On hermitian-holomorphic classes related to uniformization, the dilogarithm, and the Liouville Action

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Abstract

Metrics of constant negative curvature on a compact Riemann surface are critical points of the Liouville action functional, which in recent constructions is rigorously defined as a class in a Čech-de Rham complex with respect to a suitable covering of the surface.

We show that this class is the square of the metrized holomorphic tangent bundle in hermitian-holomorphic Deligne cohomology. We achieve this by introducing a different version of the hermitian-holomorphic Deligne complex which is nevertheless quasi-isomorphic to the one introduced by Brylinski in his construction of Quillen line bundles. We reprove the relation with the determinant of cohomology construction.

Furthermore, if we specialize the covering to the one provided by a Kleinian uniformization (thereby allowing possibly disconnected surfaces) the same class can be reinterpreted as the transgression of the regulator class expressed by the Bloch-Wigner dilogarithm.

Contents

1 Introduction

Metrics of constant negative curvature play a very important role in uniformization problems for compact Riemann surfaces of genus $g > 1$. The condition that the scalar curvature associated to a conformal metric on a Riemann surface X be equal to -1 is equivalent to the fact that the associated conformal factor satisfies a nonlinear partial differential equation known as the Liouville equation.

The Liouville equation appears as early as in one of the approaches considered by Poincaré to attack the uniformization theorem [27]. In relatively recent times, it has received considerable attention in Theoretical and Mathematical Physics due to the key role it plays in Polyakov's approach to String Theory [28], especially from the point of view of non-critical strings and two-dimensional quantum gravity. In this context one refers to the conformal factor of the metric as the Liouville "field."

As usual in the context of differential equations with a physical motivation, one normally would like to formulate a variational principle to express the Liouville equation as an extremum condition. Namely, given a Riemann surface X and the space $\mathscr{CM}(X)$ of all conformal metrics on it, the metric of constant negative curvature should be a critical point of a functional defined over $\mathscr{CM}(X)$. This functional is the Liouville action. As it happens, action functionals may turn out to be even more relevant than the equations they are associated to. The Liouville action is no exception in this sense: it has deep connection with the geometry of Teichmüller spaces [34, 35], and in Physics it describes the conformal anomaly in String Theory.

Providing a rigorous mathematical definition of the Liouville action functional is however far from trivial. The very geometric properties of the Liouville field itself prevent expressing the corresponding functional as a plain integral of a 2-form on a Riemann surface. Correction terms are required, typically in the form of integration of lower degree forms over the 1-skeleton of an appropriate simplicial realization of X. (One should notice that this behavior is not specific to the Liouville equation, and it is by now possible to give a characterization, in terms of homological algebra, of these type of functionals, see ref. [1].)

It is possible to directly determine the necessary correction terms by requiring that the variational problem be well defined. This, however, is not completely satisfactory from the point of view of certain applications to deformation theory, where a consistent definition across a *family* of surfaces is required. Quite recently, a more systematic construction, based on the homological algebra techniques developed by the author and L. A. Takhtajan in [2], was carried out by L. A. Takhtajan and L.-P. Teo in ref. [31], generalizing the earlier results of [34, 35]. The authors of ref. [31] constructed a Čech cocycle with respect to the étale cover of X associated to a quasi-Fuchsian (and more generally Kleinian) uniformization. Since their construction works across (Kleinian) deformations, it could be exploited to obtain results of global nature on the analytic geometry of Kleinian deformation spaces. As a further result, the authors of loc. cit. were able to rigorously prove the validity of the "holography principle" for the Liouville action corresponding to a large class of Kleinian (in particular Fuchsian and quasi-Fuchsian) uniformizations. Specifically, they proved that given a second kind Kleinian group, the corresponding Liouville action can be obtained as the regularized limit of the hyperbolic volume of the corresponding associated 3-manifold. This extends to the general Kleinian case a previous formula obtained by Krasnov [24] for classical Schottky groups.

Our interest in this matter is two-fold. From the perspective of the newer methods adopted in [3], the covering map $U \to X$ associated to the uniformization by a discrete group $\Gamma = \mathrm{Deck}(U/X)$ is but one of the many possible covers comprising an appropriate category C of, say, local diffeomorphisms $U \to X$ — the most obvious choice being that of standard open cover $\mathfrak{U} = \{U_i\}$ with associated space $U = \coprod_i U_i$. In particular one expects to be able to apply the methods of [2] and [3] uniformly on a class of reasonably behaved covers of X.

Second, the focus of ref. [3] was on the rigorous definition of a functional for quasi-conformal deformation of the Riemann surface X and its application to the study of projective structures. A main result is that the construction of the action is possible thanks to the vanishing of the "tame symbol" (see refs. [13] and [9] for the relevant definitions) $\big(T_X, T_X\big]$, where T_X is the holomorphic tangent line bundle of X. The vanishing determines local choices (with respect to a cover) of a Bloch-type dilogarithm which then allow for a cohomological construction of the action. There are many indication that the Liouville action ought to be the hermitian square of a functional of the type studied in [3].[1] Thus it is natural to ask whether there is an analogous mechanism as the one in loc. cit. to obtain a general construction of the Liouville action by replacing the holomorphic symbol maps and dilogarithms with corresponding real objects.

In this paper we answer this question in the affirmative. More precisely, we show that the Liouville action (up to the area term which is given by an ordinary 2-form) can be computed as a symbol map taking values in hermitian holomorphic Deligne cohomology, first introduced by Brylinski and McLaughlin in their study of degree four characteristic classes [9]. (By way of comparison, the tame symbols used in ref. [3] used holomorphic and smooth Deligne cohomology.) We will present a modified version of hermitian holomorphic cohomology that better suits our needs than the original definition—the resulting cohomology groups are however the same. In particular we show that the dilogarithm type terms are replaced here by the Bloch-Wigner function, the real valued counterpart of the dilogarithm (see refs. [7], and [23, 16] for a review.)

The appearance of the Bloch-Wigner function ties very well with the holography property of the Liouville function proved in [31] in the following sense. As mentioned before, the Liouville action (up to the area term) relative to a Kleinian uniformization[2] $U \to X$ can also be computed as the "regularized volume" of the associated 3-manifold $N = \Gamma\backslash(U \cup \mathbb{H}^3)$, where $\Gamma = \mathrm{Deck}(U/X)$ as before, $U \subset \mathbb{P}^1$ is the domain of discontinuity for Γ, and $\mathbb{H}^3$ is the standard hyperbolic 3-space. (To define the regularized volume would lead us too far afield. It suffices to mention that the conformal factor of a metric on $X = \partial N$ can be used to select a compact submanifold N_ϵ whose volume is finite. One then subtracts from the volume of N_ϵ the areas of the boundary components and other carefully chosen constants independent of the metric structure, so that the resulting quantity will have a finite limit as $\epsilon \to 0$.) On the other hand, the hyperbolic volume in three dimensions corresponds to a three dimensional (purely imaginary) class on $\mathrm{PSL}_2(\mathbb{C})$ expressible through the Bloch-Wigner dilogarithm, the so-called *regulator class.* We show that the regulator is precisely the obstruction to solving the cohomological descent conditions required to calculate the Liouville action for a covering map $U \to X$ with covering group a Kleinian group Γ. Indeed, since for a second kind Kleinian group the quotient $\mathbb{H}^3/\Gamma$ — being non-compact — carries no cohomology in dimension three, the class represented by the Bloch-Wigner function, pulled back to Γ via the imbedding $\Gamma \hookrightarrow \mathrm{PSL}_2(\mathbb{C})$, vanishes.

Returning to the cohomological interpretation of the construction of the Liouville action, it should be also noted that leaving aside the area term, our results show that the cohomologically non trivial part is indeed a square. Namely, for a conformal metric $\rho \in \mathscr{CM}(X)$ we consider the pair (T_X, ρ) as a holomorphic line bundle equipped with an hermitian metric. Then, using that hermitian holomorphic Deligne cohomology has a cup product, we show that the Liouville action is just the square of the class of (T_X, ρ). In fact this identification holds at the level of cocycles, rather than only for the corresponding classes.

Again leaving aside the area term, it immediately follows from the properties of hermitian holomorphic Deligne cohomology that most of the story carries over to the case of a *pair* of holomorphic line bundles L and L' equipped with hermitian metrics ρ and ρ', respectively. Furthermore, Brylinski shows in [11] that the pairing of two such holomorphic line bundles with metrics corresponds to the pairing defined by Deligne on the determinant of cohomology in [12]. Without introducing the machinery of 2-gerbes, we reobtain this result in our setting. Specifically, we directly obtain O. Gabber's formula for the hermitian metric on the determinant line from the explicit cocycle for the cup product of two metrized line bundles. In turn this shows that the Liouville action is a multiple of the determinant of cohomology, thereby generalizing earlier results (cf. [33]) — without assuming criticality.

1.1 Organization of the paper

This paper is organized as follows. Section 2 is devoted to the explanation of the main feature of Deligne cohomology, paying special attention to the product structures and the cone constructions. We provide some examples and collect some facts about the dilogarithm from the point of view of Deligne cohomology. Due to the definition of hermitian holomorphic Deligne cohomology we have adopted, certain constructions available in the literature and recalled in section 2 need to be slightly modified in order to obtain a (graded) commutative product. The necessary arguments, being somewhat outside the line of development of the paper have been presented in detail in Appendix A. We present our approach to hermitian holomorphic Deligne cohomology in section 3. While our definition is computationally more complex, it has the advantage for us of keeping the metric structure explicit, while the definition in refs. [9] and [11] works from the point of view of reduction of the structural group. We have explicitly proved the isomorphism in Lemma 3.2. For the sake of completeness, we give an explicit description of

[1] From a physical point of view this originates in the modular geometry approach to Conformal Field Theory advocated by Friedan and Shenker in [19]. Mathematically speaking, it is one of the many proposed forms of the holomorphic factorization property for determinant line bundles.

[2] Note that X is allowed to be disconnected.

the cocycle determined by a holomorphic line bundle with hermitian metric, and in sect. 3.3 we explicitly compute the cup product of two metrized line bundles for later usage.

