_{J_\alpha}(X, Y) = [J_\alpha X, J_\alpha Y] - J_\alpha[J_\alpha X, Y] - J_\alpha[X, J_\alpha Y] - [X, Y].$$

   (c) The subalgebra of $\mathrm{End}(\mathfrak{g})$ generated by $\{J_\alpha\}_{\alpha\in I}$ has dimension 2^n.

3. A real Lie algebra $\mathfrak{g}$ is a 2-step nilpotent Lie algebra if $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$.

Remarks.

1. Item 2(b) in the above list is the integrability constraint on the Clifford structure. Oftentimes, the *commuting integrability condition* $[JX, JY] = [X, Y]$ for all $X, Y \in \mathfrak{g}$ is used in its place. This new condition is even stronger, as it implies that $[JX, Y] + [X, JY] = 0$ and hence that $N_J \equiv 0$.

2. In the case of $m \in \{1, 2\}$, condition 2(c) is guaranteed. This goes out the window for $m \geq 3$.

Much of the focus of Clifford structures on Lie groups in [16] concern such structures on the family of 2-step nilpotent Lie algebras defined in item 3 of definition 3.5 above. In that case, one defines an associated space $d\mathfrak{g} \stackrel{\text{def}}{=} \mathfrak{g} \oplus \mathfrak{g}$ as well as a bracket on $d\mathfrak{g}$:

$$[\boldsymbol{X}, \boldsymbol{Y}] = [(X_1, X_2), (Y_1, Y_2)] \stackrel{\text{def}}{=} ([X_1, Y_1] + [X_2, Y_2], 0). \tag{3.2.1}$$

Finally, let J denote the endomorphism of $d\mathfrak{g}$ given by $J(X_1, X_2) = (-X_2, X_1)$ and satisfying $J^2 = -1$. From these, a very nice result emerges.

Proposition 3.6. If $\mathfrak{g}$ is 2-step nilpotent, then $d\mathfrak{g}$ is 2-step nilpotent and J is integrable. Moreover, for all $n \geq 1$, $d^n\mathfrak{g} = d(d^{n-1}\mathfrak{g})$ is 2-step nilpotent and carries a $\mathcal{C}\ell_{0,n}$ structure.

Proof Sketch. The first two claims follow immediately from the definitions of $d\mathfrak{g}$ and J, along with definitions 3.5 above. The final claim follows almost immediately by induction on n. □

To show the usefulness of proposition 3.6:

Example 3.7. Let $\mathfrak{g} = \mathfrak{h}_n$ be the $(2n+1)$-dimensional Heisenberg Lie algebra associated to the corresponding simply connected nilpotent Lie group H_n as discussed briefly in section 2.5.2 above. $\mathfrak{h}_n$ has a basis of the form $\{Z, X_1, Y_1, \ldots, X_n, Y_n\}$ with bracket satisfying $[X_i, Y_j] = \delta_{i,j} Z$ and $[Z, X_j] = [Z, Y_j] = 0$, $1 \leq i, j \leq n$. One easily verifies that $d\mathfrak{h}_n = \mathbb{R} \oplus \mathfrak{h}_{2n}$ and so

$$d^k\mathfrak{h}_n = \mathbb{R}^{2^k-1} \times \mathfrak{h}_{2^k n}.$$

It follows from proposition 3.6 above (along with arguments similar to those in section 2.5 showing that geometric structures on Lie algebras induce structures on the corresponding Lie groups) that the associated Lie group $\mathbb{R}^{2^k-1} \times H_{2^k n}$ admits a $\mathcal{C}\ell_{0,k}$-structure.

3.2.2 Clifford Structures on Compact Flat Manifolds

In much the same way that certain 2-step nilpotent Lie groups/algebras possess natural Clifford structures, so too do some compact connected flat Riemannian manifolds. The purpose of the exposition below is to shed some light on that fact. Worth noting is that if (M, g) is a compact connected flat Riemannian manifold—referred to imprecisely as a"Riemannian manifold" from this point forward unless otherwise mentioned—then the universal covering

space of M is a Euclidean space $\mathbb{R}^n$ while the fundamental group of M is a so-called *Bieberbach group*[28]. These properties, well-known as they may be, will be central to the discussion that follows.

Throughout, let L_v denote translation by an arbitrary element $v \in \mathbb{R}^n$, noting that for a crystallographic group (see footnote 28) Γ, the collection $\Lambda = \{v : L_v \in \Gamma\}$ is a lattice in $\mathbb{R}^n$ commonly (albeit imprecisely) identified according to $\Lambda \sim \{L_v : v \in \Lambda\}$. Though sloppy, this identification has its advantages, namely that the set $\{L_v : v \in \Lambda\}$ is a normal and maximal Abelian subgroup of Γ; in particular, this fact allows one to consider the quotient $F = \Lambda/\Gamma$ which in turn is a finite group known as *the point group of* Γ. The point group F takes on a special interpretation geometrically whenever Γ is torsion-free, namely that of the linear holonomy group of M. Much of the discussion that follows centers on results in holonomy theory including several new results in [16] itself. In particular, the goal moving forward will be to prove an analogue related to $\mathcal{C}\ell_{0,n}$-structures of a theorem of Auslander and Kuranishi [10] stating that any finite group arises as the holonomy group of a compact flat manifold.

The main piece of machinery needed to conclude this result is stated with proof in [16] as a theorem[29]; for this line of investigation, the sought-after result is a corollary of the aforementioned theorem, whereby it makes sense to state said theorem as a *lemma* and to present what's most pressing here as a theorem with proof. Before doing so, however, some terminology is in order. Throughout, let Γ be a crystallographic subgroup of $I(\mathbb{R}^n)$ (see footnote 28 again) with translation lattice Γ.

Definitions 3.8.

1. A finite group F which occurs as the holonomy group of a compact flat manifold having first Betti number zero is said to be *primitive*.

2. Writing $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ and noting that $\Lambda \oplus \Lambda$ is then a lattice of $\mathbb{R}^{2n}$, define $\Delta : I(\mathbb{R}^n) \to I(\mathbb{R}^{2n})$ as diagonal embedding. Let $\langle S \rangle$ denote the subgroup generated by a subset $S \subset I(\mathbb{R}^k)$ for k arbitrary. With this in place, define *the double* $d\Gamma \subset I(\mathbb{R}^{2n})$ of Γ by
$$d\Gamma = \langle \Delta\Gamma, L_{\Lambda\oplus\Lambda} \rangle$$
and likewise define $dM = d\Gamma/\mathbb{R}^{2n}$. Here, L_K denotes the collection $L_K = \{L_v : v \in K\}$ for a subset $K \subset \mathbb{R}^d$.

With this in place, the previously-mentioned lemma can be presented.

Lemma 3.9. If Γ is a crystallographic subgroup of $I(\mathbb{R}^n)$ with translation lattice Λ, holonomy group F, and holonomy representation τ, then $d\Gamma$ is a crystallographic subgroup of $I(\mathbb{R}^{2n})$ with translation lattice $\Lambda \oplus \Lambda$, holonomy group F, and holonomy representation $\tau \oplus \tau$. Moreover, $d\Gamma$ is torsion-free if and only if Γ is, and so the following hold:

[28] A Bieberbach group is a discrete cocompact group of isometries on $\mathbb{R}^n$ which is torsion-free. Subgroups of the isometry group $I(\mathbb{R}^n)$ which are discrete and cocompact are said to be *crystallographic groups*.

[29] Theorem 3.1.

1. If $\mathbb{R}^n$ carries a translation invariant $\mathcal{C}\ell_{0,n}$- structure which commutes with the action of F, then $d\Gamma/\mathbb{R}^{2n}$ carries a $\mathcal{C}\ell_{0,n+1}$-structure. This implies that for any Γ, $d^2\Gamma/\mathbb{R}^{4n}$ is necessarily hyperkähler.

2. $H_1(d^m M, \mathbb{Z}) \simeq (\Lambda/\Lambda_0)^{2^n-1} \times H_1(M, \mathbb{Z})$ and $\beta_1(d^n M) = 2^n \beta_1(M)$. Here,
$$\Lambda_0 = [\Gamma, \Lambda] = \langle r(\gamma)\lambda - \lambda \,:\, \gamma \in \Gamma, \lambda \in \Lambda \rangle$$
for a specific transformation $r(\gamma) \in \mathrm{O}(n)$.

The proof of lemma 3.9 is neither intuitive nor insightful, but its result links the class of Riemannian manifolds having Cliffordian structures to those manifolds characterized by Auslander-Kuranishi. In particular:

Theorem 3.10. Any finite group F occurs as the holonomy group of a Clifford $\mathcal{C}\ell_{0,n}$-flat manifold. Moreover, if F is primitive, then F occurs as the holonomy of a Clifford $\mathcal{C}\ell_{0,n}$-flat manifold with $\beta_1 = 0$.

Proof Sketch. By the original theorem of Auslander and Kuranishi [10], there exists a Bieberbach group $\Gamma \subset I(\mathbb{R}^n)$ with prescribed point group F. Lemma 3.9 implies that $d^m M$ is a flat manifold of dimension $2^m n$ with holonomy group F and having a Clifford structure of order m, thus proving the first assertion. The second assertion comes from the second item of lemma 3.9. □

And now, a lá proposition 3.6 above, consider examples constructed from the results of lemma 3.9 and theorem 3.10 above.

Examples 3.11.

1. If $\mathcal{K}$ denotes the Klein bottle, then $\mathcal{K} = \Gamma/\mathbb{R}^2$ where $\Gamma = \langle \gamma, L_\lambda \,:\, \lambda \in \Lambda \rangle$ with Λ the cannonical lattice in $\mathbb{R}^2$, $\gamma = \sigma L_{e_2/2}$, and $\sigma(x, y) = (-x, y)$. The translation lattice of Γ is Λ and the holonomy group is $F = \mathbb{Z}_2$; moreover, $\gamma^2 = L_{e_2}$ and $[\Gamma, \Lambda] = 2\mathbb{Z}e_1$, whereby it follows that $\Lambda/[\Gamma, \Lambda] \simeq \mathbb{Z} \oplus \mathbb{Z}_2$. Thus, setting $\Gamma_m = d^m\Gamma$, $\mathcal{K}_m = d^m\mathcal{K}$, then the lemma above yields that
$$H_1(\mathcal{K}_m, \mathbb{Z}) = \mathbb{Z}^{2^m} \times \mathbb{Z}_2^{2^m}.$$
One can also prove a general Betti number formula of the form
$$\beta_k(\mathcal{K}_m) = \sum_{j=0}^{2^{m-1}} \binom{2^m}{k-2j}\binom{2^m}{2j}, \qquad \binom{l}{m} \stackrel{\text{def}}{=} 0 \text{ when } m < 0.$$

2. A second example begins with a far more exotic manifold. Let $\mathcal{H}$ denote the so-called *Hantzsche-Wendt manifold*, defined qualitatively[30] to be the only flat closed

[30] Its quantitative definition is a bit more difficult [44]: It's the orbit space
$$\mathcal{H} = \mathbb{R}^3/\langle x, y, z \,:\, xy^2x^{-1}y^2 = yx^2y^{-1}x^2 = 1,\, z = xy \rangle$$
where $x = (\frac{1}{2}e_1, X)$, $y = (\frac{1}{2}(e_2 - e_3), Y)$, and $z = (\frac{1}{2}(e_1 - e_2 + e_3), Z)$ for $X = \mathrm{diag}[1, -1, 1]$, $Y = \mathrm{diag}[-1, 1, -1]$, and $Z = \mathrm{diag}[-1, -1, -1]$. In particular, $\mathcal{H}$ is the quotient of $\mathbb{R}^3$ by a subgroup of the group $\mathrm{Aff}(\mathbb{R}^3)$ of affine motions of 3-space.

