Understanding Market Price of Risk

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1 Introduction

Below I try to hash out the ever-confusing market price of risk (MPOR). I show how it naturally appears when the underlying is not tradable, and why it is absent exactly when the underlying is tradable.

I start with a derivation of the Black-Scholes PDE using the hedging portfolio approach. I don't show how to solve the PDE as that is fairly standard (see [2]). Continuing with the Black-Scholes model, I then derive the option price from a totally different approach - that of risk-neutral pricing. I should mention the original, "hedging portfolio" approach was the one used by Black, Scholes and Merton in 1973, along with Vasicek in 1977. Harrison and Pliska are usually noted for coming up with the risk-neutral approach in 1983, which is far more general for pricing. It is in this risk-neutral approach that the MPOR appears, but does so only by "forcing" it to - it is only a rearranging of an SDE, which is done to change the measure.

In contrast, in interest rate derivatives the MPOR naturally appears, and this is the topic I take up in the next section. Following this interest rate derivative section, I compare it with Black-Scholes to see that the MPOR vanishes in the latter, and this is exactly due to the underlying (stock) being a tradable asset.

2 Black-Scholes

Goal: Price a derivative written on a stock whose price is governed by the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

2.1 PDE Approach

Method: Explicitly construct a self-financing portfolio that perfectly hedges a short position in the derivative. A perfect hedge is riskless, so the portfolio must instantaneously earn exactly the risk-free instantaneous rate r, else arbitrage. To do this, we explicitly construct (that is, we actually find the number of shares Δ of underlying to own in terms of a computable quantity) a trading strategy by trading in the underlying (stock) where for all $0 \le t \le T$, we own just the right amount of stock such that the portfolio implementing this strategy is riskless. We will see this then gives a PDE for the option price.

Implementation: construct a self-financing portfolio short 1 option worth $V(t, S(t))$ and long $\Delta(t)$ shares of the stock, each worth $S(t)$. The value $\Pi(t)$ of this portfolio is $\Pi(t) = \Delta(t)S(t) - V(t)$. The self-financing assumption implies that the change in the portfolio value is

$$d\Pi(t) = \Delta(t)dS(t) - dV(t, S(t)).$$

Suppressing the arguments, note

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dS + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}dSdS \\ &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}\left[\mu S dt + \sigma S dW\right] + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\left[\sigma^2 S^2 dt\right] \\ &= \left[\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial x} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial x^2}\right]dt + \sigma S\frac{\partial V}{\partial x}dW, \end{aligned}$$

so

$$\begin{aligned} d\Pi &= \Delta\mu S dt + \Delta\sigma S dW - \left[\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial x} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial x^2}\right]dt - \sigma S\frac{\partial V}{\partial x}dW \\ &= \left[\Delta\mu S - \frac{\partial V}{\partial t} - \mu S\frac{\partial V}{\partial x} - \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial x^2}\right]dt + \left[\Delta\sigma S dW - \sigma S\frac{\partial V}{\partial x}\right]dW. \end{aligned}$$

Here's the strategy: at each time $0 \le t \le T$, long $\Delta(t) = \frac{\partial V}{\partial x}(t, S(t))$ shares of stock. Then

$$\begin{aligned} d\Pi &= \left[-\frac{\partial V}{\partial t} - \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial x^2}\right]dt \\ &= r\Pi dt \\ &= r\left(\frac{\partial V}{\partial x}S - V\right)dt, \end{aligned}$$

where the second equality is true to rule out arbitrage. Hence

$$\begin{aligned} &-\frac{\partial V}{\partial t} - \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial x^2} = rS\frac{\partial V}{\partial x} - rV \\ \iff &\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial x^2} + rS\frac{\partial V}{\partial x} - rV = 0. \end{aligned}$$

Ignoring boundary and initial conditions, this is our "pricing" PDE for $V = V(t, x)$,

$$\boxed{\frac{\partial V}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV = 0}$$

2.2 Risk-Neutral Pricing Approach

Method: Construct a self-financing strategy that replicates the derivative payoff $X(S(T))$. By no-arbitrage, the value $\Pi(t)$ of the portfolio implementing such a strategy must be the price of the derivative. Self-financing replicating strategies are guaranteed to exist by the Martingale Representation Theorem (MRT), so we won't explicitly construct the strategy as in the PDE approach but instead call on the MRT to give us one. Then, the FTAP states that if the discounted stock price processes $e^{-rt}S(t)$ is a Q-martingale, then the value $\Pi(t)$ of a portfolio implementing any self-financing replicating strategy is

$$\Pi(t) = E_Q[e^{-r(T-t)}X(S(T))|\mathcal{F}_t]$$

Then, again to rule out arbitrage, the RHS must be the value of the derivative for all $0 \le t \le T$.