Sections 4 and 6 form the core of the paper. The direct construction of the Liouville action according to the techniques of refs. [2, 3, 31] is presented in section 4. Since explicit calculations have been presented in great detail in ref. [31], and the calculation we need are quite straightforward, we keep details to a minimum. In Proposition 4.4 and Corollary 4.5 we show that the Liouville action functional computed via descent theory does solve the variational problem. These results have appeared also in ref. [31] and are presented for here completeness. We go into more details to present the connection with the dilogarithm, both for its own sake and also to facilitate the transition to the material of section 6, where our main results are contained. Before doing so, in section 5 we illustrate in some detail how the genuine Bloch-Wigner function appears as an obstruction to closing the descent equations relative to a cover associated to a Kleinian uniformization. We point out that this obstruction is in fact zero due to the non-compactness of the 3-manifold $\mathbb{H}^3/\Gamma$. This is the content of Theorem 5.2. Finally, in section 6 we compare the descent calculations in section 4 with the cup products computed in section 3 to conclude that the quadratic part of the Liouville action is indeed a square, see Theorem 6.1, Corollary 6.2, and Proposition 6.4. Finally, we prove the relation with the determinant of cohomology in Theorem 6.5 and Corollary 6.6.

1.2 Notation and conventions

If z is a complex number, then $\pi_p(z) \stackrel{\text{def}}{=} \frac{1}{2}(z + (-1)^p \bar{z})$, and similarly for any other complex quantity, e.g. complex valued differential forms. If A is a subring of $\mathbb{R}$, we will use the "twist" $A(j) = (2\pi\sqrt{-1})^j\, A$.

If X is a complex manifold, $\underline{A}^\bullet_X$ and $\underline{\Omega}^\bullet_X$ denote the de Rham complexes of smooth $\mathbb{C}$-valued and holomorphic forms, respectively. We denote by $\underline{\mathcal{E}}^\bullet_X$ the de Rham complex of sheaves of real valued differential forms and by $\underline{\mathcal{E}}^\bullet_X(j)$ the twist $\underline{\mathcal{E}}^\bullet_X \otimes_{\mathbb{R}} \mathbb{R}(j)$. We set $\mathcal{O}_X \equiv \underline{\Omega}^0_X$ as usual. When needed, $\underline{A}^{p,q}_X$ will denote the sheaf of smooth (p,q)-forms. We use the standard decomposition $d = \partial + \bar\partial$ according to types. Furthermore, we introduce the differential operator $d^c = \partial - \bar\partial$ (contrary to the convention, see, e.g. [25], we omit the factor $1/(4\pi\sqrt{-1})$). We have $2\partial\bar\partial = d^c d$. The operator d^c is an imaginary one, and accordingly, we have the rules

$$d\pi_p(\omega) = \pi_p(d\omega)$$
$$d^c\pi_p(\omega) = \pi_{p+1}(d^c\omega)$$

for any complex form ω.

An open cover of X will be denoted by $\mathcal{U}_X$. If $\{U_i\}_{i\in I}$ is the corresponding collection of open sets, we denote $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$, and so on. We can also consider more general covers $\mathcal{U}_X = \{U_i \to X\}_{i\in I}$ where the maps are regular coverings in the appropriate category. In this case intersections are replaced by $(n+1)$-fold fibered products $U_{i_0 i_1 \dots i_n} = U_{i_0} \times_X \cdots \times_X U_{i_n}$. Open coverings fit this more general description, since if U_i and U_j are two open sets, then $U_i \cap U_j = U_i \times_X U_j$, where the fiber product is taken with respect to the inclusion maps. As another example, one can consider regular covering maps $U \to X$ with $\Gamma = \mathrm{Deck}(U/X)$, and in this case, taking the fiber product over X $(n+1)$-times, one gets $U \times_X \cdots \times_X U = U \times \Gamma \times \cdots \times \Gamma$, where the group factor is repeated n-times. This includes the cases of Kleinian (and in particular Fuchsian) covers of Riemann surfaces.

The *nerve* of the cover $\mathcal{U}_X$ is the simplicial object $n \mapsto N_n(\mathcal{U}_X) = \coprod U_{i_0} \times_X \cdots \times_X U_{i_n}$ where $N_n(\mathcal{U}_X)$ maps into $N_{n-1}(\mathcal{U}_X)$ in $(n+1)$ ways by forgetting in turn each factor. For open covers this just yields the expected inclusion maps.

If $\underline{F}^\bullet$ is a complex of sheaves on X, its Čech resolution with respect to a covering $\mathcal{U}_X \to X$ is the double complex $\mathsf{C}^{p,q}(\underline{F}) \stackrel{\text{def}}{=} \check{C}^q(\mathcal{U}_X, \underline{F}^p)$, where the q-cochains with values in $\underline{F}^p$ are given by $\prod \underline{F}^p(U_{i_0} \times_X \cdots \times_X U_{i_n})$. The Čech coboundary operator is denoted $\check\delta$. The sign convention we are going to use is that the index along the Čech resolution is the *second* one, so if we denote by d the differential in the complex $\underline{F}^\bullet$, the total differential in the total simple complex of $\check{C}^q(\mathcal{U}_X, \underline{F}^p)$ will be $D = d \pm \check\delta$. For open covers we just get the familiar Čech (hyper)cohomology. The other interesting example is that of a regular covering map $U \to X$: Čech cohomology with respect to this cover is the same as group cohomology for $\Gamma = \mathrm{Deck}(U/X)$ with coefficients in the Γ-module $\underline{F}^p(U)$.

The Koszul sign rule that results in a sign being picked whenever two degree indices are formally exchanged is applied. In particular, for Čech resolutions of complexes of sheaves, it leads to the following conventions. If $\underline{G}^\bullet$ is a second complex of sheaves on X, then one defines the cup product

$$\cup : \mathsf{C}^{p,q}(\underline{F}) \otimes \mathsf{C}^{r,s}(\underline{G}) \longrightarrow \check{C}^{q+s}(\mathcal{U}_X, \underline{F}^p \otimes \underline{G}^r) \subset \mathsf{C}^{p+r,q+s}(\underline{F} \otimes \underline{G})$$

of two elements $\{f_{i_0,\dots,i_q}\} \in \mathsf{C}^{p,q}(\underline{F})$ and $\{g_{j_0,\dots,j_s}\} \in \mathsf{C}^{r,s}(\underline{G})$ by

$$(-1)^{qr}\, f_{i_0,\dots,i_q} \otimes g_{i_q,i_{q+1},\dots,i_{q+s}}\,.$$

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2 Deligne complexes

2.1 Cup products on cones

Recall that the cone of a map $f : X^\bullet \to Y^\bullet$ between two complexes is the complex $C^\bullet(f) = X^\bullet[1] \oplus Y^\bullet$ with differential $d_f(x, y) = (-d\,x, f(x) + d\,y)$, where $[k]$ denotes the shift functor. The cone fits into the exact sequence

$$0 \longrightarrow Y^\bullet \longrightarrow C^\bullet(f) \longrightarrow X^\bullet[1] \longrightarrow 0\,.$$

The following constructions are a special case of those considered by Beĭlinson in ref. [5]. Suppose we are given complexes $X_i^\bullet$, $Y_i^\bullet$, and $Z_i^\bullet$ and maps $f_i : X_i^\bullet \to Z_i^\bullet$, $g_i : Y_i^\bullet \to Z_i^\bullet$, for $i = 1, 2, 3$. Suppose also that we have product maps $X_1^\bullet \otimes X_2^\bullet \xrightarrow{\cup} X_3^\bullet$, and similarly for $Y_i^\bullet$, and $Z_i^\bullet$, strictly compatible with the f_i, g_i in the obvious sense. Then we can consider the cones

$$\operatorname{Cone}\big(f_i - g_i\big)[-1] \equiv \operatorname{Cone}\big(X_i^\bullet \oplus Y_i^\bullet \xrightarrow{f_i - g_i} Z_i^\bullet\big)[-1]\,.$$

For a real parameter α, there is a family of products

$$\operatorname{Cone}\big(f_1 - g_1\big)[-1] \otimes \operatorname{Cone}\big(f_2 - g_2\big)[-1] \xrightarrow{\cup_\alpha} \operatorname{Cone}\big(f_3 - g_3\big)[-1] \tag{2.1}$$

determined as follows. For $(x_i, y_i, z_i) \in X_i^\bullet \oplus Y_i^\bullet \oplus Z_i^{\bullet-1}$, $i = 1, 2$, one defines

$$\begin{aligned}(x_1, y_1, z_1) \cup_\alpha (x_2, y_2, z_2) = \Big(&x_1 \cup x_2, y_1 \cup y_2,\\ &(-1)^{\deg(x_1)}\big((1-\alpha) f_1(x_1) + \alpha g_1(y_1)\big) \cup z_2\\ &\quad + z_1 \cup \big(\alpha f_2(x_2) + (1-\alpha) g_2(y_2)\big)\Big)\,.\end{aligned} \tag{2.2}$$

Note that $\deg(x_1) = \deg(x_1, y_1, z_1)$. Checking that $\cup_\alpha$ is a map of complexes is a straightforward routine calculation. Different products for different values α and β of the real parameter are homotopic. Explicitly, we have

$$(x_1, y_1, z_1) \cup_\alpha (x_2, y_2, z_2) - (x_1, y_1, z_1) \cup_\beta (x_2, y_2, z_2) = \big(d\,h_{\alpha,\beta} + h_{\alpha,\beta} d\big)((x_1, y_1, z_1) \otimes (x_2, y_2, z_2))\,,$$

where the homotopy

$$h_{\alpha,\beta} : \operatorname{Tot}\Big(\operatorname{Cone}\big(f_1 - g_1\big)[-1] \otimes \operatorname{Cone}\big(f_2 - g_2\big)[-1]\Big)^\bullet \longrightarrow \operatorname{Cone}\big(f_3 - g_3\big)[-1]^{\bullet-1}$$

is given by the formula

$$h_{\alpha,\beta}((x_1, y_1, z_1) \otimes (x_2, y_2, z_2)) = (\alpha - \beta)\,(-1)^{\deg(x_1)-1}(0, 0, z_1 \cup z_2)\,. \tag{2.3}$$

If the products $X_1^\bullet \otimes X_2^\bullet \xrightarrow{\cup} X_3^\bullet$, etc., are graded commutative, then the swap functor on the tensor product maps the $\cup_\alpha$ product structure on the cones into the $\cup_{1-\alpha}$ structure. Using the homotopy (2.3) it follows at once that there is a well defined graded commutative product in cohomology.