3-manifold (out of 6) whose holonomy *isn't* cyclic. Setting $\mathcal{H}_m = d^m\mathcal{H}$, one can show (though it's completely unclear how) that $\Lambda/[\Gamma,\Lambda] \simeq \mathbb{Z}_2^3$ and that $H_1(\mathcal{H},\mathbb{Z}) \simeq \mathbb{Z}_4^2$. In particular, lemma 3.9 shows that $\mathcal{H}_m$ is a compact flat manifold of dimension $2^m \cdot 3$ with $H_1(\mathcal{H}_m,\mathbb{Z}) \cong \mathbb{Z}_4^2 \times \mathbb{Z}_2^{3(2^m-1)}$, holonomy group $\mathbb{Z}_2^2$, admitting a Clifford structure of order m.

What's more, $\mathcal{K}_2$ and $\mathcal{H}_2$ are hyperkähler manifolds of dimensions 8 and 12, respectively, with holonomy groups $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$, respectively.

The remainder of [16] focuses on Cliffordian structures on what're known as solvable extensions of so-called H-type groups. In that section[31], the authors construct explicitly these Clifford structures using a number of Lie algebra techniques and a significant amount of machinery from differential geometry. Because even the basis of that material is beyond the scope of this paper, the author has chosen to omit coverage of the remainder of [16], thus marking the end of the coverage on this particular brand of Cliffordian structures. Moving forward, other so-called Clifford structures will be considered (at best) topically with details (at best) sparsely distributed as necessary.

3.3 Clifford Structures on Riemannian Manifolds

One of the most thorough treatments of this brand of Clifford geometry comes from [56], and so a majority of what follows comes from exposition within that article. Before even stating the core definitions from [56], a bit of background is necessary; for that, ideas borrowed from [37] are presented first.

A vector bundle ξ is called an *algebra bundle* if both each fiber F_x and the typical fiber F are (perhaps non-associative) algebras and if ξ admits a coordinate representation $\{(U_\alpha,\psi_\alpha)\}$ such that each map

$$\psi_{\alpha,x} : F \xrightarrow{\cong} F_x$$

is an isomorphism of algebras. In the event that the fibers F_x, F have the structure of Clifford algebras $\mathcal{C}\ell_{p,q}$ and the maps $\psi_{\alpha,x}$ respect this algebraic structure, the bundle ξ is said to be a *Clifford algebra bundle*.

This construction can be framed in a way more naturally-associated to Riemannian geometry by considering an arbitrary Riemannian vector bundle[32] $\pi : E \to X$. Here, the fiber $E_x = \pi^{-1}(x)$ corresponding to each $x \in X$ admits a quadratic form $g_x \overset{\text{def}}{=} \|\cdot\|^2 = \langle\,\cdot\,,\,\cdot\,\rangle$ which can then be used to construct a pointwise Clifford algebra $\mathcal{C}\ell(E_x,g_x)$ for each $x \in X$, structures which can then be glued smoothly to form an overall Clifford bundle[33] $\mathcal{C}\ell(E) \to X$ where $\mathcal{C}\ell(E) = \coprod_x \mathcal{C}\ell(E_x,g_x)$. Of course, the inclusion of the adjective "Riemannian" really means unneeded specificity: Indeed, one can consider for any smooth vector bundle $\pi : E \to X$ a smooth map $\langle\,\cdot\,,\,\cdot\,\rangle : E \oplus E \to \mathbb{F}$ whose restriction to each fiber $E_x \oplus E_x$ is an

[31] Section 4.

[32] Every real vector bundle admits a Riemannian metric [76].

[33] Sometimes, $\mathcal{C}\ell(E)$ is written $\mathcal{C}\ell(E,g)$ where here, g is the glued-together form $g = \coprod_{x\in X} g_x$. The g will generally be omitted unless referencing it is particularly useful.

inner product on E_x. When the inner product is real, it's bilinear, symmetric, and positive definite, and the pair $(E, \langle \cdot\,, \cdot \rangle)$ is said to be a Euclidean vector bundle; in the complex case, $\langle \cdot\,, \cdot \rangle|_{E_x \oplus E_x}$ is positive definite and sesquilinear and the pair $(E, \langle \cdot\,, \cdot \rangle)$ is called a Hermitian vector bundle. The Hermitian case will be mostly absent from the conversation moving forward.

Now that this machinery is in-place, consider the following definitions for $M = (M, g)$ a Riemannian manifold of dimension n.

Definitions 3.12.

1. A *rank r Clifford structure* on M is an oriented rank r Euclidean bundle (E, h) over M together with a non-vanishing algebra bundle morphism $\varphi : \mathcal{C}\ell(E, h) \to \mathrm{End}(TM)$ called *the Clifford morphism* which maps E into the bundle of skew-symetric endomorphisms $\mathrm{End}^-(TM)$.

2. A Clifford structure (M, g, E, h) is called *parallel* if the subbundle $\varphi(E)$ of $\mathrm{End}^-(TM)$ is parallel with respect to the Levi-Civita connection $\nabla = \nabla^g$ of (M, g).

3. For $n \geq 2$, a *rank r even Clifford structure* on (M, g) is an oriented rank r Euclidean bundle (E, h) over M toether with an algebra bundle morphism $\varphi : \mathcal{C}\ell^0(E, h) \to \mathrm{End}(TM)$ called (again) *the Clifford morphism* which maps $\Lambda^2 E$ into the bundle $\mathrm{End}^-(TM)$ of skew-symmetric endomorphisms.

4. An even Clifford structure (M, g, E, h) is called *parallel* if there exists a metric connection ∇^E on (E, h) such that φ is connection preserving, i.e. so that

$$\varphi(\nabla^E_X \sigma) = \nabla^g_X \varphi(\sigma)$$

for every tangent vector $X \in TM$ and for every section σ of $\mathcal{C}\ell^0(E, h)$.

A number of the structures discussed elsewhere in the present exposition are Clifford structures of a certain rank r. For example, there is a one-to-one correspondence between rank 1 Clifford structures and almost Hermitian structures whereby a rank 1 Clifford structure is parallel if and only if the corresponding almost Hermitian structure is Kähler. In addition, every hyperkähler manifold (M^n, g, I, J, K) carries parallel rank 2 Clifford structures given by the subbundle of $\mathrm{End}^-(TM)$ generated by I and J. It seems obvious, then, that a classification of n-dimensional Riemannian manifolds admitting some kind of Clifford structure is something of a generalization of a lot of the discussion that's happened heretofore. Hashing out the particulars of such a classification will be the goal for the remainder of the section.

First, consider the following remarks. By and large, these concern structures presented in definition 3.12 and issues related thereto, and will be worthwhile things to have in mind moving forward.

Remarks.

1. The $\mathcal{C}\ell^0(E,h)$ in the third item of definition 3.12 is in reference to the splitting of an arbitrary Clifford algebra $\mathcal{C}\ell_{p,q}$ into even and odd parts, denoted $\mathcal{C}\ell^0_{p,q}$ and $\mathcal{C}\ell^1_{p,q}$, respectively. In particular, the Clifford bundle $\mathcal{C}\ell(E,g) \to X$ splits:
$$\mathcal{C}\ell(E,g)^0 \oplus \mathcal{C}\ell(E,g)^1 \longrightarrow X.$$

2. Item 3 of definition 3.12 above utilizes the fact that $\Lambda^2 E$ can be viewed as a subbundle of $\mathcal{C}\ell^0(E,h)$ by way of by identifying $e \wedge f$ with $e \cdot f + h(e,f)$ for every $e, f \in E$.

3. Two even Clifford structures (E_1, h_1, φ_1) and (E_2, h_2, φ_2) are isomorphic if there exists an algebra bundle isomorphism $\lambda : \mathcal{C}\ell^0(E_1,h_1) \to \mathcal{C}\ell^0(E_2,h_2)$ such that $\varphi_2 \circ \lambda = \varphi_1$.

4. Since the definition of even Clifford structures in item 3 of 3.12 above only involves the exterior power $\Lambda^2 E$, the bundle E itself is not part of an even Clifford structure. What's more, there exist isomorphic even Clifford structures with non-isomorphic bundles E.

5. The authors of [56] make it a point to emphasize that (even) Clifford structures are equivalent to reductions of the orthonormal frame bundle of M. Details are spelled out in that article but are omitted from this discussion.

6. For r even, the notion of an even Clifford structure of rank r admits a slight extension to the case where E a so-called *projective bundle*, that is, a locally defined vector bundle associated to some G-principal bundle via a projective representation $\rho : G \to \mathrm{PSO}(r)$. Because the extension of the standard representation of $\mathrm{SO}(r)$ from $\mathbb{R}^r$ to $\Lambda^2\mathbb{R}^r$ factors through $\mathrm{PSO}(r)$, the second exterior power of any projective vector bundle is a well-defined vector bundle[34]. The corresponding structure is often referred to as *the projective even Clifford structure*.

Now that these issues have been addressed, it makes sense to begin presenting some actual classification results. A good starting place for that endeavor is the following technical lemma.

Lemma 3.13. Let (E,h) be a rank r even Clifford structure and let $\{e_1, \ldots, e_r\}$ be a local h-orthonormal frame on E. The local endomorphisms $J_{ij} \stackrel{\text{def}}{=} \varphi(e_i e_j) \in \mathrm{End}(TM)$ are skew-symmetric for $i \neq j$ and satisfy:

$$\begin{cases} J_{ii} = -\,\mathrm{id} & \text{for all } 1 \leq i \leq r, \\ J_{ij} = -J_{ji} \text{ and } J_{ij}^2 = -\,\mathrm{id} & \text{for all } i \neq j \\ J_{ij} \circ J_{ik} = J_{jk} & \text{for all } i,j,k \text{ mutually distinct,} \\ J_{ij} \circ J_{k\ell} = J_{k\ell} \circ J_{ij} & \text{for all } i,j,k,\ell \text{ mutually distinct.} \end{cases}$$

In addition, for $r \neq 4$, one has that

$$\langle J_{ij}, J_{k\ell} \rangle = 0 \text{ unless } i = j, k = \ell, \text{ or } i = k \neq j = \ell, \text{ or } i = \ell \neq k = j.$$

[34] On the other hand, the projective vector bundle *is not*, in general, a vector bundle.

The proof of this lemma comes almost entirely from the multiplicative properties of Clifford algebra bases and so the details are skipped for succinctness. Yet another corollary of the structure of Clifford algebras is that every rank r Clifford structure E induces an even Clifford structure of the same rank. This follows by the fact that any local orthonormal frame $\{e_1, \ldots, e_n\}$ of E satisfies $\varphi(e_i \wedge e_j) = \varphi(e_i) \circ \varphi(e_j)$ whenever $i \neq j$, thus showing that $\varphi(e_i \wedge e_j)$ is indeed skew-symmetric. One can now easily verify part 3 of definition 3.12. The authors of [56] show that the converse holds if the rank of the bundle E is congruent to 3 modulo 4, but this is much harder to see and is much less useful overall.

What *is* useful overall is to examine aspects of this topic which will aid the overall goal of classification. The examples of Clifford structures of low rank discussed above are beneficial in that regard due to the fact that they help provide some intuition; this intuition is further developed in [56] by reframing these examples into the language of *even* Clifford structures[35]. In order to convert this intuition into explicit results, the following definition is needed.

Definition 3.14. A parallel even Clifford structure (M, E, ∇^E) is called *flat* if the connection ∇^E is a flat connection.

The first facet of the classification comes with respect to flat even Clifford structures of high rank.