Implementation: With $S(t)$ governed by $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$ and a constant interest rate r, the discounted stock price $e^{-rt}S(t)$ is governed by (suppressing arguments)

$$\begin{aligned} d\left(e^{-rt}S\right) &= -re^{-rt}Sdt + e^{-rt}dS \\ &= -re^{-rt}Sdt + e^{-rt}\left[\mu Sdt + \sigma SdW\right] \\ &= e^{-rt}S\left[(\mu - r)dt + \sigma dW\right] \qquad (1) \\ &= e^{-rt}S\sigma\left[\frac{\mu - r}{\sigma}dt + dW\right]. \qquad (2) \end{aligned}$$

Note eqn (1) shows the instantaneous expected return on the discounted stock is $\mu - r$, but the volatility is the same. In eqn (2) we see for the first time the market price of risk, which we'll call $\lambda = \frac{\mu - r}{\sigma}$.

Finally, let $W(t)$ be our BM on a probability space $(\Omega, \mathcal{F}, P)$, and let $\lambda(t)$ be adapted to the filtration for $W(t)$. Define the processes

$$\begin{aligned} Z(t) &= \exp\left(-\int_0^t \lambda(s)dW(s) - \frac{1}{2}\int_0^t \lambda^2(s)ds\right), \\ \tilde{W}(t) &= W(t) + \int_0^t \lambda(s)ds \end{aligned}$$

and define the random variable $Z = Z(T)$ (Z is a Radon-Nikodym derivative). Finally, define a probability measure Q by

$$Q(A) = \int_A Z(\omega)dP(\omega), \quad \text{for all } A \in \mathcal{F}.$$

Then by Girsanov's theorem, $\tilde{W}(t)$ is a Q-BM.

So, let $\lambda(t) = \lambda = \frac{\mu - r}{\sigma}$ in Girsanov's theorem. Define $Z(t) = \exp\left(-\lambda W(t) - \frac{1}{2}\lambda^2 t\right)$ and $\tilde{W}(t) = W(t) + \lambda t$. Define the random variable $Z = Z(T)$ and define the probability Q by $Q(A) = \int_A Z\,dP$. Then $\tilde{W}(t)$ is a Q-BM, and thus

$$d\left(e^{-rt}S\right) = e^{-rt}S\sigma d\tilde{W},$$

so $e^{-rt}S$ is a Q-martingale. Since we've found a measure for which $e^{-rt}S$ is a martingale, the value $\Pi(t)$ of any portfolio using a self-financing replicating strategy is $\Pi(t) = E_Q[e^{-r(T-t)}X(S(T))|\mathcal{F}_t]$ by the FTAP. By no-arbitrage,

$$E_Q[e^{-r(T-t)}X(S(T))|\mathcal{F}_t]$$

is the price of the derivative with payoff $X(S(T))$.

Some items worth noticing at this point:

- In the "hedging portfolio" (first) approach, we didn't say, "the price of the *portfolio* must be the same as the option, else arbitrage." This is the argument used in the risk-neutral approach, where instead of a *hedging* portfolio, we have a *replicating* portfolio. Instead, we simply use the hedging portfolio as a tool to get to the pricing PDE.
- The MPOR was the adapted process we used in Girsanov's theorem to change the drift of our discounted underlying process.
- the μ and σ seen in the MPOR were the μ and σ from the asset used to hedge the derivative, where the asset price S was governed by an SDE of the form $dS = \mu S dt + \sigma S dW$. The coefficients μ and σ may depends on (t, S) as well, but *the asset used as a hedge was written as an exponential SDE.*
- The only source of randomness anywhere is the stock, and hence this is the only source of randomness in the derivative. We then hedge this single source of randomness with the stock.
- MPOR parameters are for the *tradable* asset. Think: talk about *returns* over risk-free per unit of *risk* - these are terms for tradable assets only!