If we do not assume the product structures $X_1^\bullet \otimes X_2^\bullet \xrightarrow{\cup} X_3^\bullet$, etc., are strictly compatible with the maps f_i, g_i, some of the preceding constructions must be slightly modified. With an eye toward certain constructions to be carried out later in this paper, let us assume we have compatibility up to homotopy, namely there exist maps

$$\begin{aligned} h &\colon \big(X_1 \otimes X_2\big)^\bullet \longrightarrow Z_3^{\bullet-1}\\ k &\colon \big(Y_1 \otimes Y_2\big)^\bullet \longrightarrow Z_3^{\bullet-1}\end{aligned}$$

such that

$$f_3 \circ \cup - \cup \circ (f_1 \otimes f_2) = d\,h + h\,d \qquad (2.4)$$
$$g_3 \circ \cup - \cup \circ (g_1 \otimes g_2) = d\,k + k\,d\,,$$

with obvious meaning of the symbols.

Lemma 2.1. *Let X_i, Y_i, Z_i and the maps f_i, g_i be as above. Let α be a real parameter. We have a product of type* (2.1) *for the cones* $\mathrm{Cone}(f_i - g_i)[-1]$ *defined by the following modification of formula* (2.2)*:*

$$\begin{aligned}(x_1, y_1, z_1) \cup_\alpha (x_2, y_2, z_2) = \Big(&x_1 \cup x_2, y_1 \cup y_2,\\ &(-1)^{\deg(x_1)}\big((1-\alpha) f_1(x_1) + \alpha g_1(y_1)\big) \cup z_2\\ &\quad + z_1 \cup \big(\alpha f_2(x_2) + (1-\alpha) g_2(y_2)\big)\\ &\quad\quad - h(x_1 \otimes x_2) + k(y_1 \otimes y_2)\Big)\,.\end{aligned} \qquad (2.5)$$

The product (2.5) *is a map of complexes and two products $\cup_\alpha$ and $\cup_\beta$ are related by the same homotopy formula* (2.3).

Proof. Direct verification. □

This modified framework carries over to the full structure considered by Beĭlinson in ref. [5]. We will still refer to this modified product as the Beĭlinson product. It is also necessary to relax the assumption that the products $X_1^\bullet \otimes X_2^\bullet \xrightarrow{\cup} X_3^\bullet$, etc., be graded commutative. It is possible to complete all the diagrams so that the permutation of factors in the tensor products still yields a homotopy commutative product (2.5) for the cones. As a consequence the induced product in cohomology will still be graded commutative. Explicit formulas are not needed except to ensure this latter fact, therefore we shall not discuss this matter any further and refer the reader to the appendix, where a brief but explicit treatment can be found.

2.2 Deligne complexes

Let X be a complex manifold. Recall the standard Hodge filtration of $\underline{\Omega}_X^\bullet$:

$$F^p \underline{\Omega}_X^\bullet : 0 \longrightarrow \cdots \longrightarrow \underline{\Omega}_X^p \longrightarrow \cdots \longrightarrow \underline{\Omega}_X^n\,, \qquad (2.6)$$

where $n = \dim_{\mathbb{C}} X$.

The corresponding filtration for the complex of smooth $\mathbb{C}$-valued forms is defined as follows: denote by $F^p \underline{A}_X^\bullet$ the subcomplex of $\underline{A}_X^\bullet$ comprising forms of type (r, s) where r is at least p, so that $F^p \underline{A}_X^k = \oplus_{r \geq p} \underline{A}_X^{r,k-r}$. Then (cf. [14]) the inclusion $\underline{\Omega}_X^\bullet \hookrightarrow \underline{A}_X^\bullet$ is a quasi-isomorphism respecting the filtrations, namely $F^p \underline{\Omega}_X^\bullet \hookrightarrow F^p \underline{A}_X^\bullet$, and the latter inclusion induces an isomorphism in cohomology.

If A is a subring of $\mathbb{R}$, and $\imath$ and $\jmath$ denote the inclusions of $A(p)$ and $F^p \underline{\Omega}_X^\bullet$ into $\underline{\Omega}_X^\bullet$ respectively, the $p-$th Deligne complex of sheaves is defined by

$$A(p)_{\mathcal{D}}^\bullet = \mathrm{Cone}\big(A(p)_X \oplus F^p \underline{\Omega}_X^\bullet \xrightarrow{\imath - \jmath} \underline{\Omega}_X^\bullet\big)[-1]\,. \qquad (2.7)$$

It is quasi-isomorphic to the complex:

$$\mathrm{Cone}\big(A(p)_X \oplus F^p \underline{A}_X^\bullet \xrightarrow{\imath - \jmath} \underline{A}_X^\bullet\big)[-1]\,, \qquad (2.8)$$

where $\imath$ and $\jmath$ have the same meaning. $A(p)_{\mathcal{D}}^\bullet$ is quasi-isomorphic to:

$$\underline{A(p)}_{\mathcal{D}}^\bullet = A(p)_X \xrightarrow{\imath} \mathcal{O}_X \xrightarrow{d} \underline{\Omega}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{\Omega}_X^{p-1}\,, \qquad (2.9)$$

and an explicit quasi-isomorphism

$$\phi_p : \underline{A(p)}_{\mathcal{D}}^\bullet \xrightarrow{\simeq} A(p)_{\mathcal{D}}^\bullet$$

is given in [18]. For completeness, we recall the explicit formula. We have

$$\begin{array}{ccccccccccccc} A(p)_X & \xrightarrow{\imath} & \mathcal{O}_X & \xrightarrow{d} & \underline{\Omega}_X^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \underline{\Omega}_X^{p-1} & & & & \\ \big\downarrow{\scriptstyle \phi_0} & & \big\downarrow{\scriptstyle \phi_1} & & \big\downarrow{\scriptstyle \phi_2} & & & & \big\downarrow{\scriptstyle \phi_p} & & & & \\ A(p)_X & \xrightarrow{-\imath} & \mathcal{O}_X & \xrightarrow{-d} & \underline{\Omega}_X^1 & \xrightarrow{-d} & \cdots & \xrightarrow{(0,-d)} & \underline{\Omega}_X^p \oplus \underline{\Omega}_X^{p-1} & \xrightarrow{(d,\jmath-d)} & \underline{\Omega}_X^{p+1} \oplus \underline{\Omega}_X^p & \xrightarrow{(d,\jmath-d)} & \cdots \end{array} \qquad (2.10)$$

with $\phi_i(x) = (-1)^i x$, for $0 \le i < p$ and $\phi_p(x) = (-1)^p(dx, x)$. The complex (2.9) is perhaps more widely used (or known) as the complex used to define Deligne cohomology.

When $A = \mathbb{R}$ there are further quasi-isomorphisms, namely

$$\mathbb{R}(p)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \mathrm{Cone}\left(F^p\underline{\Omega}^\bullet_X \to \underline{\mathcal{E}}^\bullet_X(p-1)\right)[-1] \xrightarrow{\simeq} \mathrm{Cone}\left(F^p\underline{A}^\bullet_X \to \underline{\mathcal{E}}^\bullet_X(p-1)\right)[-1]$$

since the maps

$$\left(\mathbb{R}(p) \to \underline{\Omega}^\bullet_X\right) \xrightarrow{\simeq} \mathbb{R}(p) \to \mathbb{C} \xrightarrow{\simeq} \mathbb{R}(p-1) \xrightarrow{\simeq} \underline{\mathcal{E}}^\bullet_X(p-1)$$

are all quasi-isomorphisms in the derived category, cf. [18]. Here we have used $\mathbb{C} \cong \mathbb{R}(p) \oplus \mathbb{R}(p-1)$. Following op. cit., we set:

$$\widetilde{\mathbb{R}(p)}^\bullet_{\mathcal{D}} \overset{\text{def}}{=} \mathrm{Cone}\left(F^p\underline{A}^\bullet_X \xrightarrow{-\pi_{p-1}} \underline{\mathcal{E}}^\bullet_X(p-1)\right)[-1]\,. \tag{2.11}$$

Again, there is an explicit quasi-isomorphism ([18]):

$$\begin{gathered}\rho_p : \mathbb{R}(p)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \widetilde{\mathbb{R}(p)}^\bullet_{\mathcal{D}} \\ \rho_p|_{\mathbb{R}(p)} = 0\,, \quad \rho_p|_{F^p\underline{\Omega}^\bullet_X} = incl\,, \quad \rho_p|_{\underline{\Omega}^\bullet_X} = \pi_{p-1}\end{gathered} \tag{2.12}$$

The *Deligne cohomology groups* of X with coefficients in $A(p)$ are the hypercohomology groups

$$H^\bullet_{\mathcal{D}}(X, A(p)) = \mathbf{H}^\bullet(X, A(p)^\bullet_{\mathcal{D}})\,.$$

and clearly, any complex quasi-isomorphic to $A(p)^\bullet_{\mathcal{D}}$ would do. In order to perform calculations with these cohomology groups we shall normally resort to a Čech resolution with respect to an open cover $\mathcal{U}_X$ of X or, later in the paper, an étale map $\mathcal{U}_X \to X$, e.g. a regular cover with deck group Γ. However, when $A = \mathbb{R}$, we can exploit the above quasi-isomorphisms and the fact that $\underline{A}^\bullet_X$ is a complex of fine sheaves to conclude that $H^\bullet_{\mathcal{D}}(X, \mathbb{R}(p))$ can also be computed as the homology of the complex of global sections

$$\mathrm{Cone}\left(F^pA^\bullet(X) \longrightarrow \mathcal{E}^\bullet(X)(p-1)\right)[-1]\,.$$