Theorem 3.15. A complete simply connected Riemmanian manifold (M^n, g) carrying a flat even Clifford structure E of rank $r \geq 5$ is flat and hence is isometric with a $\mathcal{C}\ell^0_{0,r}$ representation space.

Outline of the Proof. Let $J_{ij} = \varphi(e_i e_j)$. The case of M irreducible follows from (i) the Ricci flatness of M stemming from M being hyperkähler with respect to J_{12}, J_{31}, and J_{23}, and (ii) Berger's classification of holonomy (see appendix 2.1 below). The general case follows by contradiction by applying the Bianchi identity to the de Rham decomposition $M = M_0 \times M_1 \times \cdots \times M_k$ into irreducible non-flat components M_i, $i > 0$. □

The classification is further advanced by the following proposition concerning non-flat even Clifford structures of various dimensions.

Proposition 3.16. Assume that the complete simply connected Riemannian manifold (M, g) carries a parallel non-flat even Clifford structure (E, ∇^E) of rank $r \geq 3$. Then the following results hold:

1. If $r = 4$, then (M, g) is a Riemannian product of two quaternion-Kähler manifolds (see appendix 2.3 below).

2. If $r \neq 4$ and $n \neq 8$, then:

   (a) The curvature of ∇^E as a map $\Lambda^2 M \to \mathrm{End}^-(E) \simeq \Lambda^2 E$ is a non-zero constant times the metric adjoint of the Clifford map φ.

[35] Examples 2.6 and 2.7.

(b) M is Einstein (see appendix 2.1 below) with non-vanishing scalar curvature and irreducible holonomy.

3. If $r \neq 4$ and $n = 8$, then (a) implies (b).

What's more, for any Riemannian manifold (M^n, g) satisfying the above hypotheses, the Lie algebra $\mathfrak{h}$ associated to the holonomy group H (which associated to some holonomy bundle P) splits as a direct sum of Lie algebras of the form $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{so}(r)$ for some Lie subalgebra $\mathfrak{g} < \mathfrak{h}$.

The proof of proposition 3.16 begins with two pages of hard differential geometry focusing on a number of properties of Riemannian curvature, connections on Riemannian manifolds, etc., and ending with a number of highfalutin results on the structure of Lie algebras. Interested readers should examine the proposition[36] in [56]. For the goal at hand, however, this proposition should be viewed as simply a tool to aid future results.

One other tool that will be beneficial in proving the objectively more interesting results that follow is the duality between the geometric and algebraic interpretations of the framework being examined presently. For example, a parallel rank $r \geq 3$ even Clifford structure, $r \neq 4$, on a simply connected Riemannian manifold (M^n, g) with holonomy group $H = \mathrm{Hol}(M)$ acting on $\mathbb{R}^n$ is equivalent to an orthogonal representation $\rho : H \to \mathrm{SO}(r)$ of H on $\mathbb{R}^r$ together with an H-equivariant algebra morphism $\phi : \mathcal{C}\ell^0_{0,r} \to \mathrm{End}(\mathbb{R}^n)$ which maps $\mathfrak{so}(r) \subset \mathcal{C}\ell^0_{0,r}$ into $\mathfrak{so}(n) \subset \mathrm{End}(\mathbb{R}^n)$. This result also holds for projective even Clifford structures by replacing all instances of orthogonal representations with projective ones, a fact that's particularly relevant due to the increased flexibility which comes with these projective structures.

With this added bit of perspective, consider the following objectively-significant[37] result focusing on the classification of non-flat even Clifford manifolds of high rank.

Theorem 3.17. A Riemannian manifold (M^n, g) carrying a parallel non-flat even Clifford structure (E, ∇^E) of rank $r \geq 5$ is either locally symmetric or is 8-dimensional.

After yet another colossal block of intense differential geometry, the authors obtain the conclusion demonstrated above. They then go one step farther, cross-referencing the complete classification of compact locally symmetric spaces (see [18], e.g.) to conclude a complete classification of Riemannian manifolds carrying first parallel *even* Clifford structures and then parallel Clifford structures in general. The complete results are as follows.

Theorem 3.18. The list of complete simply connected Riemannian manifolds M carrying a parallel rank r *even* Clifford structure is given in the tables below[38]. Here, $q = \dim(M)$,

[36] Proposition 2.10 + Corollary 2.12.

[37] This theorem—Theorem 2.13 of [56]—is said to be the first "important result" by the authors themselves.

[38] As noted by the authors of [56]: "In this table we adopt the convention that the QK condition is empty in dimension 4. For the sake of simplicity we have omitted the non-compact duals of the symmetric spaces in Table 2. The meticulous reader should add the spaces obtained by replacing $\mathrm{Sp}(k+8)$, $\mathrm{SU}(k+4)$, $\mathrm{SO}(k+8)$, $\mathbb{F}_4$, E_6, E_7 and E_8 in the last seven rows with $\mathrm{Sp}(k,8)$, $\mathrm{SU}(k,4)$, $\mathrm{SO}_0(k,8)$, $\mathbb{F}_4^{-20}$, E_6^{-14}, E_7^{-5} and E_8^{8} respectively."

$q_i = \dim(M_i)$ in the case of reducible manifold products, and $N_0(r)$ is the dimension of an irreducible $\mathcal{C}\ell^0_{0,r}$ representation.

r	M	dimension of M
2	Kähler	$2m,\ m \geq 1$
3	Hyperkähler	$4q,\ q \geq 1$
4	Reducible hyperkähler $M_1 \times M_2$	$4(q_1 + q_2),\ q_1, q_2 \geq 1$
Arbitrary	$\mathcal{C}\ell^0_r$ representation space	Multiple of $N_0(r)$

Table 1

Manifolds with flat even Clifford structure

r	Type of E	M	Dimension of M
2		Kähler	$2m,\ m \geq 1$
3		Quaternion-Kähler (QK)	$4q,\ q \geq 1$
4	Projective	Product of two QK manifolds	$4q,\ q \geq 1$
5		QK	8
6	Projective if M non-spin	Kähler	8
7		Spin(7) holonomy	8
8	Projective if M non-spin	Riemannian	8
5		$\mathrm{Sp}(k+2)/\,\mathrm{Sp}(k) \times \mathrm{Sp}(2)$	$8k,\ k \geq 2$
6	Projective	$\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(4))$	$8k,\ k \geq 2$
8	Projective if k odd	$\mathrm{SO}(k+8)/\,\mathrm{SO}(k) \times \mathrm{SO}(8)$	$8k,\ k \geq 2$
9		$\mathbb{OP} = \mathbb{F}_4/\,\mathrm{Spin}(9)$	16
10		$(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_6\,/\,\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	32
12		$(\mathbb{H} \otimes \mathbb{C})\mathbb{P}^2 = \mathrm{E}_7\,/\,\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$	64
16		$(\mathbb{C} \otimes \mathbb{C})\mathbb{P}^2 = \mathrm{E}_8\,/\,\mathrm{Spin}(16)$	128

Table 2

Manifolds with parallel non-flat even Clifford structure

And now, the result regarding parallel rank r Clifford structures in the general case:

Theorem 3.19. A simply connected Riemannian manifold (M^n, g) carries a parallel rank r Clifford structure if and only if one of the following (non-exclusive) cases occur:

1. $r = 1$ and M is Kähler.
2. $r = 2$ and either $n = 4$ and M is Kähler or $n \geq 8$ and M is hyper-Kähler.
3. $r = 3$ and M is quaternion-Kähler.
4. $r = 4$, $n = 8$ and M is a product of two Ricci-flat Kähler surfaces.
5. $r = 5$, $n = 8$ and M is hyper-Kähler.
6. $r = 6$, $n = 8$ and M is Kähler Ricci-flat.
7. $r = 7$ and M is an 8-dimensional manifold with Spin(7) holonomy.
8. r is arbitrary and M is flat, isometric to a nontrivial representation of the Clifford algebra $\mathcal{C}\ell_{0,r}$.

After stating the above results[39], a great deal of additional work is done by the authors of [56] proving results which are largely differential geometric. In particular, the last section of that paper[40] focuses on Riemannian submersions $\pi : (Z^{k+n}, g_Z) \to (M^n, g_M)$ and on bundle-theoretic properties thereof. Despite seeming completely unrelated to the previous work, this particular line of reasoning culminates in a number of results linking such submersions to the admittance of a parallel (even) Cliffordian structure. The end result of the exposition serves as a desirable conclusion for the end of the current section, though special care must be shown to avoid getting bogged down in deep theoretical results beyond the scope of this paper. Therefore, what follows will be an attempt at presenting the aforementioned classification while still omitting as many unnecessary technical details as possible.

Throughout, consider a Riemannian submersion $\pi : (Z^{k+n}, g_Z) \to (M^n, g_M)$. Let $Z_x = \pi^{-1}(x)$ be the fiber of π over $x \in M$. Well-known results show that all such fibers are isometric to some fixed Riemannian manifold (F, g_F) and that π is a locally trivial fibration with structure group the Lie group $G = \mathrm{Iso}(F)$ of isometries of F. Next, define the G-principal fiber bundle P over M as the set of isometries from F to the fibers of π:

$$P \stackrel{\text{def}}{=} \{u : F \to Z \,:\, u \text{ maps } F \text{ isometrically onto } Z_x \text{ for some } x \in M\}.$$

Let $p : P \to M$ be the natural projection and let P_x be the fiber of p over x:

$$P_x = \{u : F \to Z_x \,:\, u \text{ is an isometry}\}.$$

[39] Theorems 3.17, 3.18, and 3.19 here correspond to theorems 2.13, 2.14, and 2.15, respectively, of [56].
[40] Section 3.

Let $x \in M$, $u \in p^{-1}(x)$, and denote by H_u the image of the map $T_xM \to T_uP$ which sends an arbitrary vector field $X \in T_xM$ to the associated vector field $\widetilde{X} \in T_uP$. The collection $\{H_u : u \in P\}$ is called the *horizontal distribution*. Define the adjoint bundle $\mathrm{ad}(P) \stackrel{\text{def}}{=} P \times_{\mathrm{ad}} \mathfrak{g}$ of P to be the vector bundle associated to P via the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Then, for each $x \in M$, the fiber $\mathrm{ad}(P)_x$ of $\mathrm{ad}(P)$ over x has a Lie algebra structure in which every element $\alpha \in \mathrm{ad}(P)_x$ induces a Killing vector field α^* on the corresponding fiber Z_x. Finally, if $\alpha = [u, A]$ and $z = [u, f]$ for some frame $u \in P_x$, write $\alpha^*_z = uAf$ and define the collection $\{\alpha^*_z : u \in P\}$ to be the so-called *vertical distribution*.

With these definitions in place, the remainder of the classification is almost ready to be stated. In what remains, let R^Z denote the Riemannian curvature tensor on a Riemannian manifold (Z, g_Z) and define for each $z \in Z$ the *curvature constancy* at z by

$$\mathcal{V}_z = \{V \in T_zZ \,:\, R^Z_{V,X}Y = g_Z(X,Y)V - g_Z(V,Y)X \text{ for every } X, Y \in T_zZ\}.$$

The authors of [56] note that a number of alternative formulations of $\mathcal{V} \stackrel{\text{def}}{=} \coprod_{z \in Z} \mathcal{V}_z$ in terms of horizontal and vertical vector fields (that is, elements of the horizontal and vertical distributions defined above) but these aren't particularly useful for the results that follow. What *is* important is the assumption that $\mathcal{V}$ be the vertical distribution of some submersion $(Z^{n+k}, g_Z) \to (M^n, g)$, whereby the following result can be proven.