3 Interest Rate Derivatives

Goal: Price a "derivative" "written on" an interest rate whose value is governed by a general SDE

$$dr(t) = \alpha(t, r(t))dt + \beta(t, r(t))dW(t).$$

3.1 Overview of SDE Assumption

Where do SDEs come from? Below, Vasicek's argument from his seminal paper [1] is summarized, where he gives explicit assumptions that lead to an SDE for the underlying r (spot rate). Assumptions:

A.1 The spot rate r follows a continuous Markov process (probability distribution of $\{r(s), s \geq t\}$ is completely determined by the value of $r(t)$).

A.2 The price $P(t,T)$ of a discount bond is determined by the assessment, at time t, of the segment $\{r(s), t \leq s \leq T\}$ of the spot rate over the term of the bond.

A.3 The market is efficient. In particular,

- There are no transaction costs,
- Information is available to all investors simultaneously,
- Every investor acts rationally (prefers more wealth to less and uses all available info).

Continuous Markov processes are called diffusion processes, and can be described by an SDEs. Fix a probability space $(\Omega, \mathcal{F}, P)$. then by [A.1], the evolution of $r(t)$ is governed by

$$dr(t) = \alpha(t, r(t))dt + \beta(t, r(t))dW(t), \tag{3}$$

where W is a P-BM and α, β are given functions.

By [A.1] the development of the spot rate over an interval (t,T), given its values prior to t, depends only on the current value $r(t)$. Then, [A.2] implies $P(t,T) = P(t,T,r(t))$; i.e., the current spot rate is the *only* state variable for the entire term structure.

3.2 PDE Approach

Method: Explicitly construct a self-financing portfolio that perfectly hedges a short position in the derivative (bond). A perfect hedge is riskless, so the portfolio must instantaneously earn exactly the risk-free instantaneous rate r (spot rate), else arbitrage. To do this, we explicitly construct a trading strategy by trading in another derivative, which will be a bond of another maturity, where for all $0 \leq t \leq T$, we own just the right amount of the "hedging" derivative such that the portfolio implementing this strategy is riskless. Just like for the stock derivative, this riskless assumption with lead to a pricing PDE.

Implementation: Note first that since r follows the SDE (3) and since $P = P(t,T,r(t))$,

$$\begin{aligned}
dP &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial r}dr + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}drdr \\
&= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial r}\left[\alpha dt + \beta dW(t)\right] + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\left[\beta^2 dt\right] \\
&= \left(\frac{\partial P}{\partial t} + \alpha\frac{\partial P}{\partial r} + \frac{1}{2}\beta^2\frac{\partial^2 P}{\partial r^2}\right)dt + \beta\frac{\partial P}{\partial r}dW(t) \\
&= \mu(t,T)Pdt + \sigma(t,T)PdW(t),
\end{aligned}$$

where

$$\mu(t,T,r) = \frac{1}{P(t,T,r)}\left(\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial r} + \frac{1}{2}\beta^2\frac{\partial^2}{\partial r^2}\right)P(t,T,r),$$
$$\sigma(t,T,r) = \frac{1}{P(t,T,r)}\beta\frac{\partial}{\partial r}P(t,T,r).$$

Above, we emphasize the dependencies of the parameters by including and excluding some function arguments. E.g. for Ito's lemma we view $P = P(t, r(t))$ as T is just a parameter. In $dP = \mu(t,T)Pdt - \sigma(t,T)PdW(t)$ we emphasize that the coefficients depend on maturity T and of course the current time t, although they also depend on r as well. The dependence on maturity is the relevant dependence later, so we emphasize it now.

