Elementary Row Operations, REF, and RREF

Now that we know a little bit about matrices, we're going to learn how to use matrices to solve problems!

One of the most useful things we can do to a matrix is to "row reduce" it.

Row reduction is a process by which a matrix is simplified into an "equivalent" matrix which is easier to use overall. In order to make this procedure more canonical, we'll perform our reduction using a very precise collection of operations known as elementary row operations.

Throughout, we'll refer to the matrix

$$\mathsf{M} = \begin{pmatrix} 4 & 1 & -2 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

for all of our examples.

Elementary Row Operations

There are **three** elementary row operations that we can perform on a matrix to get a new matrix which is considered "row equivalent" to it:[1]

1. Interchange two rows.

   Example: Interchanging rows 2 and 4 (shorthand: R2 $\leftrightarrow$ R4) in M yields the following:

$$\begin{pmatrix} 4 & 1 & -2 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{\text{R2}\leftrightarrow\text{R4}} \begin{pmatrix} 4 & 1 & -2 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & 3 & -1 \\ 1 & 0 & 0 & -5 \end{pmatrix}$$

2. Multiply all entries in a row by a nonzero constant.

   Example: We can multiply row 1 of M by 3 (shorthand: R1 = 3R1) to get the following:

$$\begin{pmatrix} 4 & 1 & -2 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{\text{R1}=\text{3R1}} \begin{pmatrix} 12 & 3 & -6 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

[1] **Definition:** Two matrices M and N are said to be row equivalent if there is a series of elementary row operations which transforms M into N (and vice versa).

3. Add (or subtract) a nonzero multiple of one row to another row.

   Example: Let's say we wanted to add 4 times row 2 to row 3, i.e. we leave every row the same *except* row 3, and we change row 3 by adding to it 4R2 (shorthand: R3 = R3 + 4R2).

   We could do this all at once, but to split it into steps, we could:

   (a) compute 4R2, i.e. $4 \cdot \langle 1, 0, 0, -5 \rangle = \langle 4, 0, 0, -20 \rangle$;

   (b) compute R3 plus 4R2, i.e. $\langle 0, 1, 3, -1 \rangle + \langle 4, 0, 0, -20 \rangle = \langle 4, 1, 3, -21 \rangle$; and

   (c) form the new matrix having the same entries as M in rows 1, 2, and 4, and having $\mathrm{R3} + 4\mathrm{R2} = \langle 4, 1, 3, -21 \rangle$ as its third row.

   Hence, the result is:

$$\begin{pmatrix} 4 & 1 & -2 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{\mathrm{R3}=\mathrm{R3}+4\mathrm{R2}} \begin{pmatrix} 4 & 1 & -2 & 0 \\ 1 & 0 & 0 & -5 \\ 4 & 1 & 3 & -21 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

Unsurprisingly, we can perform these three elementary row operations in succession to provide additional simplification. With a little foresight, this can yield a *much simpler* matrix which is row-equivalent to the matrix we started with:

Example:

$$\underbrace{\begin{pmatrix} 4 & 1 & -2 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}}_{\mathsf{M}} \xrightarrow{\mathrm{R1}\leftrightarrow\mathrm{R2}} \begin{pmatrix} 1 & 0 & 0 & -5 \\ 4 & 1 & -2 & 0 \\ 0 & 1 & 3 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{\mathrm{R2}\leftrightarrow\mathrm{R3}} \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 4 & 1 & -2 & 0 \\ 1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{\mathrm{R3}=\mathrm{R3}-4\mathrm{R1}} \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & -2 & 20 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$\xrightarrow{\mathrm{R3}=\mathrm{R3}-\mathrm{R2}} \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -5 & 21 \\ 1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{\mathrm{R3}=(-\frac{1}{5})\mathrm{R3}} \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -\frac{21}{5} \\ 1 & 1 & -1 & -1 \end{pmatrix} \longrightarrow \cdots$$

Note that each of the above matrices is row-equivalent to M.

Moving forward, one of our main goals will be to perform these three elementary row operations in succession until we get to a matrix which is in Row Echelon Form (REF) and/or Reduced Row Echelon Form (RREF).

Row Echelon Form (REF)

First, the definition:

Definition: A matrix is in row echelon form (REF) if it satisfies the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading (nonzero) entry of a row is in a column to the right of the leading (nonzero) entry of the row above it.
3. All entries in a column below a leading (nonzero) entry are zeros.

As a remark, note that the entries above the leading (nonzero) entries of such a matrix may or may not equal 0. For instance, **both** of the following matrices are in REF:

$$\mathsf{A} = \begin{pmatrix} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{2} & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathsf{B} = \begin{pmatrix} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{2} & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that an entry $\boxed{\text{in a box}}$ denotes the leading nonzero entry of that particular row.

Reduced Row Echelon Form (RREF)

As it happens, *neither* A nor B are in reduced row echelon form (RREF), because both have some mild simplifications that can be done to them.

Definition: A matrix is in reduced row echelon form (RREF) if it satisfies the following three properties:

1. It is in REF;
2. The leading (nonzero) entry in each row is 1.
3. Each leading 1 is the only nonzero entry in its column.

In particular, we note that A fails both (2) and (3): The leading nonzero entry in row2 (i.e. the (2,2)-entry) is 2, and the column containing that 2 has other nonzero entries (namely, the 1 in the (1,2)-entry). However, we can put A in RREF using two elementary row operations:

$$\mathsf{A} = \begin{pmatrix} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{2} & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{R2}=(\frac{1}{2})\text{R2}} \begin{pmatrix} \boxed{1} & 1 & 1 & 2 \\ 0 & \boxed{1} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{R1}=\text{R1}-\text{R2}} \underbrace{\begin{pmatrix} \boxed{1} & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & \boxed{1} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{RREF}}.$$

For B, one elementary row operation will get it into RREF:

$$\mathsf{B} = \begin{pmatrix} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{2} & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{R2}=(\frac{1}{2})\text{R2}} \underbrace{\begin{pmatrix} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{1} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{RREF}}$$