Theorem 3.20. Assume that the curvature constancy $\mathcal{V}$ of Z is the vertical distribution of a Riemannian submersion $(Z^{k+n}, g_Z) \to (M^n, g)$. Then:

(a) (M, g) carries a parallel even Clifford structure (E, ∇^E, φ) of rank $r = k + 1$; and

(b) The curvature of E, viewed as an endomorphism $\omega : \Lambda^2(TM) \to \mathrm{End}^-(E)$, equals minus twice the metric adjoint of $\varphi : \Lambda^2 E \simeq \mathrm{End}^-(E) \to \mathrm{End}^-(TM) \simeq \Lambda^2(TM)$.

Conversely, if (M, g) satisfies these conditions, then the sphere bundle Z of E, together with the Riemannian metric induced by the connection ∇^E on Z defines a Riemannian submersion onto (M, g) whose vertical distribution belongs to the curvature constancy.

Admittedly, theorem 3.20 doesn't seem like much. To the authors of [56], however, it was a huge stepping stone towards the last main component of the paper. Therefore, this section concludes with the statement of the last result sought after herein and the last theorem proved in [56][41]. Throughout, $\mathrm{scal}(M)$ denotes the scalar curvature of M, S^r denotes an r-dimensional sphere, m, q, and k denote the dimensions of manifolds M, and $q^\pm$ denotes the dimensions of the constituent manifolds in a product $M = Q^+ \times Q^-$.

Theorem 3.21. There exists a Riemannian submersion from a complete Riemannian manifold (Z^{k+n}, g_Z) to a complete simply connected Riemannian manifold (M^n, g) whose vertical distribution belongs to the curvature constancy if and only if (Z, M) appears to the following list:

[41] Theorem 3.7.

$\boldsymbol{Z}$	$\boldsymbol{M}$	**Fiber**	$\mathbf{dim}(\boldsymbol{M})$	$\mathbf{scal}(\boldsymbol{M})$
Sasakian	Hodge	S^1	$2m,\ m \geq 1$	
Twistor space Z	Quaternion-Kähler (QK)	S^2	$4q,\ q \geq 1$	$8q(q+2)$
Quaternion-Sasakian	Product of two QK manifolds	$\mathbb{RP}^3$	$4q,\ q \geq 1$ $q = q^+ + q^-$	$16q^+(q^+ + 2)$ $+16q^-(q^- + 2)$
$\frac{\mathrm{Sp}(k+2)}{\mathrm{Sp}(k)\times\mathrm{Spin}(4)}$	$\mathrm{Sp}(k+2)/\,\mathrm{Sp}(k) \times \mathrm{Sp}(2)$	S^4	$8k,\ k \geq 1$	$32k(k+3)$
$\frac{\mathrm{SU}(k+4)}{\mathrm{S}(\mathrm{U}(k)\times(\mathrm{Sp}(2)\cdot U(1)))}$	$\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(4))$	$\mathbb{RP}^5$	$8k,\ k \geq 1$	$32k(k+4)$
$\frac{\mathrm{SO}(k+8)}{\mathrm{SO}(k)\times\mathrm{Spin}(7)}$	$\mathrm{SO}(k+8)/\,\mathrm{SO}(k) \times \mathrm{SO}(8)$	$\mathbb{RP}^7$	$8k,\ k \geq 1$	$32k(k+6)$
$\mathrm{F}_4\,/\,\mathrm{Spin}(8)$	$\mathrm{F}_4\,/\,\mathrm{Spin}(9)$	S^8	16	$2^6 \cdot 3^2$
$\mathrm{E}_6\,/\,\mathrm{Spin}(9) \cdot \mathrm{U}(1)$	$\mathrm{E}_6\,/\,\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	S^9	32	$2^9 \cdot 3$
$\mathrm{E}_7\,/\,\mathrm{Spin}(11) \cdot \mathrm{SU}(2)$	$\mathrm{E}_7\,/\,\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$	S^{11}	64	$2^9 \cdot 3^2$
$\mathrm{E}_8\,/\,\mathrm{Spin}(15)$	$\mathrm{E}_8\,/\,\mathrm{Spin}(16)$	S^{15}	128	$2^{10} \cdot 3 \cdot 5$

Table 3
Riemannian submersions with curvature constancy

With the remainder of the paper, the goal will be to simply expose the reader to a little bit of information in each of a few "miscellaneous notions of Clifford structures." As such, the presentation of *deep* results has essentially ceased.

3.4 Miscellaneous Notions

The remainder of this section consists of a hodgepodge of miscellaneous notions called or related to "Clifford structures" and/or "Clifford geometry." Each subsection will essentially present a very small amount of information from $1 \leq \alpha \leq 2$ articles related to the notion described therein.

3.4.1 Clifford-Kähler

Much of the present exposition will come from the articles [30] and [31]. Throughout, the goal will be to focus on geometric structures related to the algebra $\mathcal{O} \stackrel{\text{def}}{=} \mathcal{C}\ell_{0,3}$. Recall from the previous discussion on Bott periodicity (see (2.4.7) above) that $\mathcal{O} \cong \mathbb{H} \oplus \mathbb{H}$ is an

8-dimensional algebra consisting of three generators $\{e_1, e_2, e_3\}$ subject to the relations:

$$\begin{aligned} e_0 e_i = e_i e_0 = e_i \quad & i = 0, 1, \ldots, 7 \\ e_i^2 = -e_0, e_7^2 = e_0 \quad & i = 1, 2, \ldots, 6 \\ e_i e_j + e_j e_i = 0 \quad & i \neq j, i, j = 1, 2, \ldots, 6, i + j \neq 7, \\ e_i e_j = e_j e_i \quad & i = 0, 1, \ldots, 7, i \neq j, i + j = 7 \\ e_1 e_2 = e_4, e_1 e_3 = e_5, \; & e_2 e_3 = e_6, e_1 e_6 = e_7. \end{aligned}$$

At this point, one can say that a smooth $8n$-dimensional real manifold M equipped with the action of $\mathcal{O}$ on its tangent bundle is *an almost-Cliffordian manifold*; alternatively, an *almost-Cliffordian manifold* M^{8n} is a smooth $8n$-dimensional real manifold M equipped with a rank-6 subbundle $Q \subset \mathrm{End}(TM)$ which is locally spanned by almost hypercomplex structures $\{J_\alpha\}_{\alpha=1,\ldots,6}$. Using the second notion, one can easily extend the almost-Cliffordian structure to a *Cliffordian structure* whenever there is a torsionless connection ∇ on TM which preserves Q in the sense that $\nabla_X \sigma \in \Gamma(Q)$ for all vector fields X and smooth sections $\sigma \in \Gamma(Q)$. This is equivalent to imposing an integrability condition on the $\mathcal{O}$-action from the first notation.

Now, a *Clifford-Kähler manifold* is a Riemannian manifold (M^{8n}, g) whose holonomy group $\mathrm{Hol}(g)$ is isomorphic to a subgroup $\mathrm{Op}(n) \cdot \mathrm{Op}(1) \subset \mathrm{SO}(8n)$ where here, $\mathrm{Op}(n)$ is the group consisting of all matrices $A \in M_{n\times n}(\mathcal{O})$ so that $\langle Ap, Aq\rangle = \langle p, q\rangle$ for all $p, q \in \mathcal{O}^n$. Here, $\langle p, q\rangle$ denotes the quasi-inner product of the form

$$\langle p, q\rangle = \frac{1}{2}\sum_{i=1}^{n}(p_i \overline{q}_i + q_i \overline{p}_i)$$

for all $p, q \in \mathcal{O}^n$ and $\overline{a}$ denotes the conjugate of the element $a_0 e_0 + \cdots a_7 e_7 \in \mathcal{O}$ which has the form

$$\overline{a} = a_0 e_0 - a_1 e_1 - \ldots - a_6 e_6 + a_7 e_7.$$

In the current context, similar to what was mentioned in section 3.3 above, an almost-Cliffordian structure on M is equivalent to a reduction of the structural group of the principal bundle of M to—in this case—$\mathrm{Op}(n) \cdot \mathrm{Op}(1)$. Note, too, that the identification of $\mathcal{O}^n \simeq \mathbb{R}^{8n}$ gives two very immediate examples of Clifford manifolds, namely (i) $\mathbb{R}^{8n}$ with J_1, J_2, J_3 as defined in the example[42] of [31], and (ii) The tangent bundle of any quaternionic-like manifold endowed with a linear connection. Both of these results echo sentiments discussed above.

Also echoing the sentiments above are the various geometrical structures one can define related to this particular brand of Clifford structure. For example, one can define *an almost Clifford connection* on the almost-Cliffordian manifold (M, V) to be a linear connection ∇ on M which preserves parallel transport to the vector bundle V. In particular, this means that for any cross-section Φ of the bundle V, $\nabla_X \Phi$ is also a cross-section of V for X an arbitrary vector field. A result[43] give a full characterization of linear connections ∇ which

[42] Example 3.1.

[43] Proposition 4.1 in [31].

are almost Clifford connections in terms of the covariant derivatives of the local canonical base and 1-forms defined on the domains of J_α.

There are several other key geometric tidbits described throughout [30] and [31]. For example, by defining a generalized Nijenhuis tensor (see definition A1.7 in appendix 1.3 below) for any pair A, B of endomorphisms defined on any pair X, Y of vector fields to have the form

$$N(A,B)(X,Y) = N(AX,BY) - AN(BX,Y) - BN(X,AY) + N(BX,AY) - BN(AX,Y) \\ - AN(X,BY) + (AB+BA)[X,Y]$$

and by analyzing $N_{ab} = N(J_a, J_b)$ for various pairs (a,b), [30] is able to systematically characterize vanishing behavior in pairs of tensor fields N_{ab}. Though not entirely fruitful in its own right, this analysis allows characterization of symmetric affine almost-Cliffordian connections ∇ in terms of vanishing Nijenhuis tensors N_{ab} for various pairs (a,b). Many of these results mimic earlier results.

On the other hand, [31] considers a so-called *almost Cliffordian Hermitian manifold* to be a Riemannian manifold (M^{8n}, g) for which the almost hypercomplex structures $\{J_\alpha\}_{\alpha=1,\ldots,6}$ satisfy

$$g(J_\alpha X, J_\alpha Y) = g(X,Y) \text{ for } \alpha = 1,\ldots,6 \text{ and for all } X,Y \in \mathfrak{X}(M).$$

From here, one can re-define the notion[44] of Clifford-Kähler to be describe a manifold for which $\nabla\Omega = 0 = \nabla\Lambda$ for ∇ the connection induced by g and for specifically defined 4-forms, respectively (2,2)-tensor fields Ω, respectively Λ.

Unsurprisingly, the addition of a Riemannian structure in the latter half of [31] allows an entirely different sort of exposition. Indeed, later sections[45] consider coordinate-wise formulae for curvature tensors on Clifford-Kähler manifolds (M, V, g), thereby allowing the definitions of Ricci tensors, etc., as well as various identities related thereto. The paper ends with a collection of summarizing results with proof for the various structures considered throughout. In particular, given a Clifford-Kähler manifold (M^{8n}, V, g): (i) The Ricci tensor is parallel, (ii) (M, V, g) is an Einstein manifold (see appendix 2.1 below), (iii) The restricted holonomy group is a subgroup of $\mathrm{Op}(n)$ if and only if the Ricci tensor is identically vanishing, and (iv) V is locally parallelizable if and only if the Ricci tensor is identically vanishing. Proofs of these statements are scattered throughout the last section[46] of [31].

3.4.2 Para-Hypercomplex Structures

I ran out of time, but my plan was to summarize some details from [19] here.

3.4.3 (Almost) Split Quaternion & Split Quaternion Kähler Structures

I ran out of time, but my plan was to summarize some details from [8] and [7] here.