One of the important properties of Deligne cohomology is the existence of a graded commutative cup product

$$H^i_{\mathcal{D}}(X, A(p)) \otimes H^j_{\mathcal{D}}(X, A(q)) \xrightarrow{\cup} H^{i+j}_{\mathcal{D}}(X, A(p+q))\,, \tag{2.13}$$

which follows from the existence of the Beĭlinson cup product at the level of Deligne complexes whose construction was recalled above. There are products $A(p) \otimes A(q) \to A(p+q)$ and $F^p\underline{\Omega}^\bullet_X \otimes F^q\underline{\Omega}^\bullet_X \to F^{p+q}\underline{\Omega}^\bullet_X$, plus the obvious (wedge) product on $\underline{\Omega}^\bullet_X$, thus it follows from the cone version (2.7) that the Deligne complexes come equipped with the Beĭlinson product, and therefore the Deligne cohomology groups inherit the graded commutative cup product (2.13). The explicit form, that is, the translation of (2.2) to the case at hand can be found in [18]. Furthermore, the product formula (cf. [10],[18])

$$f \cup g = \begin{cases} f \cdot g & \deg f = 0\,, \\ f \wedge dg & \deg f > 0 \quad \text{and} \quad \deg g = q\,, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

on the complex (2.9) maps via (2.10) to the product $\cup_0$ on the cone version (2.7). The explicit form of the cup product for the complex (2.11) as computed in [4] (see also [18]) will be needed in the sequel. Let (ω_1, η_1) be an element of degree n in $\widetilde{\mathbb{R}(p)}^\bullet_{\mathcal{D}}$—this means that $\omega_1 \in F^p\underline{\Omega}^n_X$ and $\eta_1 \in \underline{\mathcal{E}}^{n-1}_X(p-1)$—and (ω_2, η_2) any element in $\widetilde{\mathbb{R}(q)}^\bullet_{\mathcal{D}}$. The product is defined by the formula:

$$(\omega_1, \eta_1)\,\tilde{\cup}\,(\omega_2, \eta_2) = \left(\omega_1 \wedge \omega_2, (-1)^n\,\pi_p\omega_1 \wedge \eta_2 + \eta_1 \wedge \pi_q\omega_2\right)\,. \tag{2.15}$$

The product $\tilde{\cup}$ is a morphism of complexes and (modulo the quasi-isomorphisms ρ_p) is homotopic to the Beĭlinson products $\cup_\alpha$ on the complexes $\mathbb{R}(p)^\bullet_{\mathcal{D}}$. Specifically, if we denote an element of $\mathbb{R}(p)^\bullet_{\mathcal{D}}$ by the triple (r, f, ω), where $r \in \mathbb{R}(p)_X$, $f \in F^p\underline{\Omega}^\bullet_X$, and $\omega \in \underline{\Omega}^\bullet_X$, the homotopy between $\tilde{\cup} \circ (\rho_p \otimes \rho_q)$ and $\rho_{p+q} \circ \cup_\alpha$ is given by

$$\tilde{h}((r, f, \omega) \otimes (r', f', \omega')) = (-1)^{\deg\omega}\left(0, (1-\alpha)\,\pi_p\omega \wedge \pi_{q-1}\omega' - \alpha\,\pi_{p-1}\omega \wedge \pi_q\omega'\right)\,. \tag{2.16}$$

2.3 Examples

2.3.1

Let $A=\mathbb{Z}$. Then $\mathbb{Z}(1)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \mathcal{O}^\times_X[-1]$ via the standard exponential sequence, so that $H^k_{\mathcal{D}}(X,\mathbb{Z}(1)) \cong H^{k-1}(X,\mathcal{O}^\times_X)$. In particular $H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) \cong H^0(X,\mathcal{O}^\times_X)$, the global invertibles on X, and $H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) \cong \mathrm{Pic}(X)$, the Picard group of line bundles over X.

If an open cover $\{U_i\}_{i\in I}$ of X is chosen, the class of a line bundle L in $H^2_{\mathcal{D}}(X,\mathbb{Z}(1))$ can be represented via a Čech resolution by the cocycle (f_{ij},c_{ijk}), where $f_{ij}\in \mathcal{O}_X(U_{ij})$, and $c_{ijk}\in\mathbb{Z}(1)_X(U_{ijk})$. Here we have used the Deligne complex in the form (2.9). The functions f_{ij} should be interpreted as *logarithms* of the corresponding transition functions for L. Then, the collection $c_{ijk}=(\check{\delta} f)_{ijk}$ provides a representative for the first Chern Class $c_1(L)$. Similarly, an invertible function f would be described by the collection f_i of its logarithms on each open U_i, plus the "integers" $m_{ij}=f_j-f_i\in\mathbb{Z}(1)$ on each U_{ij}.

2.3.2

Still using the exponential sequence, $\mathbb{Z}(2)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \big(\mathcal{O}^\times_X \xrightarrow{d\log} \underline{\Omega}^1_X\big)[-1]$. Thus $H^2_{\mathcal{D}}(X,\mathbb{Z}(2))$ is the group of isomorphism classes of holomorphic line bundles with (holomorphic) connection. Using the (in fact, any) product $\mathbb{Z}(1)^\bullet_{\mathcal{D}}\otimes\mathbb{Z}(1)^\bullet_{\mathcal{D}}\to\mathbb{Z}(2)^\bullet_{\mathcal{D}}$, the cup product of two global invertible holomorphic functions f and g on X yields a line bundle with connection—the tame symbol—denoted by $\big(f,g\big]$ whose class is in $H^2_{\mathcal{D}}(X,\mathbb{Z}(2))$, see [13, 6, 29]. Higher cup products in this spirit have been studied in [9].

2.3.3

If $A=\mathbb{R}$, we have $H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p)) = H^{2p}(X,\mathbb{R}(p))\cap H^{p,p}(X)$. The Čech representative (c_{ijk},f_{ij}) of a class in $H^2_{\mathcal{D}}(X,\mathbb{Z}(1))$ mentioned above maps to the cocycle $(-df_{ij},-|f_{ij}|)$ under (2.12). Taking into account that the f_{ij} are the logarithms of the transition functions, the corresponding $(1,1)$ class would be given by the associated canonical connection, see sec. 3.2.

2.3.4

$H^1_{\mathcal{D}}(X,\mathbb{R}(1))$ is the group of real valued functions f on X such that there exists a holomorphic one-form ω such that $\pi_0\omega=df$. In other words it is the group of those smooth functions f such that ∂f is holomorphic, which amounts to say that such an f itself is harmonic. Going to higher values, we have $H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p)) = H^{2p-2}(X,\mathbb{R}(p-1))\cap H^{p-1,p-1}(X)$ if X is the complex manifold corresponding to a projective variety, cf. [30]. In general, e.g. X quasi-projective, $H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p))$ will correspond to $\partial\bar\partial$-closed $\mathbb{R}(p-1)$-valued $(p-1,p-1)$-forms modulo those forms in $\mathrm{Im}\,\partial+\mathrm{Im}\,\bar\partial$.

2.4 Remarks on the dilogarithm

It is convenient to consider the case of the cup product of two invertible functions f and g in various complexes in more detail, and to introduce some related notions we shall need later.

As observed, $\mathbb{Z}(1)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \mathcal{O}^\times_X[-1]$ and an invertible function f can be considered as an element of $H^1_{\mathcal{D}}(X,\mathbb{Z}(1))$. Therefore, via (2.12), it induces $\rho_1(f)\in H^1_{\mathcal{D}}(X,\mathbb{R}(1))$ represented by $(d\log f,\log|f|)$. (Note that $\log|f|$ fits the description of $H^1_{\mathcal{D}}(X,\mathbb{R}(1))$ in 2.3.4.) Given two such f and g, the expression for the cup product (2.15) gives the following element of $H^2_{\mathcal{D}}(X,\mathbb{R}(2))$:

$$\rho_1(f)\,\tilde\cup\,\rho_1(g) = \big(d\log f\wedge d\log g, -\pi_1(d\log f)\,\log|g| + \log|f|\,\pi_1(d\log g)\big)\,. \tag{2.17}$$

The first term is obviously zero when X is a curve. Given f and g, invertible on X, let us define the imaginary 1-form:

$$r_2(f,g) = \pi_1(d\log f)\,\log|g| - \log|f|\,\pi_1(d\log g)\,. \tag{2.18}$$

On the other hand, the cup product of f and g as elements of $H^1_{\mathcal{D}}(U,\mathbb{Z}(1))$ yields an element $f\cup g$ of $H^2_{\mathcal{D}}(U,\mathbb{Z}(2))$ represented by $\big(d\log f\wedge d\log g, \log f\; d\log g\big)$ (if we use the $\cup_0$ product) and this maps via ρ_2 to the element

$$\big(d\log f\wedge d\log g, -\pi_1(\log f\;d\log g)\big)\,.$$

This is equal to (2.18) only up to homotopy. Indeed, using $\pi_{p+q-1}(a \wedge b) = \pi_p(a) \wedge \pi_{q-1}(b) + \pi_{p-1}(a) \wedge \pi_q(b)$, we find the expression

$$r_2(f, g) = d\big(\pi_1(\log f)\, \log|g|\big) - \pi_1\big(\log f\; d\log g\big)\,, \tag{2.19}$$

where the first term is just the explicit homotopy as computed from (2.16).