Now construct a portfolio short one bond (a derivative) with price $P_1 := P(t,T_1)$ and long $\Delta(t)$ units of another bond with price $P_2 := P(t,T_2)$ where $t \le T_1 < T_2$. Again, by the self-financing assumption the change in value Π of this portfolio is

$$\begin{aligned} d\Pi &= \Delta dP_2 - dP_1 \\ &= \Delta\left[\mu_2 P_2 dt - \sigma_2 P_2 dW\right] - \left[\mu_1 P_1 dt - \sigma_1 P_1 dW\right] \\ &= \left(\Delta\mu_2 P_2 - \mu_1 P_1\right)dt - \left(\Delta\sigma_2 P_2 - \sigma_1 P_1\right)dW, \end{aligned}$$

where, e.g., $\mu_1 := \mu(t,T_1)$. Here's the strategy: at each time $0 \le t \le T_1$, long $\Delta(t) = \frac{\sigma_1 P_1}{\sigma_2 P_2}$ units of the T_2-bond. Then

$$d\Pi = \left(\frac{\sigma_1 P_1}{\sigma_2}\mu_2 - \mu_1 P_1\right)dt = r\Pi dt = r\left(\frac{\sigma_1 P_1}{\sigma_2} - P_1\right)dt,$$

where the second equality is to rule out arbitrage. Hence

$$\begin{aligned} &\frac{\sigma(t,T_1)P_1}{\sigma(t,T_2)}\mu(t,T_2) - \mu(t,T_1)P_1 = r\frac{\sigma(t,T_1)P_1}{\sigma(t,T_2)} - rP_1 \\ \iff\ &\frac{\mu(t,T_2)-r}{\sigma(t,T_2)} = \frac{\mu(t,T_1)-r}{\sigma(t,T_1)}. \end{aligned} \tag{4}$$

Since (4) is valid for arbitrary maturity dates it follows that the ratio $\frac{\mu(t,T)-r}{\sigma(t,T)}$ is independent of T. Let $\lambda(t,r)$ denote the common value of this ratio for any T, given the current spot rate of $r(t) = r$:

$$\begin{aligned} \lambda(t,r) &= \frac{\mu(t,T,r)-r}{\sigma(t,T,r)}, \qquad T \ge t \\ \iff\ &\sigma(t,T,r)\lambda(t,r) = \mu(t,T,r) - r. \end{aligned}$$

Substituting the value for μ and σ we arrive at a PDE for the bond price:

$$\begin{aligned} &\frac{1}{P}\beta\frac{\partial}{\partial r}P\lambda = \frac{1}{P}\left(\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial r} + \frac{1}{2}\beta^2\frac{\partial^2}{\partial r^2}\right)P - r \\ \iff\ &\frac{\partial P}{\partial t} + \frac{1}{2}\beta^2\frac{\partial^2 P}{\partial r^2} + (\alpha - \beta\lambda)\frac{\partial P}{\partial r} - rP = 0. \end{aligned}$$

Again ignoring boundary and initial conditions we have a "pricing" PDE for $P(t, r)$:

$$\boxed{\frac{\partial P}{\partial t} + \frac{1}{2}\beta^2 \frac{\partial^2 P}{\partial r^2} + (\alpha - \beta\lambda)\frac{\partial P}{\partial r} - rP = 0.}$$

Some items worth noticing at this point:

- We used an arbitrage argument (setting $d\Pi = r\Pi dt$) to arrive at the conclusion that *the expected return on any bond in excess of the spot rate is proportional to its standard deviation* (Eq. (4)). This property was then used to derive a PDE for the bond prices.

- The MPOR appears explicitly in the pricing PDE. Compare this with the BS PDE, where it is absent. We'll see in the next section that this is exactly due to the underlying (r in the interest rate derivative case) not being tradable.

4 Comparing Interest Rate and Stock Derivatives

Takeaway: The fundamental difference between interest rate and stock derivatives is that the underlying for stock derivatives is tradable, and we use the underlying itself to trade in for the hedging portfolio, while the underlying for the interest rate derivatives is NOT tradable (interest rates aren't directly tradable), so we instead trade in another derivative in the hedging portfolio. We'll see below that when the underlying is tradable, the MPOR disappears.

Note: being careful about "hedging" and "replicating" portfolios. So far, we haven't shown these hedging portfolio actually replicate the payoff - all we care about is if the *hedge* is just right (choosing Δ just right), the return on the portfolio must be risk-free, which then gives us a PDE for the option price. Replicating portfolios take a different approach.