[44] Theorem 4.2 of [31].
[45] Starting in section 5.
[46] Section 6.

Appendix 1: Complex Geometry

1.1 General Preliminaries

Classically, the notion of a complex manifold was studied in differential geometric terms, i.e. in terms of atlases of suitably-compatible holomorphic charts. As machinery developed, the study changed direction so that a complex manifold M was defined in terms of an almost-complex structure I on its tangent bundle TM where I satisfies certain "nicety" properties. For the sake of fluidity, the latter approach is the one primarily adhered to in this paper though, for the sake of completeness, both perspectives will be addressed and discussed.

First, consider a definition:

Definition A1.1. An *almost-complex manifold* is a real-differentiable manifold M whose (real) tangent bundle TM is equipped with a vector bundle endomorphism $I : TM \to TM$ for which

$$I^2 = -\operatorname{id}_{TM} . \tag{A1}$$

In general, an endomorphism I defined on a certain vector space V and satisfying (A1) is called *an almost-complex structure* on V.

Definition A1.1 dates back to the 1940s to the works of Hopf and Ehresmann. There is much literature on the topic including [41], [74], etc., where it may be noted that the term *almost-complex manifold* is applied to the ordered pair (M, I) where M, I are as above. This notation is particularly helpful when one wants to explicitly describe the endomorphism I; as such, the ordered pair notation will be used later when considering the presence of multiple almost-complex structures on a single manifold.

At this point, it's logical to discuss the different notions of a *complex manifold*, though it's anyone's preference about which way to best proceed. For the sake of fluidity, the historical (differential geometry) definition will be presented first followed by some results in section 1.3 relating it to the modern definition.

Throughout, let $D \subset \mathbb{C}^n$ be an open subset and let $\mathcal{O}(D)$[47] denote the complex-valued holomorphic functions on D, i.e. the collection of all functions $f(\boldsymbol{z}) = f(z_1, z_2, \ldots, z_n)$ which can be represented by convergent power series of the form

$$f(\boldsymbol{z}) = \sum_{\alpha_1,\ldots,\alpha_n=0}^{\infty} a_{\alpha_1,\ldots,\alpha_n}(z_1 - z_1^0)^{\alpha_1} \cdots (z_n - z_n^0)^{\alpha_n}$$

near every point $\boldsymbol{z}^0 = (z_1^0, \ldots, z_n^0) \in D$. This notation follows that found in [74], from which the following definitions are borrowed.

Definition A1.2. An $\mathcal{O}$-structure $\mathcal{O}_M$ on a manifold M is a family $\mathcal{F} = \{f_\alpha\}$ of complex-valued functions defined on the open sets of M satisfying two conditions, namely that

[47] The $\mathcal{O}$ notation used here and in definitions A1.2, A1.3 should remind the reader of notation commonly used in sheaf theory, the perspective from which much of [74] is written. Equivalent definitions flavored more differential geometrically can be found in [41] among others.

1. For every point $p \in M$, there exists an open neighborhood U of p and a homeomorphism $h : U \to U'$, $U' \subset \mathbb{C}^n$ open, such that for any open set $V \subset U$,

$$f : V \to \mathbb{C} \text{ is in } \mathcal{O}_M \text{ if and only if } f \circ h^{-1} \in \mathcal{O}(h(V)).$$

2. If $f : U = \bigcup_i U_i \to \mathbb{C}$, U_i open in M for all i, then $f \in \mathcal{O}_M$ if and only if $f|_{U_i} \in \mathcal{O}_M$ for all i.

Definition A1.3. Given a manifold M with an $\mathcal{O}$-structure $\mathcal{O}_M$, the ordered pair $(M, \mathcal{O}_M)$ is called a *complex manifold*. An open subset $U \subset M$ and a homeomorphism $h : U \to U' \subset \mathbb{C}^n$ as in part 1 of definition A1.2 is called a *holomorphic coordinate system*, and the functions of $\mathcal{O}_M$ are called *holomorphic functions*.

Succinctly, definition A1.2 describes an atlas of charts, the transition functions of which are holomorphic, and—according to definition A1.3—a complex manifold is an ordered pair $(M, \mathcal{O}_M)$ whose constituent manifold M can be covered by an atlas of charts with holomorphic transition functions. Such charts are commonly called *holomorphic charts* and atlases of holomorphic charts are typically called *holomorphic atlases*, from which definition A1.3 can be rewritten to say that a complex manifold is an ordered pair whose base manifold M endowed with an equivalence class of holomorphic atlases.

As it stands, there are a number of equivalent (yet often very different-looking) definitions of a complex manifold. Moreover, it's worth noting that the use of the term "almost-complex" in definition A1.1 is hardly coincidental, and in fact, the relation between almost-complex manifolds and complex manifolds becomes more apparent as different definitions are utilized. Moving forward, the goal is to discuss some of the ideas behind these other notations and to examine the relationships between those notions and the ones discussed thus far.

1.2 Algebraic Preliminaries

The idea moving forward will be to devise enough machinery to finally link almost-complex structures with complex structures. One way to do this is by way of differential forms, and in order to elaborate, one must consider some algebraic tools defined on tangent bundles and their duals. This section comes almost entirely from [41], though some exposition is based on [49].

Throughout, assume that any manifold M has dimension $\dim M = n$ unless noted otherwise, let $T_{\mathbb{C}}M$ denote the *complexified tangent space* $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ of M, and note that the eigenvalues of the endomorphism I of an almost-complex manifold (M, I) are $\lambda = i$ and $\lambda = -i$, $i = \sqrt{-1}$. Next, define the i-component, respectively the $-i$-component of $T_{\mathbb{C}}M$ to be the kernel of $I - i \cdot \mathrm{id}$, respectively $I + i \cdot \mathrm{id}$; denote these spaces $T^{1,0}M = \{v \in T_{\mathbb{C}}M \ : \ I(v) = i \cdot v\}$, respectively $T^{0,1}M = \{v \in T_{\mathbb{C}}M \ : \ I(v) = -i \cdot v\}$[48].

[48] By abuse of notation, we write I to be the $\mathbb{C}$-linear extension of the almost complex structure I to $T_{\mathbb{C}}M$.

Clearly, $T^{1,0}M \cap T^{0,1}M = \{0\}$, and straightforward computation confirms[49]

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M. \tag{A1}$$

Moving forward, the bundle $T^{1,0}M$, respectively $T^{0,1}M$, is called the *holomorphic tangent bundle*, respectively the *antiholomorphic tangent bundle*, of the almost-complex manifold M. Moreover, elements of $T^{1,0}M$, respectively of $T^{0,1}M$, will be called *vector fields of type* $(1,0)$ (or *holomorphic vector fields*), respectively *vector fields of type* $(0,1)$ (or *antiholomorphic vector fields*).

Next, note that the existence of an almost-complex structure I on TM necessarily induces an almost-complex structure on the cotangent bundle $T^*M = \mathrm{Hom}_{\mathbb{R}}(TM, \mathbb{R})$ which has the form $I(f)(v) = f(I(v))$ for all $v \in TM$. Furthermore, one notes that

$$(T^*M)_{\mathbb{C}} = T^*M \otimes \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(TM, \mathbb{R}) \otimes \mathbb{C} \cong \mathrm{Hom}_{\mathbb{R}}(TM, \mathbb{C}) = (T_{\mathbb{C}}M)^*,$$

i.e. the $\mathbb{C}$-linear extension to $T_{\mathbb{C}}M$ of the almost complex structure $I : TM \to TM$ induces a $\mathbb{C}$-linear extension to the complexified cotangent bundle $(T_{\mathbb{C}}M)^*$. For notational simplicity, write $T^*_{\mathbb{C}}M$ for $(T_{\mathbb{C}}M)^* = (T^*M)_{\mathbb{C}}$. Applying to $T^*_{\mathbb{C}}M$ a decomposition analogous to (A1) yields the existence of spaces

$$\begin{aligned}
(T^*M)^{1,0} &= \{\alpha \in \mathrm{Hom}_{\mathbb{R}}(TM, \mathbb{C}) \,:\, \alpha(I(v)) = i\alpha(v)\} &&= (TM^{1,0})^* \\
(T^*M)^{0,1} &= \{\alpha \in \mathrm{Hom}_{\mathbb{R}}(TM, \mathbb{C}) \,:\, \alpha(I(v)) = -i\alpha(v)\} &&= (TM^{0,1})^*
\end{aligned}$$

satisfying

$$T^*_{\mathbb{C}}M = (TM^{1,0})^* \oplus (TM^{0,1})^*. \tag{A2}$$

Now, denote by $\Lambda^k(T^*M)$ the vector space of all differential k-forms on M, i.e. the collection of all alternating covariant tensors of rank k on T^*M. Similarly, define $\Lambda^k_{\mathbb{C}}(T^*M) = \Lambda^k(T^*_{\mathbb{C}}M)$ to be the space of all differential k-forms on the complexified cotangent space $T^*_{\mathbb{C}}M$ and let the exterior algebra[50] of the vector spaces T^*M and $T^*_{\mathbb{C}}M$ be the spaces

$$\bigwedge^* T^*M = \bigoplus_{k=0}^{n=\dim M} \bigwedge^k T^*M \quad \text{and} \quad \bigwedge^* T^*_{\mathbb{C}}M = \bigoplus_{k=0}^{n=\dim M} \bigwedge^k T^*_{\mathbb{C}}M,$$

respectively. Let $\mathcal{A}^*_M$ denote the collection of sections of $\bigwedge^* T^*_{\mathbb{C}}M$ and let $\mathcal{A}^*(M)$ denote the collection of global sections of $\mathcal{A}^*_M$.

[49]Indeed, for any real vector space V with almost-complex structure $I : V \to V$, $V^{1,0} \cap V^{0,1} = \{0\}$ implies that the canonical map $\varphi : V^{1,0} \oplus V^{0,1} \to V_{\mathbb{C}}$ defined by $\varphi : x \oplus y \mapsto x + iy$ is injective. Moreover, one can verify that the map

$$\vartheta : v \mapsto \frac{1}{2}(v - iI(v)) \oplus \frac{1}{2}(v + iI(v))$$

is the inverse of φ.

[50]Recall that the exterior algebra $\Lambda(V)$ over a vector space V is defined to be the quotient of the tensor algebra $T(V)$ over V by the two-sided ideal I generated by all products $v \otimes v$, $v \in V$.

Finally, for an arbitrary vector space V endowed with an almost-complex structure I, define a là equations (A1) and (A2) the decomposition

$$\bigwedge^{p,q} V \stackrel{\text{def}}{=} \bigwedge^{p} V^{1,0} \bigotimes_{\mathbb{C}} \bigwedge^{q} V^{0,1}, \tag{A3}$$

and note that one can derive a natural direct sum decomposition of k-forms as (p,q)-forms:

$$\bigwedge^{k} T^*_{\mathbb{C}}M = \bigoplus_{p+q=k} \bigwedge^{p,q} T^*_{\mathbb{C}}M. \tag{A4}$$

Considering the collection $\mathcal{A}^k_M$, respectively $\mathcal{A}^{p,q}_M$, of smooth sections of $\bigwedge^k T^*_{\mathbb{C}}M$, respectively $\bigwedge^{p,q} T^*_{\mathbb{C}}M$, define $\mathcal{A}^k(M)$, respectively $\mathcal{A}^{p,q}(M)$, to be the collection of global sections thereof. One easily verifies the existence of both a direct sum decomposition

$$\mathcal{A}^k_M = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_M, \tag{A5}$$

stemming from equation (A4) as well as a projection $\Pi^{p,q} : \mathcal{A}^*(M) \to \mathcal{A}^{p,q}(M)$ induced by the decomposition in (A5). Therefore, if one denotes by $d : \mathcal{A}^k_M \to \mathcal{A}^{k+1}_M$ the natural $\mathbb{C}$-linear extension of exterior differentiation (sending k-forms to $(k+1)$-forms), one can define two operators ∂, $\overline{\partial}$ in terms of the projection $\Pi^{p,q}$:

$$\partial \stackrel{\text{def}}{=} \Pi^{p+1,q} \circ d : \mathcal{A}^{p,q}_M \to \mathcal{A}^{p+1,q}_M \quad \text{and} \quad \overline{\partial} \stackrel{\text{def}}{=} \Pi^{p,q+1} \circ d : \mathcal{A}^{p,q}_M \to \mathcal{A}^{p,q+1}_M. \tag{A6}$$

The operators in (A6) send (p,q)-forms to $(p+1,q)$- and $(p,q+1)$-forms, respectively, where $p+q=k$.