Recall that the *tame symbol* ([13, 6]) $\big(f, g\big]$ associated to f and g is the line bundle with connection determined (up to isomorphism) by the class $f \cup g$. A Bloch dilogarithm ([18]) is (the logarithm of) a horizontal trivializing section, namely a function L on U satisfying the equation

$$dL = -\log f\; d\log g\,.$$

Thus L realizes the isomorphism $\big(f, g\big] \cong \mathcal{O}_X$. The Steinberg relation satisfied by the Tame symbol says in particular this is true when $g = 1-f$: $\big(1-f, f\big]$ is trivial [6, 18] in the sense specified above. The universal case corresponding to $f = z$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ allows to identify L with the classical Euler dilogarithm Li_2, namely

$$Li_2(z) = -\int_0^z \log(1-t)\frac{dt}{t}\,,$$

see [13]. The classical dilogarithm has a single valued parter, denoted $\mathscr{D}_2$, introduced by Bloch and Wigner:

$$\mathscr{D}_2(z) = \arg(1-z)\log|z| + \operatorname{Im} Li_2(z)\,. \tag{2.20}$$

$\mathscr{D}_2$ is real-analytic on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and extends continuously to $\mathbb{P}^1$. If we introduce

$$\mathscr{L}_2(z) = \sqrt{-1}\,\mathscr{D}_2(z)\,,$$

then clearly

$$d\mathscr{L}_2 = r_2(1-z, z)\,.$$

More generally, if L trivializes $\big(f, g\big]$ in the sense explained above, we can associate a function $\mathscr{L}_2(f, g)$ such that

$$d\mathscr{L}_2(f, g) = r_2(f, g)$$

via the position

$$\mathscr{L}_2(f, g) = \pi_1(\log f)\, \log|g| + \operatorname{Im} L\,. \tag{2.21}$$

3 Constructions in hermitian holomorphic Deligne cohomology

In this section we construct a set of complexes to compute hermitian holomorphic Deligne cohomology, as in ref. [11]. The complexes we introduce here are tailored to the description of a metrized line bundle in terms of local representatives of the hermitian fiber metric, whereas those in loc. cit. work best from a structure group reduction point of view. The two constructions are related by a quasi-isomorphism, so the cohomology groups are the same, and therefore it is justified to retain the same name for the resulting cohomology groups.

3.1 Hermitian holomorphic Deligne cohomology

Brylinski introduced in ref. [11] the complexes $C(l)$ in order to compare the Beĭlinson-Chern classes of a holomorphic vector bundle E with the Cheeger-Chern-Simons classes determined by (E, ∇), where ∇ is the *canonical connection,* namely the unique connection compatible with both the holomorphic and hermitian structures. The cohomology groups determined by these complexes were called *Hermitian Holomorphic Deligne cohomology groups.* We will recall Brylinski's definition below; we introduce now a different set of complexes with a slightly different motivation in mind: as it has been mentioned above, and it will be expanded in further detail in sec. 3.2, a holomorphic line bundle L on X determines a class in $H^2_{\mathcal{D}}(X, \mathbb{Z}(1))$, and this class has an image in real Deligne cohomology via the obvious map $\mathbb{Z} \to \mathbb{R}$ inducing a map between the corresponding Deligne complexes. The image in real Deligne cohomology correspond to a class of pure Hodge type $(1, 1)$. If L is equipped with a hermitian metric ρ, this class will be represented by a concrete 2-form built from the canonical connection associated to ρ, and furthermore this form is an imaginary $(1, 1)$-form. So we want a complex whose cohomology classes in degree $2p$ correspond to

(p,p)-forms representing hodge classes of the corresponding pure type, coming from integer Deligne cohomology classes. To this end, consider the complex

$$D(l)^\bullet_{h.h.} = \mathrm{Cone}\big(\mathbb{Z}(l)^\bullet_{\mathcal{D}} \oplus (F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l)) \longrightarrow \widetilde{\mathbb{R}(l)}^\bullet_{\mathcal{D}}\big)[-1]\,, \tag{3.1}$$

where, as usual, σ^p denotes the (sharp) truncation in degree p, namely for a complex $\underline{F}^\bullet$, $\sigma^p\underline{F}^\bullet$ is zero in degrees less than p. The map $\mathbb{Z}(l)^\bullet_{\mathcal{D}} \to \widetilde{\mathbb{R}(l)}^\bullet_{\mathcal{D}}$ is the composite of the obvious map $\mathbb{Z}(l)^\bullet_{\mathcal{D}} \to \mathbb{R}(l)^\bullet_{\mathcal{D}}$ with the quasi-isomorphism ρ_l defined by (2.12). We will simply denote it by ρ_l in the sequel, suppressing the first morphism in the notation. The map $(F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l)) \to \widetilde{\mathbb{R}(l)}^\bullet_{\mathcal{D}} = \mathrm{Cone}\big(F^l\underline{A}^\bullet_X \to \underline{\mathcal{E}}^\bullet_X(l-1)\big)[-1]$ is induced by the inclusion of $(F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l))$ into $F^l\underline{A}^\bullet_X$. In (3.1) we take the cone of the difference between these two maps. We have:

Definition 3.1. The complex (3.1) is the *hermitian holomorphic Deligne* complex. The hermitian holomorphic Deligne cohomology groups are the corresponding hypercohomology groups

$$H^p_{\mathcal{D}_{h.h.}}(X,l) \stackrel{\mathrm{def}}{=} \mathbf{H}^p(X, D(l)^\bullet_{h.h.})\,. \tag{3.2}$$

The specific choice of the complexes in Definition 3.1 is geared towards the computation of certain explicit cocycles representing cup products of metrized line bundles to be considered below. In order to justify our choice of the name, let us compare this definition with the one in ref. [11], where Brylinski introduces the complexes:

$$C(l)^\bullet = \mathrm{Cone}\big(\mathbb{Z}(l)_X \oplus (F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l)) \longrightarrow \underline{\mathcal{E}}^\bullet_X(l)\big)[-1]\,. \tag{3.3}$$

Lemma 3.2. *The complexes $C(l)^\bullet$ and $D(l)^\bullet_{h.h.}$ are quasi-isomorphic.*

Proof. By elementary manipulation of cones

$$D(l)^\bullet_{h.h.} = \mathrm{Cone}\Big(F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l) \to \mathrm{Cone}\big(\mathbb{Z}(l)^\bullet_{\mathcal{D}} \to \widetilde{\mathbb{R}(l)}^\bullet_{\mathcal{D}}\big)\Big)[-1]$$

and clearly: $\mathrm{Cone}\big(\mathbb{Z}(l)^\bullet_{\mathcal{D}} \to \widetilde{\mathbb{R}(l)}^\bullet_{\mathcal{D}}\big) \xrightarrow{\simeq} \mathrm{Cone}\big(\mathbb{Z}(l)_X \to \mathbb{R}(l)_X\big) \xrightarrow{\simeq} \mathrm{Cone}\big(\mathbb{Z}(l)_X \to \underline{\mathcal{E}}^\bullet_X(l)\big)$, where all arrows are quasi-isomorphisms. Thus

$$\begin{aligned} D(l)^\bullet_{h.h.} &\xrightarrow{\simeq} \mathrm{Cone}\Big(F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l) \to \mathrm{Cone}\big(\mathbb{Z}(l)_X \to \underline{\mathcal{E}}^\bullet_X(l)\big)\Big)[-1] \\ &= \mathrm{Cone}\big(\mathbb{Z}(l)_X \oplus (F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l)) \to \underline{\mathcal{E}}^\bullet_X(l)\big)[-1] \\ &\equiv C(l)\,, \end{aligned}$$

as wanted. ☐

It follows from Lemma 3.2 that the hypercohomology groups of the two complexes are the same and therefore the hermitian holomorphic Deligne cohomology groups as per Definition 3.1 are the same as those defined by Brylinski in ref. [11].

From (3.1) and the exact sequence satisfied by the cone construction, we can derive the exact sequence

$$\cdots \longrightarrow H^{2l-1}_{\mathcal{D}}(X,\mathbb{R}(l)) \longrightarrow H^{2l}_{\mathcal{D}_{h.h.}}(X,l) \longrightarrow H^{2l}_{\mathcal{D}}(X,\mathbb{Z}(l)) \oplus A^{(l,l)}(X)_{\mathbb{R}(l)} \longrightarrow H^{2l}_{\mathcal{D}}(X,\mathbb{R}(l)) \longrightarrow \cdots \tag{3.4}$$

where $A^{(l,l)}(X)_{\mathbb{R}(l)}$ denotes the space of smooth $\mathbb{R}(l)$-valued global (l,l)-forms on X. In view of the examples, we can rewrite the exact sequence as

$$\begin{aligned} \cdots \longrightarrow H^{2l-1}_{\mathcal{D}}(X,\mathbb{R}(l)) \longrightarrow H^{2l}_{\mathcal{D}_{h.h.}}(X,l) \longrightarrow \\ \longrightarrow H^{2l}_{\mathcal{D}}(X,\mathbb{Z}(l)) \oplus A^{(l,l)}(X)_{\mathbb{R}(l)} \longrightarrow H^{2l}(X,\mathbb{R}(l)) \cap H^{l,l}(X) \longrightarrow \cdots \end{aligned}$$

We see that the group $H^{2l}_{\mathcal{D}_{h.h.}}(X,l)$ map onto those (l,l)-forms representing the Hodge classes corresponding to $H^{2l}_{\mathcal{D}}(X,\mathbb{Z}(l))$.

Now recall that we have cup products for $\mathbb{Z}(l)^\bullet_{\mathcal{D}}$, $\widetilde{\mathbb{R}(l)}^\bullet_{\mathcal{D}}$, and $F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l)$, the latter being induced by the standard wedge product. Since the complex $D(l)^\bullet_{h.h.}$ is a cone, we have the Beĭlinson family of products

$$D(l)^\bullet_{h.h.} \otimes D(k)^\bullet_{h.h.} \xrightarrow{\cup_\alpha} D(l+k)^\bullet_{h.h.}\,, \tag{3.5}$$

and therefore a cup product

$$H^i_{\mathcal{D}_{h.h.}}(X,l) \otimes H^j_{\mathcal{D}_{h.h.}}(X,k) \xrightarrow{\cup} H^{i+j}_{\mathcal{D}_{h.h.}}(X,l+k)\,. \tag{3.6}$$

Remark 3.3. A note of warning: in the definition (3.1) of $D(l)^\bullet_{h.h.}$ the complexes $\mathbb{Z}(l)^\bullet_{\mathcal{D}}$ and $\widetilde{\mathbb{R}(l)}^\bullet_{\mathcal{D}}$ have product structures that are compatible with the map ρ_l only up to homotopy (given by formula (2.16)), therefore the Beĭlinson product in (3.5) is to be intended in the modified version of formula (2.5). Similarly, the product structures of the individual cones in (3.1) are already graded commutative up to homotopy, thus that the product (3.5) is graded commutative up to homotopy follows from Proposition A.4 in the appendix. That allows to conclude that the induced product on cohomology (3.6) is genuinely graded commutative as intended.