Let's construct the BS PDE with the general SDE we used for the interest rate. We'll make a slight change to (3) in that we're going to write the value S explicitly, as opposed to implicitly in the coefficients α and β. This results in a slight simplification down the road, but for now we make the remark that if you're going to be trading in it, make the SDE exponential.

The dynamics for the underlying stock are given by

$$dS(t) = \alpha(t)S(t)dt + \beta(t)S(t)dW(t). \tag{5}$$

Again, first note that

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dSdS \\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\left[\alpha S dt + \beta S dW\right] + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left[\beta^2 S^2 dt\right] \\
&= \left(\frac{\partial V}{\partial t} + \alpha S\frac{\partial V}{\partial S} + \frac{1}{2}\beta^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \beta S\frac{\partial V}{\partial S}dW \\
&= \mu(t,V)V dt + \sigma(t,V)V dW,
\end{aligned}
$$

where

$$
\begin{aligned}
\mu(t,S(t)) &= \frac{1}{V(t,S(t))}\left(\frac{\partial}{\partial t} + \alpha(t)S(t)\frac{\partial}{\partial r} + \frac{1}{2}\beta(t)^2 S(t)^2\frac{\partial^2}{\partial S^2}\right)V(t,S(t)), \\
\sigma(t,S(t)) &= \frac{1}{V(t,S(t))}\beta(t)S(t)\frac{\partial}{\partial S}V(t,S(t)).
\end{aligned}
$$

The dynamics of the self-financing hedging portfolio are

$$
\begin{aligned}
d\Pi &= \Delta dS - dV \\
&= \Delta[\alpha S dt + \beta S dW] - [\mu V dt + \sigma V dW] \\
&= (\Delta\alpha S - \mu V)dt + (\Delta\beta S - \sigma V)dW.
\end{aligned}
$$

Again, set $\Delta(t,S(t)) = \frac{\sigma(t)V(t,S(t))}{\beta(t)S(t)}$. Then the portfolio is riskless for all $0 \le t \le T$, so

$$
d\Pi = \left(\frac{\sigma V\alpha}{\beta} - \mu V\right)dt = r\left(\frac{\sigma V}{\beta} - V\right)dt.
$$

Hence we have

$$
\begin{aligned}
&\frac{\sigma V\alpha}{\beta} - \mu V = r\frac{\sigma V}{\beta} - rV \\
\iff\ &\frac{\alpha - r}{\beta} = \frac{\mu - r}{\sigma}. \qquad (6)
\end{aligned}
$$

Eqn. (6) is the key difference between the underlying being tradable or not. Because the underlying was tradable, the MPOR for the derivative $\frac{\mu-r}{\sigma}$ is the same as the MPOR for the *underlying* $\frac{\alpha-r}{\beta}$. We derived this result (6) by a no-arbitrage argument, namely that the portfolio hedging the derivative must earn the risk-free rate, and hence all tradables in a market should have the same MPOR.

As before set $\lambda(t,S(t)) = \frac{\mu(t,S(t))-r}{\sigma(t,S(t))} \iff \sigma(t,S(t))\lambda(t,S(t)) = \mu(t,S(t)) - r$. We again get our PDE from this equation:

$$
\begin{aligned}
&\frac{1}{V}\beta S\frac{\partial V}{\partial S}\lambda = \frac{1}{V}\left(\frac{\partial V}{\partial t} + \alpha S\frac{\partial V}{\partial r} + \frac{1}{2}\beta^2 S^2\frac{\partial^2 P}{\partial S^2}\right) \\
\iff\ &\frac{\partial V}{\partial t} + \frac{1}{2}\beta^2 S^2\frac{\partial^2 V}{\partial S^2} + (\alpha - \lambda\beta)S\frac{\partial V}{\partial S} - rV = 0.
\end{aligned}
$$

The key difference now is that the MPOR λ vanishes. Indeed, since $\lambda = \frac{\alpha - r}{\beta}$, plugging this in above gives the usual BS PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\beta^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

References

[1] Oldrich Vasicek (1977). "An Equilibrium Characterization of the Term Structure." Journal of Financial Economics 5, 177-188.

[2] Steven Shreve (2010). "Stochastic Calculus for Finance II".

[3] Fabrice Douglas Rouah Four Derivations of the Black Scholes PDE. `http://www.frouah.com/financenotes/BlackScholesPDE.pdf`