Although it may not have been clear throughout, the above exposition was made to allow a crucial piece of machinery needed to connect almost-complex structures with complex ones. To bring things back down from the universe of abstraction, consider the following definition.

Definition A1.4. An almost-complex structure I on M is said to be *integrable* if M admits local holomorphic coordinates for I around every point $x \in M$. In this case, these local coordinates can be "patched" together to form a holomorphic atlas for M, thus giving M a complex structure which naturally induces I.

Said differently, an almost-complex structure on I coinides with a complex structure on M if and only if I is integrable; moreover, one can easily see that the complex structure constructed on (M, I) via the "patching" argument in A1.4 is unique. This definition will be expanded and utilized in the next section, where some basic results dependent upon the algebraic machinery derived here will be collected.

1.3 Some Results

The purpose of this section is to collect a number of preliminary results on almost-complex and complex manifolds. Except when beneficial to the overall exposition, facts will be stated

without proof; unproven nontrivial results will be accompanied by references as appropriate. When unsaid, the assumption is that (M, I) is an almost-complex manifold of some dimension.

Proposition A1.5. Every complex manifold M is an almost-complex manifold.

Proof. Let M be a complex manifold and let $(x_1, y_1, \ldots, x_\mu, y_\mu)$ be the associated holomorphic basis for M. Next, let $(\partial x_1, \partial y_1, \ldots, \partial x_\mu, \partial y_\mu)$ be the associated basis of TM where here, ∂x_i and ∂y_j are shorthand for

$$\frac{\partial}{\partial x_i}, \text{ respectively } \frac{\partial}{\partial y_j}.$$

Define a map $J : TM \to TM$ by its action on the basis elements:

$$J(\partial x_i) = \partial y_i \quad \text{and} \quad J(\partial y_i) = -\partial x_i. \tag{A1}$$

One easily verifies that the map J is an automorphism, while the fact that $J^2 = -\,\mathrm{id}$ is abundantly clear. □

Worth noting is that the map J in the proof above may seem canonical but it *isn't* the only almost-complex structure one can define given a complex manifold. Indeed, it isn't uncommon for the tangen bundle of such a manifold to have many automorphisms squaring to -1. Something that *isn't* in flux, however, is that the holomorphic basis elements used in proposition A1.5 can be paired, a fact that perhaps makes intuitive sense and is verified as follows.

Proposition A1.6. Every almost-complex manifold (M, I) is even-dimensional.

Proof. Let $\dim M = n$ and suppose that $I : TM \to TM$ is an almost-complex structure. Necessarily, then, $p(x) := \det(I - x \cdot \mathrm{id})$ is a polynomial of degree n where here, id denote the $n \times n$ identity matrix. If n were odd, then p has a real root x_0 satisfying $\det(I - x_0 \cdot \mathrm{id}) = 0$, whereby it follows that there exists a vector $v \in TM$ for which $Iv = x_0 v$. By definition, though, $-v = I^2 v = x_0^2 v$, i.e. $x_0^2 = -1$. Because $x_0 \in \mathbb{R}$, $x_0^2 \neq -1$ and so it must happen that n is even. □

Next, consider the following definition which, at first, seems like a detour.

Definition A1.7. For any linear map A on each tangent space of M, one defines the associated *Nijenhuis Tensor* N_A to be the tensor field of rank $(1, 2)$ given by the formula

$$N_A(X, Y) = -A^2[X, Y] + A([AX, Y] + [X, AY]) - [AX, AY],$$

$X, Y \in \mathfrak{X}(M)$. Here, $\mathfrak{X}(M)$ is the collection of all vector fields on M and $[*, *]$ is the usual Lie bracket.

Now, consider the following result which makes obvious how all the above-stated ideas are interrelated.

Theorem A1.8. An almost-complex structure I is integrable in the sense of definition A1.4 if it satisfies any (and hence all) of the following equivalent conditions:

1. For all $\alpha \in \mathcal{A}^*(M)$, $d\alpha = \partial(\alpha) + \overline{\partial}(\alpha)$.
2. On $\mathcal{A}^{1,0}(M)$, $\Pi^{0,2} \circ d = 0$.
3. $\overline{\partial}^2 = 0$.
4. $\partial^2 = 0$.
5. The Lie bracket of two $(1,0)$-vector fields is again of type $(1,0)$.
6. $N_I(X,Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$.

The equivalence of the first two statements in definition A1.4 is proven in [41]. The equivalence of the third and forth statements is discussed in [74] and follows from the fact that $Q\overline{\partial}(Q\alpha) = \partial\alpha$, $\alpha \in \mathcal{A}^*(M)$, where Q denotes complex conjugation. The third statement follows directly from the first along with the fact $d^2 = 0$:

$$d^2 = \underbrace{\partial^2}_{\text{Type (2,0)}} + \underbrace{\partial\overline{\partial} + \overline{\partial}\partial}_{\text{Each of type (1,1)}} + \underbrace{\overline{\partial}^2}_{\text{Type (0,2)}} = 0,$$

where now type-considerations yield the result (see [74]). Note that the fifth statement of definition A1.4 is nothing more than a rewording of the second statement; moreover, it's an involution-type statement reminiscent of the Frobenius theorem, whereby several additional equivalences related to foliation theory, etc., can also be derived (see, e.g., [49]). The equivalence of the sixth and final statement with integrability is a classical result sketched, e.g., in [52].

Many times, in practice, the Nijenhuis tensor is the easiest to use condition for verifying integrability. To conclude this section, consider the following (repetitive, though perhaps) worthwhile observation which uses conditions from theorem A1.8 to prove something that follows immediately from definition A1.4.

Corollary A1.9. The induced almost-complex structure on a complex manifold (M, I) is integrable.

Proof. This is proved in detail in [74], where one shows that the canonically-induced basis elements for TM and T^*M satisfy the first condition of Theorem A1.8. □

Appendix 2: Various Other Preliminaries

The purpose of this appendix is to collect a variety of (mostly) definitions, (very few) results, and (occasional bits of) miscellany that didn't fit well throughout the main body of the article. The presentation here will be much more list-centric with much less exposition in-between.

2.1 Some Differential Geometry

2.1.1 G-Structures

First, recall that a *frame bundle* is a principal fiber bundle $F(E)$ associated to a vector bundle $E \to X$ so that the fibers $F_x = F(E)_x$ over a point $x \in X$ is the set of all ordered bases or *frames* for E_x. In particular, $F(E) = \coprod_{x \in X} F_x$. In the event that the space X is a smooth manifold, say $X = M$, the frame bundle of M is the one associated to the tangent bundle TM. In this case, $F = F(TM)$ is called *the tangent frame bundle* and is often denoted FM or $\mathrm{GL}(M)$.

Definition B2.1. For a given structure group G, *a G-structure* on an n-manifold M^n is a G-subbundle of the tangent frame bundle FM.

The notion in definition B2.1 is fundamental to the contents of this paper. For example, a Riemannian structure corresponds to $G = \mathrm{O}(n)$, an almost complex structure is a $\mathrm{GL}(n, \mathbb{C})$-structure (requiring real dimension $2n$), and an almost hypercomplex structure is a $\mathrm{GL}(n, \mathbb{H})$-structure (requiring real dimension $4n$). What's more, the reduction of the orthonormal frame bundles discussed throughout sections 3.3 and 3.4.1 is a fundamental component of G-structures defined on manifolds for various values G. This notion was discussed in a bit more generality in section 3.2 in relation to flat Cliffordian structures and—in more generality—in terms of Joyce's "geometric structures" (see definition 3.1 in section 3.2 plus exposition in [47]). Additional details of a different flavor can be found in [4] as well.

2.1.2 Einstein Manifolds

Definition B2.2. An *Einstein manifold* is a Riemannian manifold (M, g) whose Ricci tensor Ric is proportional to g, i.e. $\mathrm{Ric} = kg$ for some constant k.

In the event that $k = 0$, the Einstein manifold M is said to be *Ricci-flat*. Though not of particular importance throughout this article, [18] describes the importance of Einstein manifolds by saying that they're among the "nicest" of all Riemannian manifolds.

2.1.3 Weyl Manifolds

Much like the Einstein manifolds described above, Weyl manifolds don't play a particularly significant role in the current exposition. Because of their semi-regular mention throughout, and because defining them in-text seemed anachronistic, the definition from [71] is given here.

Definition B2.3. Let (M, g) be a Riemannian manifold with ∇ a torsion-free connection on M. If ∇ preserves the conformal class of g, i.e. if

$$\nabla g = g \otimes \theta$$

for some 1-form θ, then (M, g, ∇, θ) is said to be a *Weyl manifold*.

2.1.4 Generalized Hopf Manifolds

The necessity of considering generalized Hopf manifolds comes, in part, because of theorem 2.6 in section 2.2 above. The exposition here comes from [69].

Definition B2.4. A Hermitian manifold (M, J, g) is *a generalzied Hopf manifold* if its Kähler form Ω satisfies $d\Omega = \omega \wedge \Omega$ for a parallel form ω.

Note, in particular, that every generalized Hopf manifold is locally conformally Kähler due to the decomposition of $d\Omega$ *without* the imposition of the parallelism of ω. What's more, note that every Hopf manifold $H_n \simeq S^1 \times S^{2n-1}$ is necessarily a generalized Hopf manifold. Other formulations of this definition, as well as results thereon, can be found in [69].

2.1.5 The Obata Connection

The following definition comes from [64].

Definition B2.5. On every hypercomplex manifold (M, I, J, K), there is a unique torsion-free connection ∇ satisfying $\nabla I = \nabla J = \nabla K = 0$. This connection is called *the Obata connection*.

In addition to the obvious relations to hypercomplex geometry which can be found scattered throughout the present volume, [70] shows that a converse of the existence criterion mentioned in definition B2.5 above. Indeed, given a smooth manifold M with operators I, J, K defining a quaternionic structure on TM, then the existence of a torsion-free affine connection ∇ preserving each of I, J, and K ensures the integrability of all three of these structures, thus making (M, I, J, K) into a hypercomplex manifold. Additional properties of the Obata connection including algebraic definitions, curvature arguments, and discussions on holonomy can be found in [64].

2.1.6 Holonomy

Because pertinent holonomy-related results extend far beyond the mere definition of holonomy, this subsection will be considerably longer than the others in this section of the appendix. Before getting too far ahead, however, it's important to start with the definition. Throughout, let ∇ be a connection on a principal bundle P with structure group G.

Definition B2.6. The *holonomy group* of ∇ on $P = (P, G)$ is the subgroup of all $a \in G$ such that a fixed $u \in P$ can be joined to $u \cdot a$ by a horizontal curve.