To conclude this introduction, we briefly mention the case of a proper submersion of complex manifolds $\pi \colon X \to S$. It follows from [20] that Deligne cohomology has an integration along the fiber $\int_\pi$, hence in analogy with [11, 9] there is a mapping

$$\mathbf{R}\pi_* D(l)^\bullet_{h.h.} \longrightarrow D(l-d)^\bullet_{h.h.}[-2d]$$

where d is the complex dimension of the fiber, inducing

$$H^i_{\mathcal{D}_{h.h.}}(X,l) \longrightarrow H^{i-2d}_{\mathcal{D}_{h.h.}}(S,l-d) \tag{3.7}$$

and commutative diagrams analogous to [11, Theorem 5.1]. We shall be interested in the case of complex relative dimension 1 and degree 4, namely that of the map: $H^4_{\mathcal{D}_{h.h.}}(X,2) \to H^2_{\mathcal{D}_{h.h.}}(S,1)$, inducing a complex hermitian line bundle on S (cf. below).

3.2 Hermitian holomorphic line bundles

A hermitian holomorphic line bundle or, equivalently, a metrized line bundle, cf. [25], is a holomorphic line bundle L over X together with a hermitian fiber metric $\rho : L \to \mathbb{R}_{\geq 0}$. As a rule, we will not distinguish L and its sheaf of holomorphic sections. We will also use the alternate notation $|s|_\rho$ to denote the length of a local section s of L with respect to ρ. Metrized line bundles can be tensor multiplied and an inverse is defined, see. op. cit. An isomorphism of metrized line bundles (L,ρ) and (L',ρ') is defined in the obvious way, namely it is a map $\phi : L \to L'$ such that $|s|_\rho = |\phi(s)|_{\rho'}$ for some local section s of L. We denote by $\widehat{\mathrm{Pic}(X)}$ the group of isomorphism classes of metrized line bundles.

If L is trivialized over a Čech cover $\mathcal{U}_X = \{U_i\}_{i\in I}$ by sections s_i, then as usual we obtain the cocycle of transition functions $g_{ij} \in \mathcal{O}^\times_X(U_i \cap U_j)$ by writing $s_j = s_i\, g_{ij}$. Then if (L,ρ) is a metrized line bundle, we can define the positive function $\rho_i = |s_i|^2_\rho$, namely the local representative of the hermitian structure with respect to the given trivialization. It follows that the various local representatives satisfy

$$\rho_j = \rho_i\, |g_{ij}|^2 \,. \tag{3.8}$$

Let us work out the local version of the isomorphism introduced above. Let s'_i be a local section of L' over U_i. Introduce analogous (primed) quantities for L' as we just did for L. Given the isomorphism $\phi : L \to L'$ we have $\phi(s_i) = s'_i\, f_i$, for some $f_i \in \mathcal{O}^\times_X(U_i)$. Then we find $f_i\, g_{ij} = g'_{ij}\, f_j$ and $\rho_i = \rho'_i\, |f_i|^2$.

Still working with respect to the chosen cover $\mathcal{U}_X$, a *connection* compatible with the holomorphic structure is the datum of a collection of $(1,0)$-forms $\xi_i \in \underline{A}^{1,0}_X(U_i)$ satisfying

$$\xi_j - \xi_i = d\log g_{ij} \,. \tag{3.9}$$

Note for future reference that $\underline{A}^{1,0}_X = F^1\underline{A}^1_X$. The connection is compatible with the hermitian metric if

$$\pi_0(\xi_i) = \frac{1}{2} d\log\rho_i \,. \tag{3.10}$$

Using $d = \partial + \bar\partial$ and decomposition with respect to (p,q)-types, we find the familiar relation

$$\xi_i = \partial \log \rho_i \tag{3.11}$$

for the *unique* connection compatible with both the complex and hermitian structures [22].

The global 2-form

$$c_1(\rho) = \bar\partial\partial\log\rho_i \tag{3.12}$$

represents the first Chern class of L in $H^2(X,\mathbb{R}(1))$. Actually, the class of $c_1(\rho)$ is a pure Hodge class in $H^{1,1}(X)$ and, according to the examples, it is the image of the first Chern class of L under the map $H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) \to H^2_{\mathcal{D}}(X,\mathbb{R}(1))$ induced by $\mathbb{Z}(1) \to \mathbb{R}(1)$. Observe that $c_1(\rho) = c_1(\rho')$ under the isomorphism considered above. In the same way as in [9, 11] we have the following:

Proposition 3.4.

$$\widehat{\operatorname{Pic}(X)} \cong H^2_{\mathcal{D}_{h.h.}}(X,1)$$

Proof. As noted in the examples before, the class of L in $H^2_{\mathcal{D}}(X,\mathbb{Z}(1))$ is represented by the Čech cocycle $(\log g_{ij}, c_{ijk})$, where $c_{ijk} = \log g_{jk} - \log g_{ik} + \log g_{ij}$ represents $c_1(L)$. Moreover, from (3.8) we have the relation

$$\frac{1}{2}\log\rho_j - \frac{1}{2}\log\rho_i = \pi_0(\log g_{ij}) \equiv \log|g_{ij}|\,,$$

thus we obtain a degree 2 total cocycle $(\frac{1}{2}\log\rho_i, \log g_{ij}, c_{ijk})$ in the Čech resolution of the complex

$$0 \longrightarrow \mathbb{Z}(1)_X \xrightarrow{\imath} \mathcal{O}_X \xrightarrow{-\pi_0} \underline{\mathcal{E}}^0_X \longrightarrow 0\,. \tag{3.13}$$

By a standard computation isomorphic metrized line bundles determine equivalent cocycles, therefore a metrized line bundle (L,ρ) determines a class in the second (hyper)cohomology group of the complex (3.13). Furthermore, two such classes can be added by component-wise adding the corresponding cocycles. It is readily verified that for metrized bundles (L,ρ) and (L',ρ') this yields the cocycle corresponding to $(L\otimes L', \rho\rho')$. Thus, passing to classes, this corresponds to the group operation in $\widehat{\operatorname{Pic}(X)}$.

We now proceed to show that the complexes (3.13) and $D(1)^\bullet_{h.h.}$ are quasi-isomorphic. To begin with, note that the complex (3.13) is quasi-isomorphic to

$$\operatorname{Cone}\big(\mathbb{Z}(1)^\bullet_{\mathcal{D}} \xrightarrow{\pi_0} \underline{\mathcal{E}}^0_X[-1]\big)[-1]\,. \tag{3.14}$$

On the other hand, we want to consider the cone of the morphism

$$F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1) \xrightarrow{-incl\oplus 0} \widetilde{\mathbb{R}(1)}^\bullet_{\mathcal{D}}\,.$$

This morphism is mono, so its cone can be replaced (in the derived category) by

$$\operatorname{Cone}\big(F^1\underline{A}^\bullet_X / F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1) \xrightarrow{-\pi_0} \underline{\mathcal{E}}^\bullet_X\big)[-1]\,, \tag{3.15}$$

since

$$\operatorname{Cone}\big(A^\bullet/K^\bullet \to B^\bullet\big)[-1] \xrightarrow{\simeq} \operatorname{Cone}\big(A^\bullet \to B^\bullet\big)[-1]/K^\bullet \xrightarrow{\simeq} \operatorname{Cone}\big(K^\bullet \to \operatorname{Cone}\big(A^\bullet \to B^\bullet\big)[-1]\big)$$

when $K^\bullet \to A^\bullet$ is mono. By direct inspection, the complex (3.15) is quasi-isomorphic to $\underline{\mathcal{E}}^0_X[-1]$. Therefore we have the following quasi-isomorphisms:

$$\begin{aligned}
(3.14) &\xrightarrow{\simeq} \operatorname{Cone}\Big(\mathbb{Z}(1)^\bullet_{\mathcal{D}} \to \operatorname{Cone}\big(F^1\underline{A}^\bullet_X / F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1) \xrightarrow{-\pi_0} \underline{\mathcal{E}}^\bullet_X\big)[-1]\Big)[-1] \\
&\xrightarrow{\simeq} \operatorname{Cone}\Big(\mathbb{Z}(1)^\bullet_{\mathcal{D}} \to \operatorname{Cone}\big(F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1) \to \widetilde{\mathbb{R}(1)}^\bullet_{\mathcal{D}}\big)\Big)[-1] \\
&\xrightarrow{\simeq} D(1)^\bullet_{h.h.}
\end{aligned}$$

so that (3.13) and $D(1)^\bullet_{h.h.}$ are indeed quasi-isomorphic. Thus (3.13) or (3.14) also compute $H^\bullet_{\mathcal{D}_{h.h.}}(X,1)$ and this proves the statement, since we have already seen that a metrized line bundle determines a degree two hypercohomology class with coefficients in (3.13). ☐

There is another—indeed, more direct—proof to the above proposition, which is simply to exhibit a cocycle in a Čech resolution of $D(1)^\bullet_{h.h.}$. Explicit representatives will be needed for the calculation of the cup products (3.6) and subsequent development, so we present this construction here.

Alternative proof of Proposition 3.4. Recall that $D(1)^\bullet_{h.h.}$ is the cone of the map $\rho_1 - \jmath$, where $\rho_1 : \mathbb{Z}(1)^\bullet_{\mathcal{D}} \to \widetilde{\mathbb{R}(1)}^\bullet_{\mathcal{D}}$, and $\jmath : F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1) \to \widetilde{\mathbb{R}(1)}^\bullet_{\mathcal{D}}$. By unraveling the structure of all the cones involved we have:

$$\begin{array}{ccccccccc}
\mathbb{Z}(1)_X & \longrightarrow & \underline{\Omega}^1_X \oplus \mathcal{O}_X & \longrightarrow & \underline{\Omega}^2_X \oplus \underline{\Omega}^1_X & \longrightarrow & \underline{\Omega}^3_X \oplus \underline{\Omega}^2_X & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \imath\oplus\pi_0} & & \downarrow{\scriptstyle \imath\oplus\pi_0} & & \downarrow{\scriptstyle \imath\oplus\pi_0} & & \\
 & & F^1\underline{A}^1_X \oplus \underline{\mathcal{E}}^0_X & \longrightarrow & F^1\underline{A}^2_X \oplus \underline{\mathcal{E}}^1_X & \longrightarrow & F^1\underline{A}^3_X \oplus \underline{\mathcal{E}}^2_X & \longrightarrow & \cdots \\
 & & & & \uparrow{\scriptstyle \jmath\oplus 0} & & \uparrow{\scriptstyle \jmath\oplus 0} & & \\
 & & & & F^1\underline{A}^2_X \cap \underline{\mathcal{E}}^2_X(1) & \longrightarrow & F^1\underline{A}^3_X \cap \underline{\mathcal{E}}^3_X(1) & \longrightarrow & \cdots
\end{array}$$

With respect to this diagram, an element of total degree 2 can be written in the form:

$$
\begin{array}{c|c|c}
c_{ijk} & -\,d\log g_{ij} \oplus \log g_{ij} & 0 \\
\hline
0 & \xi_i \oplus \sigma_i & \mathsf{X} \\
\hline
0 & 0 & \eta_i
\end{array}
\tag{3.16}
$$

for $\xi_i \in F^1\underline{A}^1_X(U_i)$, $\sigma_i \in \underline{\mathcal{E}}^0_X(U_i)$ and $\eta_i \in (F^1\underline{A}^2_X \cap \underline{\mathcal{E}}^2_X(1))(U_i)$. To make sense out of (3.16), note that each entry is an element of the object in the corresponding position in the left 3×3 part of the previous diagram. Then since the total degree is 2, the degree of each element *in the complex* $D(1)^\bullet_{h.h.}$ is 2 minus the Čech degree as found in (3.16). The top line is the class of L in $\mathbb{Z}(1)^2_{\mathcal{D}}$. Finally, the entry marked X means there is no applicable element—it would have degree 3.

A totally routine calculation shows that (3.16) is a degree 2 cocycle if and only if relations (3.9), (3.10) and $\eta_i = d\xi_i = 2\,\bar\partial\partial\sigma_i$ are satisfied (with $\sigma_i = \frac{1}{2}\log\rho_i$). Thus $\eta_i = c_1(\rho)|_{U_i}$ and all the relations defining a metrized line bundle with its canonical connection are satisfied. The verification that adding an appropriate coboundary to the cocycle leads to an isomorphic metrized bundle in the sense explained above is also routine. Finally, the correspondence between component-wise addition of cocycles modulo coboundaries and the group operation in $\widehat{\mathrm{Pic}(X)}$ is again a direct verification. □

It follows from the arguments in the second proof of Proposition 3.4 that a cocycle representing (L,ρ) in the Čech resolution of $D(1)^\bullet_{h.h.}$ must have the expression

$$
\begin{array}{c|c|c}
c_{ijk} & -\,d\log g_{ij} \oplus \log g_{ij} & 0 \\
\hline
0 & \partial\log\rho_i \oplus \frac{1}{2}\log\rho_i & \mathsf{X} \\
\hline
0 & 0 & \bar\partial\partial\,\log\rho_i
\end{array}
\tag{3.17}
$$

Furthermore, if $[L,\rho]$ and $[L',\rho']$ are the classes corresponding to the metrized bundles (L,ρ) and (L',ρ'), then we write $[L\otimes L', \rho\rho'] = [L,\rho] + [L',\rho']$.

3.3 Cup product of hermitian holomorphic line bundles

If L and L' are two line bundles on X, their cup product in Deligne cohomology would produce a class in $H^4_{\mathcal{D}}(X,\mathbb{Z}(2))$ denoted $\big(L, L'\big]$. Calculations were carried out in [9] where a geometric interpretation of $\big(L, L'\big]$ as a 2-gerbe and its connection with the determinant of cohomology (when X is the total space of a family of Riemann surfaces) were put forward. The structure ensuing from the generalization to line bundles equipped with a hermitian metric was further analyzed in ref. [11] by means of hermitian holomorphic Deligne cohomology.

The cup product of two metrized line bundles (L,ρ) and (L',ρ') in hermitian holomorphic Deligne cohomology produces a class in $H^4_{\mathcal{D}_{h.h.}}(X,2)$. We explicitly compute a representative of this class in terms of the expression (3.17) for the class of a metrized line bundle and the modified Beĭlinson product (2.5) for

$$
D(1)^\bullet_{h.h.} \otimes D(1)^\bullet_{h.h.} \xrightarrow{\ \cup_\alpha\ } D(2)^\bullet_{h.h.}\,.
$$

It is convenient to use the diagram for $D(1)^\bullet_{h.h.}$ displayed in the second proof to Proposition 3.4 and to use a corresponding diagram for the complex $D(2)^\bullet_{h.h.}$, namely:

$$
\begin{array}{ccccccccccc}
\mathbb{Z}(2)_X & \longrightarrow & \mathcal{O}_X & \longrightarrow & \underline{\Omega}^2_X \oplus \underline{\Omega}^1_X & \longrightarrow & \underline{\Omega}^3_X \oplus \underline{\Omega}^2_X & \longrightarrow & \underline{\Omega}^4_X \oplus \underline{\Omega}^3_X & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \imath\oplus\pi_1} & & \big\downarrow{\scriptstyle \imath\oplus\pi_1} & & \big\downarrow{\scriptstyle \imath\oplus\pi_1} & & \\
 & & \underline{\mathcal{E}}^0_X(1) & \longrightarrow & F^2\underline{A}^2_X \oplus \underline{\mathcal{E}}^1_X(1) & \longrightarrow & F^2\underline{A}^3_X \oplus \underline{\mathcal{E}}^2_X(1) & \longrightarrow & F^2\underline{A}^4_X \oplus \underline{\mathcal{E}}^3_X(1) & \longrightarrow & \cdots \\
 & & & & & & & & \big\uparrow{\scriptstyle \jmath\oplus 0} & & \\
 & & & & & & & & F^1\underline{A}^4_X \cap \underline{\mathcal{E}}^4_X(2) & \longrightarrow & \cdots
\end{array}
\tag{3.18}
$$

The cup product of two elements of the form (3.17) in the Čech resolution of $D(1)^\bullet_{h.h.}$ would result in a 5×3 table. When computing the entries, the signs are determined by those in the Beĭlinson product (2.5) plus those arising from the Čech resolution as explained in sect. 1.2. The expression obtained is more manageable for certain specific values of the parameter α. For instance, for $\alpha = 0$ we have:

(3.19)

$c_{ijk}c'_{klm}$	$-c_{ijk}\log g'_{kl}$	$-d\log g_{ij} \wedge d\log g'_{jk} \oplus -\log g_{ij}\, d\log g'_{jk}$	0	0
0	$-\pi_1 \log g_{ij}\ \log\lvert g'_{jk}\rvert$	$-d\log g_{ij} \wedge \partial\log\rho'_j$ $\oplus\big(d^c\log\lvert g_{ij}\rvert\ \tfrac{1}{2}\log\rho'_j - \log\lvert g_{ij}\rvert\ \tfrac{1}{2}d^c\log\rho'_j\big)$	$\partial\log\rho_i \wedge \bar\partial\partial\log\rho'_i$ $\oplus\tfrac{1}{2}\log\rho_i\ \bar\partial\partial\log\rho'_i$	X
0	0	0	0	$\bar\partial\partial\log\rho_i \wedge \bar\partial\partial\log\rho'_i$

For general values of the real parameter α, the expression one obtains is rather unwieldy and not so easy to display

$(0,4)$	$c_{ijk}c'_{klm}$
$(1,3)$	$-c_{ijk}\log g'_{kl}$
$(2,2)$	$\big(-d\log g_{ij} \wedge d\log g'_{jk} \oplus -\log g_{ij}\, d\log g'_{jk}\big) \bigoplus -\pi_1 \log g_{ij}\ \log\lvert g'_{jk}\rvert$
$(3,1)$	$\big(-(1-\alpha)\, d\log g_{ij} \wedge \partial\log\rho'_j + \alpha\,\partial\log\rho_i \wedge d\log g'_{ij}\big) \oplus \Big\{(1-\alpha)\big(d^c\log\lvert g_{ij}\rvert\ \tfrac{1}{2}\log\rho'_j - \log\lvert g_{ij}\rvert\ \tfrac{1}{2}d^c\log\ \rho'_j\big)$ $+\alpha\big(-\tfrac{1}{2}d^c\log\rho_i\ \log\lvert g'_{ij}\rvert + \tfrac{1}{2}\log\rho_i\ d^c\log\lvert g'_{ij}\rvert\big)\Big\}$
$(4,0)$	$\bar\partial\partial\log\rho_i \wedge \bar\partial\partial\log\rho'_i \bigoplus \big(\alpha\,\bar\partial\partial\log\rho_i \wedge \partial\log\rho'_i + (1-\alpha)\,\partial\log\rho_i \wedge \bar\partial\partial\log\rho'_i\big)$ $\oplus\ \big(\alpha\,\bar\partial\partial\log\rho_i\ \tfrac{1}{2}\log\rho'_i + (1-\alpha)\,\tfrac{1}{2}\log\rho_i\ \bar\partial\partial\log\rho'_i\big)$

Table 1: Cup product of (L, ρ) with (L', ρ') for generic values of the parameter α.

in the same form as eq. (3.19). Therefore we have put it in table 1, where the corresponding bidegrees are explicitly indicated. (The first degree is the overall degree in the cone and the second is the Čech degree.) Only the nonzero terms have been explicitly written.