In particular, then, the holonomy of a connection measures the extent to which to which the distribution of horizontal subspaces fails to be integrable [66]. This alone shows the deep-seeded connection between holonomy and the previous topics, but again, the immensity of the notion of holonomy on manifold theory in general means that there's much more than meets the eye. One of the big results, the Ambrose-Singer theorem, relates the holonomy of a connection in a principal bundle to the curvature form of the connection. The following theorem is original, taken from [5], and so the language used is a bit more archaic than used elsewhere.

Theorem B2.7 (The Ambrose-Singer Theorem)**.** Let M be a connected smooth manifold whose fundamental group is at most countably infinite, let H be a connection on a principal bundle $\pi : B \to M$ with structure group G. Further, suppose that ω is the 1-form associated to H, that Ω is the curvature 2-form of H, that $\mathfrak{g}$ is the Lie algebra of G, and that $G(b)$, respectively $G_0(b)$, denotes the holonomy group, respectively the null-holonomy group, corresponding to each $b \in B$. Then for any $b \in B$, the subgroup of G generated by the Lie subalgebra $L(b)$ is precisely $G_0(b)$. Here, $L(b)$ is the subalgebra generated by all $\Omega(s,t)$ with s,t running through all pairs of tangent vectors to B at all points $B(b)$

Admittedly, the language of theorem B2.7 is none too illustrative. Intuitively, it can be thought that curvature is equal to the holonomy over an infinitesimal closed loop / parallelogram. In order to see this, imagine a surface $\sigma : I \times I \to M$ in M parameterized by a pair of variables x and y, $x, y \in I = [0,1]$. In this case, a vector v may be transported along $\partial\sigma$ by way of the holonomy loop

$$(x,0) \longrightarrow (1,y) \longrightarrow (x,1) \longrightarrow (0,y), \tag{B1}$$

where here, v is being acted upon by a lift of the boundary $\partial\sigma$. The parallelogram outlined (B1) can be thought of as being "shrank to zero" by traversing smaller and smaller parallelograms over $[0,x] \times [0,y]$, a process by which (a) curvature explicitly enters the picture and (b) one can think of as taking the derivative of the parallel transport maps $x = y = 0$. The result is that

$$\frac{D}{dx}\frac{D}{dy}V - \frac{D}{dy}\frac{D}{dx}V = R\left(\frac{\partial\sigma}{\partial x}, \frac{\partial\sigma}{\partial y}\right)V$$

where R is the curvature tensor. Thus, heuristically, it follows that the curvature is the differential of the holonomy action at the identity of the holonomy group so that $R(X,Y)$ is an element of the Lie algebra of $\mathrm{Hol}_p(\omega)$.

The relationship between holonomy and hypercomplex geometry hardly stops there, however. Indeed, a classical result of Berger completely classified Riemannian manifolds based on their holonomy. The succinct version from [18] can be stated as follows:

Theorem B2.8. Let (M,g) be a Riemannian manifold and assume that Hol^0, the holonomy group restricted to curves homotopic to the identity, is irreducible. Then either Hol^0 is transitive on the unit sphere or (M,g) is a locally symmetric space of rank greater than or equal to 2.

Of course, theorem B2.8 is hardly explicit. Therefore, consider the following rewrite, found also in [18].

Theorem B2.9. Let (M, g) be a Riemannian manifold of dimension n which is not locally symmetric and whose holonomy representation Hol^0 is irreducible. Then its dimension and holonomy representation Hol^0 is one of the following:

$\mathbf{Hol^0(g)}$	$\mathbf{dim\, M}$	**Type of Manifold**	**Comments**
$\mathrm{SO}(n)$	n	Orientable Manifold	
$\mathrm{U}(n)$	$2n$	Kähler Manifold	Kähler
$\mathrm{SU}(n)$	$2n$	Calabi-Yau Manifold	Ricci-Flat, Kähler
$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$	$4n$	Quaternion-Kähler Manifold	Einstein
$\mathrm{Sp}(n)$	4n	Hyperkähler Manifold	Ricci-Flat, Kähler
G_2	7	G_2 Manifold	Ricci-Flat
Spin(7)	8	Spin(7) Manifold	Ricci-Flat

Table 4
Summary of Berger's Classification

Special attention must be paid to the above table relative to Lie group inclusions. For example,

$$\mathrm{Sp}(n) \subset \mathrm{SU}(2n) \subset \mathrm{U}(2n) \subset \mathrm{SO}(4n),$$

whence it follows that every hyperkähler manifold is Calabi-Yau, which is in turn Kähler and hence is orientable. Moreover, note that $\mathrm{Sp}(n) \subset \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, and so characterizing Quaternion-Kähler[51] manifolds as Riemannian manifolds (M, g) for which $\mathrm{Hol}^0(g)$ is a *subgroup* of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ immediately yields that all hyperkähler manifolds are also Quaternion-Kähler. Generally, this is undesirable and is typically remedied by requiring a manifold (M, g) to instead satisfy $\mathrm{Hol}^0(g) = \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ in order to be Quaternion-Kähler.

An interesting historical tidbit related to the table above is that Berger's original classification looked quite different from what's presented above. In particular, Berger allowed for the possibility of Spin(9) holonomy as a subgroup of SO(16), though Riemannian manifolds with such holonomy were later shown to be locally symmetric. The original list also included non-positive-definite pseudo-Riemannian metric non-locally symmetric holonomy in a list consisting of $\mathrm{SO}(p, q)$ of signature (p, q), $\mathrm{U}(p, q)$ and $\mathrm{SU}(p, q)$ of signature $(2p, 2q)$,

[51] As mentioned elsewhere, the class of Quaternion-Kähler manifolds is one that hasn't been investigated in this particular manuscript. The fact that such manifolds are quaternion-like rather than hypercomplex has essentially excluded it from inclusion, though some authors studied above *have* done some investigation. For example, in the language of section 3.3, a Quaternion-Kähler structure is a parallel rank 3 Clifford structure [56].

$\mathrm{Sp}(p,q)$ and $\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$ of signature $(4p,4q)$, $\mathrm{SO}(n,\mathbb{C})$ of signature (n,n), $\mathrm{SO}(n,\mathbb{H})$ of signature $(2n,2n)$, split G_2 of signature $(4,3)$, $\mathrm{G}_2(\mathbb{C})$ of signature $(7,7)$, $\mathrm{Spin}(4,3)$ of signature $(4,4)$, $\mathrm{Spin}(7,\mathbb{C})$ of signature $(7,7)$, $\mathrm{Spin}(5,4)$ of signature $(8,8)$ and $\mathrm{Spin}(9,\mathbb{C})$ of signature $(16,16)$. The split and complexified $\mathrm{Spin}(9)$ are necessarily locally symmetric as above and should not have been on the list, while the complexified holonomies $\mathrm{SO}(n,\mathbb{C})$, $\mathrm{G}_2(\mathbb{C})$, and $\mathrm{Spin}(7,\mathbb{C})$ may be realized from complexifying real analytic Riemannian manifolds.

Finally, note that in the above table, the case of locally symmetric manifolds is left out entirely. As shown in [18], Riemannian manifolds with this property are often classified based on the normalizer $N(G)$ of their holonomy group $G = \mathrm{Hol}^0(g)$. As such, a similar table can be made relative to the following result from [75]:

Theorem B2.10. Let M be a Riemannian manifold, let Γ be the group of deck transformations of the universal Riemannian covering $\pi : M \to N$ of a complete locally symmetric Riemannian manifold, let $M = M_0 \times M'$ be the decomposition of M into Euclidean and non-Euclidean parts, respectively, and let V be the group of pure translations of M_0. Then there is a canonical isomorphism between

$$\Gamma \cdot (V \times I_0(M'))/(V \times I_0(M'))$$

and the group $\mathrm{Hol}(N)/\mathrm{Hol}^0(N)$ of components of the homogeneous holonomy of N.

As a final closing remark, note that several of the authors whose works were explored above make explicit mention of holonomy in their expositions. Obviously, Berger's classification is mentioned throughout. Additionally, [16] note that the quotient Γ/Λ of a crystallographic group (see footnote 28 in section 3.2, e.g.) Γ by the lattice $\Lambda \subset \mathbb{R}^n$ corresponds to the linear holonomy group of a flat Riemannian manifold possessing a Clifford structure whenever Γ is torsion free. Indeed, a great amount of information about a geometric structure can be made by observing the properties of its holonomy, thus making holonomy one of the most important concepts in these branches of geometry.

2.2 Some Topology

2.2.1 Betti Numbers

While of fundamental importance in the field of algebraic topology, the notion of Betti number isn't crucial to any of the results given throughout. However, the authors of more than a few articles examined above were able to improve arbitrary constructions of hypercomplex structures to constructions whose resulting manifolds satisfy certain Betti number criteria. Without having a working understanding of what this means, it's impossible to understand why the aforementioned constructions are *improvements*. So, consider the following definition from [38].

Definition B2.11. The nth Betti number of a topological space X is the number of $\mathbb{Z}$ summands in the decomposition of the nth homology group $H_n(X)$ into a direct sum of cyclic groups.

Among the sources cited herein which makes extensive usage of Betti numbers is [29], where a construction is outlined which produces compact 3-Sasakian 7-manifolds with arbitrary second Betti number. This second Betti number condition can be thought of as allowing a bit of freedom regarding the structure which is produced. Similar topological considerations come into play in a number of cited sources, e.g. [24], [25], [26], [27], and [28].

2.2.2 Foliations & Contact Structures

Both foliations and contact structures have presented themselves as tools used to prove other results throughout this paper. In particular, sections 2.3 and 2.4 make use of natural foliated structures on manifolds and orbifolds in order to derive information about hypercomplex structures. Contact structures are also defined in section 2.3. For the sake of completeness (in addition to the fact that repetition is often a worthwhile facet of good pedagogy), both these terms will be defined in this section. To begin, attention is focused on foliations.

Definition B2.12 (Foliations 1)**.** A *foliation* $\mathcal{F}$ on a manifold M^n is an equivalence relation on M consisting of equivalence classes of connected immersed submanifolds all of the same dimension $k \leq n$, so that locally the decomposition into equivalence classes can be modeled on the decomposition of $\mathbb{R}^n$ into cosets $x+\mathbb{R}^k$ of the standardly embedded subspace $\mathbb{R}^k \subset \mathbb{R}^n$. Here, k is called the *dimension* of the foliation while $n-k$ is called its *codimension*. The equivalence classes that make up $\mathbb{F}$ are called its *leaves*.

The theory of foliations is one of great depth, so much so that any reasonable treatment is out of the question here. On the other hand, due to its extensive recurrence throughout the exposition thus far is only a glimpse into the interrelatedness between that area and the areas studied throughout. Therefore, a number of different aspects will be touched upon before proceeding.

For that reason, it may be worthwhile to state at least one other definition equivalent to B2.12, which was borrowed from [32].

Definition B2.13 (Foliations 2)**.** A *foliation of dimension* k on a smooth manifold M^n is an integrable rank k subbundle $\mathcal{F}$ of the tangent bundle TM. Here, $n-k$ is said to be the codimension of $\mathcal{F}$.

Due to the heavy prevalence of bundle theory throughout the paper thus far, it's no surprise that definition B2.13 is so pertinent. Indeed, the theory of foliations is deeply intertwined in the theory of complex and hypercomplex geometry in a number of seemingly unrelated ways. For example, the discussion on the Frobenius theorem at the end of appendix 1: is intimately related to the theory of foliations due to the fact that it's precisely this theorem of Frobenius which connects the rather loose notion of a hyperplane distribution to the significantly more rigid notions in foliation theory by way equating integrability, complete integrability, and involutivity.

One result discussed previously which involves relates hypercomplex geometry to foliation theory is the paper [24] which states explicitly that all manifolds which are locally

conformally hyperkähler but not hyperkähler admit a natural dimension-1 foliation $\mathcal{F}$ and that, when this foliation has compact leaves, the leaf space $\mathcal{L} = M/\mathcal{F}$ of $\mathcal{F}$ is a compact 3-Sasakian orbifold. Authors of [57] take this line of exposition even farther, noting that by imposing the assumptions necessary to make $\mathcal{L}$ either a C^∞ manifold or an orbifold, the intersection between the studies of hypercomplex geometry and foliation theory can be highlighted more still. Those authors later show that examination of a select few natural foliations on locally conformally hyperkähler spaces allow classifications involving Betti number restrictions, integrability and structure compatibility, and bundle theory related to complex and hypercomplex Hopf surfaces [57].

On the other hand, there are a number of equally interesting results relating foliation theory to some of the geometry discussed elsewhere which *haven't* been considered thus far. One example comes from [63] and can be summarized as follows.

Example B2.14. Let (M, g) be a 4-dimensional Riemannian manifold and let $\mathcal{F}$ be a dimension-2 foliation on M. The foliation $\mathcal{F}$ has associated tangent and normal bundles $T_\mathcal{F}$ and $N_\mathcal{F} = T_M/T_\mathcal{F}$, respectively, consisting of vector fields which are tangent, respectively normal, to the leaves of $\mathcal{F}$. What's more, the metric g determines an embedding of the normal bundle $N_\mathcal{F}$ as a subbundle of T_M, whereby there exists a g-orthogonal splitting

$$T_M = T_\mathcal{F} \oplus N_\mathcal{F}.$$

The claim is that $\mathcal{F}$ induces an almost-complex structure $J = J_\mathcal{F}$ on M. Indeed, defining J so that it rotates the fibers of $T_\mathcal{F}$ and $N_\mathcal{F}$ by $\pi/2$ clockwise yields an endomorphism which squares to minus the identity. This can be expressed more explicitly by picking an orthonormal frame $\{\tau_1, \tau_2, \nu_1, \nu_2\}$ in T_M so that $\tau_i \in T_\mathcal{F}$ and $\nu_i \in N_\mathcal{F}$ for $i = 1, 2$, at which point J can be defined so that $J\tau_1 = \tau_2$ and $J\nu_1 = \nu_2$. It can be shown that the leaves of $\mathcal{F}$ are J-holomorphic and that the homotopy class of $J_\mathcal{F}$ is independent of g.

A number of other geometry-centric results concerning foliations, especially in 4-manifolds, can be found in [63] and [20], among others.

Next, attention shifts to the previously-discussed contact structures. The basics here will be borrowed mostly from [50].

Definition B2.15. A smooth 1-form α on a smooth manifold M^{2n-1} is said to be a *contact form* if $\alpha \wedge (d\alpha)^{n-1} \neq 0$ everywhere on M. A smooth hyperplane distribution $\mathcal{D}$ on M is called a *contact structure* if, for every point $x \in M$, there is a neighborhood U and a contact form α on U such that $\mathcal{D}|_U = \ker(\alpha)$.

As a result of the Frobenius theorem, a contact structure $\mathcal{D}$ on a manifold M^{2n-1} is nowhere integrable; this follows from a technical lemma stated, e.g., in [50][52] which states that a distribution $\mathcal{D}$ of the form $\mathcal{D} = \ker(\alpha)$ for a 1-form α is involutive (hence, integrable) if and only if $\alpha \wedge d\alpha \equiv 0$.

Among the many applications of contact geometry to the material presented thus far was the vast intersection between contact geometry and Sasakian geometry. As it happens,

[52] Lemma 1.3.

however, the applications of contact geometry (both directly and via tangentially related notions) to the geometries discussed herein is substantial, so much so that a genuine exposition thereon is impossible. A number of worthwhile considerations related to contact structures can be found throughout [50], [58], [34], and sources cited therein.

2.2.3 Orbifold Theory

The word "orbifold" comes up incredibly often in the sections above pertaining to 3-Sasakian and Stiefel structures, whereby it stands to reason that somehow these *orbifolds* allow more flexibility than their *manifold* counterparts. As a matter of fact, this is true, and in the limited exposition that follows, the goal will be to illustrate some of the major characteristics of these orbifolds.

Succinctly, an orbifold is a space locally modeled on $\mathbb{R}^n$ modulo finite group actions. This notion was first made precise by Thurston, and so it would make no sense to formally define the term using any source except [68]. To that end:

Definition B2.16. An orbifold O consists of a Hausdorff space X_O, along with a covering of X_O by a collection of open sets $\{U_\alpha\}$ closed under finite intersections so that to each U_i is associated a finite group Γ_i, an action of Γ_i on an open subset $\widetilde{U}_i \subset \mathbb{R}^n$, and a homeomorphism $\varphi_i : U_i \cong \widetilde{U}_i/\Gamma_i$ subject to the following compatibility conditions: Whenever $U_i \subset U_j$, there is to be an injective homomorphism $f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$ and an embedding $\widetilde{\varphi}_{ij} : \widetilde{U}_i \hookrightarrow \widetilde{U}_j$ equivariant with respect to f_{ij} (so that for $\gamma \in \Gamma_i, \widetilde{\varphi}_{ij}(\gamma x) = f_{ij}(\gamma)\widetilde{\varphi}_{ij}(\gamma)$) so that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{U}_i \arrow[r, "\widetilde{\varphi}_{ij}"] \arrow[d] & \widetilde{U}_j \arrow[d] \\
\widetilde{U}_i/\Gamma_i \arrow[r, "\varphi_{ij} = \widetilde{\varphi}_{ij}/\Gamma_i"] & \widetilde{U}_j/\Gamma_i \arrow[d, "f_{ij}"'] \\
 & \widetilde{U}_j/\Gamma_j \\
U_i \arrow[uu, "\varphi_i"] \arrow[r, phantom, "\subset"] & U_j \arrow[u, "\varphi_j"']
\end{tikzcd}$$

Figure 2
Orbifold Compatibility Condition

In addition to the above definition, Thurston gives a number of examples of orbifold structures occurring "naturally" in topology. First and foremost, every closed manifold is an orbifold where each group Γ_i is trivial, i.e. $\widetilde{U} = U$. Similarly, manifolds with boundary can

be given orbifold structures by treating the boundary as a "mirror" so that any point on the boundary has a neighborhood modeled on $\mathbb{R}^n/\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting by reflection in a hyperplane. What's more, for any manifold M and any group Γ acting properly discontinuously on M, M/Γ is an orbifold. A generous amount of theory pertaining to orbifolds is presented in this chapter[53] of [68] along with a number of illustrative drawings to further emphasize the notions described.

2.3 Some Miscellany

2.3.1 Compact Hypercomplex 4-Manifolds & Hyperkähler $4n$-Manifolds

Recall two major results given early in the present paper concerning dimension-4 compact manifolds admitting hypercomplex and hyperkähler structures. Indeed, in section 2.1, it's shown that any compact hypercomplex 4-manifold which isn't conformally equivalent to a flat torus is necessarily conformally equivalent to either a Hopf surface or a K3 surface; what's more, in section 2.2, it's stated that any compact hyperkähler $4n$-manifold, $n > 1$, is deformation equivalent to either the Hilbert scheme of points on a K3 surface, or to a generalized Kummer variety. Like so many of the results throughout the paper, these are stated without proof and without clearly defining some of the terms involved. The goal here is to remedy part of that.

Definitions B2.17.

1. A *Hopf surface* is the quotient manifold obtained from $\mathbb{C}^n \setminus \{0\}$ modulo the $\mathbb{Z}$-action of the form $(z_1, \ldots, z_n) \mapsto (\lambda^k z_1, \ldots, \lambda^k z_n)$, $k \in \mathbb{Z}$, $0 < \lambda < 1$.

2. A *K3 surface* is a complete smooth surface which is simply connected and which has trivial canonical bundle.

3. A *Hilbert scheme* is a scheme which is the parameter space for the closed subschemes of some projective space or general projective scheme.

4. A *generalized Kummer variety* is a quotient variety (W, ϕ) of a polarized abelian variety A with respect to $\mathfrak{G}(A, \mathfrak{X})$ such that W, ϕ are defined over a field $\mathbb{K}$ and that W is a projective variety. Here, A is an Abelian variety, $\mathfrak{X}$ is a structure on A which polarizes A, and $\mathfrak{G}(A, \mathfrak{X})$ is the group of automorphisms of the polarized variety A.

Remarks.

1. The first of these definitions is straightforward enough. Note that the Hopf surface is subsumed by the generalized Hopf surface defined in appendix 2.1 above. Moreover, a Hopf surface $X = (\mathbb{C}^n \setminus \{0\})/\Gamma$ is necessarily isomorphic to the product of spheres, $X \cong S^1 \times S^{2n-1}$.

[53] Chapter 13.

2. As noted in several sources (e.g., [41], [70], [71], etc.), there are a number of different (but equivalent) ways to define K3 surfaces. Moreover, all K3 surfaces are diffeomorphic to one another.

3. The third definition isn't insightful at all. Unfortunately, gaining insight requires a discussion of scheme theory and the algebraic geometry of Grothendieck that's far more sophisticated than what it's worth for the purpose of this paper.

4. The remarks about the third item carry true here as well, replacing "scheme" with "variety." Here, the interested reader is encouraged to read [51] and its cited references for more background.

Besides the definitions given above, the author feels it worthwhile to say something small about the structures listed in 2.2 in section 2.1 above, particularly with regard to demonstrating explicit hypercomplex structures on each of the spaces listed therein. Because both the torus and the quaternionic Hopf surface exist as quotient spaces obtained from $\mathbb{H}$, right-multiplication by imaginary quaternion units $i, j, k \in \mathbb{H}$ yield hypercomplex structures accordingly. Moreover, the fact that (i) all K3 surfaces are diffeomorphic, (ii) the sphere $S^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ is a K3 surface, and (iii) S^3 has a natural hypercomplex structure $\{I, J, K\}$ (given again by right multiplication by $i, j, k \in \mathbb{H}$) means that all K3 surfaces inherit induced hypercomplex structures from S^3.

2.3.2 A Few Words About Twistor Theory

The goal of this last appendix subsection is to consider the relationship of twistor theory with, e.g., hyperkähler manifolds. Undoubtedly, this material exists in many pieces of literature; for the purpose of this section, however, the presentation in [39] will suffice.

Given a manifold M, the *twistor space* of M is the product $Z = M \times S^2$. Note that when $M = (M, I, J, K)$ has real dimension $4n$ and is hyperkähler, there is an S^2 of integrable complex structures of the form $I_{\boldsymbol{u}} = aI + bJ + cK$ where $\boldsymbol{u} = (a, b, c) \in \mathbb{R}^3$ satisfies $\|\boldsymbol{u}\| = 1$. Moreover, the existence of a natural complex structure I_0 on S^2 implies that the structure $\boldsymbol{I}$ of the form

$$\boldsymbol{I}(X, Y) = (I_{\boldsymbol{u}} X, I_0 Y),$$

$X \in T_m M$, $Y \in T_{\boldsymbol{u}} S^2$, is an integrable almost-complex structure on the tangent space $T_m M \oplus T_{\boldsymbol{u}} S^2$ to the twistor space Z, whereby it follows that Z is a complex manifold of dimension $2n + 1$.

[39] goes on to list a number of properties related to the complex structure on Z, including an in-depth discussion of how the twistor space Z "encodes" the hyperkähler metric on M. Diligent readers are encouraged to follow [39] and sources cited therein for more details.

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