If $\pi\colon X \to S$ is a holomorphic fibration with compact connected Riemann surfaces as fibers, from eq. (3.7) we have that the cup product of (L, ρ) and (L', ρ') induces a metrized line bundle on S. From [9, Theorem 6.1] there is a corresponding map $A^{(2,2)}(X)_{\mathbb{R}(2)} \to A^{(1,1)}(S)_{\mathbb{R}(1)}$. From the explicit cocycles we have computed the corresponding representative in $H^2(S, \mathbb{R}(1)) \cap H^{1,1}(S)$ of the first Chern class of the resulting metrized line bundle will be

$$\int_\pi c_1(\rho) \wedge c_1(\rho')\,,$$

as in [12], Proposition 6.6.1.

4 Conformal metrics and the Liouville functional

Let X be a compact Riemann surface of genus $g \geq 2$. For simplicity we can assume X to be connected, although this is not necessary, and in fact this assumption will be dropped when dealing with Kleinian groups.

Let $\mathscr{CM}(X)$ be the space of conformal metrics on X. Locally on X, if z is a local analytic coordinate defined on an open set U, any metric ds^2 can be represented as

$$ds^2 = \rho\,\lvert dz\rvert^2$$

for a positive function $\rho : U \to \mathbb{R}_{>0}$. According to sec. 3.2, a conformal metric corresponds to considering the metrized line bundle (T_X, ρ), where T_X is the holomorphic tangent bundle of X. With respect to a cover $\mathfrak{U}_X$ of X, the conformal factors ρ_i and ρ_j, associated to U_i and U_j respectively, satisfy the relation

$$\rho_j = \rho_i\,\lvert z'_{ij}\rvert^2\,, \tag{4.1}$$

where $z'_{ij} = dz_i/dz_j$ and z_i is a local coordinate defined over U_i.

It follows from the uniformization theorem that there exists a unique conformal metric of scalar curvature equal to -1, the Poincaré metric. Locally on $U_i \subset X$, the condition for the metric to have curvature -1 is equivalent to the nonlinear PDE

$$\frac{\partial^2}{\partial z_i \partial \bar z_i}\phi_i = \frac{1}{2}\exp\phi_i\,, \tag{4.2}$$

known as the Liouville equation, for the smooth function $\phi_i = \log \rho_i$. Observe that equation (4.2) can be written in the form

$$c_1(\rho) = \sqrt{-1}\,\omega_\rho\,,$$

where we have used the Kähler form associated to the metric:

$$\omega_\rho|_{U_i} = \frac{\sqrt{-1}}{2}\rho_i\,dz_i \wedge d\bar{z}_i\,.$$

This representation makes it apparent that the Liouville equation is independent of the choice of the coordinate system. On the other hand, a direct verification of this fact is immediate using $\phi_j - \phi_i = \log|z'_{ij}|^2$.

4.1 Variational problem for conformal metrics

It is well known that the Liouville equation has a local variational principle in the following sense. Let D be a region in the complex plane. Then equation (4.2) is the Euler-Lagrange equation for the variational problem defined by the action functional

$$S[\phi] = \frac{\sqrt{-1}}{2}\int_D \bigl(\partial\phi \wedge \bar{\partial}\phi + e^\phi\,dz\wedge d\bar{z}\,\bigr)\,, \tag{4.3}$$

defined on smooth functions $\phi : D \to \mathbb{R}$, with the condition that variations $\frac{d}{d\alpha}(\phi_\alpha)\big|_{\alpha=0}$ of ϕ be zero on ∂D. However, it is easily seen that the functional (4.3) cannot be defined globally on X, since, as a consequence of (4.1), the first term under the integral sign would not yield a well-defined 2-form on X. (The second one would of course present no problems, it would just give the area $A_X(\rho)$ of X with respect to the given metric $\rho\,|dz|^2 = e^\phi\,|dz|^2$.) Accordingly, it is convenient to write the integrand in (4.3) as

$$\sqrt{-1}\,\omega[\phi] + \omega_\rho$$

where we have defined the 2-form[3]

$$\omega^0[\phi] = \frac{1}{2}\partial\phi \wedge \bar{\partial}\phi\,, \tag{4.4}$$

and restrict our considerations to the first term of (4.3) which we denote

$$\check{S}[\phi] = \sqrt{-1}\int_D \omega^0[\phi]\,. \tag{4.5}$$

There is by now an established procedure on how to address the problem caused by the fact that (4.4) is not globally defined. In general terms, given the choice of a conformal metric $ds^2 \in \mathscr{CM}(X)$ and a cover $\mathfrak{U}_X$, one works with the full Čech-de Rham complex $\check{C}^\bullet(\mathfrak{U}_X, \underline{\mathcal{E}}^\bullet_X(1))$ with respect to $\mathfrak{U}_X$, rather than with just differential forms. The 2-form (4.4) is then completed to a total degree 2-cocycle—to be denoted $\Omega[\phi]$. This results in a class in $H^2(X, \mathbb{R}(1))$ after taking cohomology. (Whether or not there also is a de Rham type theorem will depend on the acyclicity properties of $\mathfrak{U}_X$.)

This scheme has been previously carried out not quite for covers of X by open sets, but actually for different choices of planar coverings. For the covering associated to a Schottky uniformization of X a generalization of eq. (4.3) was written in ref. [35]. More recently, a detailed calculation of the cocycle for the general case of a covering associated to a Kleinian uniformization was carried out in ref. [31] by exploiting the homological methods developed in ref. [2]. Note, however, that from the point of view of ref. [3] these planar coverings are "étale" coordinates on X, so the group cohomology constructions required to work with the various kinds of uniformization coverings just follow from specializing the Čech formalism to the coverings at hand.

Finally, the integration in eqs. (4.3) or (4.5) should be replaced by the evaluation of $\Omega[\phi]$ against an appropriate representative Σ of the fundamental class of X. The "appropriate" form for both Σ and the evaluation will be dictated by the chosen cover $\mathfrak{U}_X$ *and* the cohomology theory being used. Typically Σ will be a cycle in a double complex of $\mathfrak{U}_X$-small simplices, where the differentials are the singular one and the one determined by the face maps induced by the cover. Thus, in the case of a Čech cover, it will be the complex determined by a triangulation of X subordinated to the open $\mathfrak{U}_X$, and in the same way, for a planar cover the singular complex of the planar domain $\mathfrak{U}_X$ tensored with an appropriate bar resolution of the group of deck transformations. These issues have been discussed at length in refs. [2, 3, 31], so we will not repeat the discussion here. Whenever we have a cocycle extending (4.4) and a cycle Σ representing X we state

[3]Note that equation (4.4) defines an imaginary form. The reason for this choice will be apparent later.

Definition 4.1. The Liouville functional (without the area term) is given by the evaluation:

$$\check{S}[\phi] = -\frac{1}{2\pi\sqrt{-1}} \langle \Omega[\phi]\,,\, \Sigma \rangle\,. \tag{4.6}$$

For the complete functional we add the area term

$$S[\phi] = \check{S}[\phi] + \frac{1}{2\pi} \int_X \omega_\rho\,. \tag{4.7}$$

Remark 4.2. $\langle \Omega[\phi]\,,\, \Sigma \rangle \in \mathbb{R}(1)$, and $\check{S}[\phi]$ (or $S[\phi]$) is real. Division by $2\pi\sqrt{-1}$ is conventional, but note that $\mathbb{R}(1) \xrightarrow{\simeq} \mathbb{R}_+$ via $\exp(\cdot/2\pi\sqrt{-1})$. In the sequel it will be more convenient to work with imaginary classes. (See also section 6.)

In next two subsections we examine these constructions in some detail. For definiteness, we initially make use of an ordinary open cover. First, we recall the direct construction of a cocycle generalizing (4.4), and we show that this way eq. (4.2) indeed is the resulting extremum condition. Then we emphasize the role played by Deligne cohomology and the tame symbol. These aspects will become important when introducing a Kleinian uniformization later in the paper, when we discuss connections with the dilogarithm function. In a later section we shall tackle the question of its geometrical significance by making full use of the hermitian-holomorphic version of Deligne cohomology presented in section 3, and we show that the cocycle constructed following refs. [35, 31] corresponds to the square of (T_X, ρ) in hermitian holomorphic Deligne cohomology.

4.2 Direct construction of the Liouville cocycle

4.2.1 Initial setup

Let X a compact Riemann surface of genus greater than 2. We shall not include the area term in our explicit calculations, therefore it makes sense to extend our considerations to a general metrized line bundle (L, ρ). Of course, whenever referring to a conformal metric or to the variational problem for the Liouville equation, it will be be assumed that $L = T_X$ and that $g_{ij} = z'_{ij}$. Upon choosing a cover $\mathfrak{U}_X$, which for now we assume to be a Čech cover by open sets, the pair (L, ρ) is described in terms of the data expounded in section 3.2. Our starting point will be the 0-cochain

$$\omega_i^0[\log \rho_i] = \frac{1}{2}\,\partial \log \rho_i \wedge \bar{\partial} \log \rho_i = -\frac{1}{2} d \log \rho_i \wedge \frac{1}{2} d^c \log \rho_i \tag{4.8}$$

with values in $\underline{\mathcal{E}}^2_X(1)(U_i)$.

Remark 4.3. A generalization for eq. (4.8) would be to consider a *pair* of metrized line bundles (L, ρ) and (L', ρ'), and then the analogue of (4.8) would be

$$\omega_i^0[\log \rho_i, \log \rho'_i] = \frac{1}{2}\Big(-\frac{1}{2} d \log \rho_i \wedge \frac{1}{2} d^c \log \rho'_i + \frac{1}{2} d^c \log \rho_i \wedge \frac{1}{2} d \log \rho'_i \Big) \tag{4.9}$$

Note, however, that the expressions are quadratic. Moreover, $L \otimes L'$ has metric $\rho\rho'$, so that there is a natural "polarization identity"

$$\omega[\log \rho, \log \rho'] = \frac{1}{4} \omega[\log \rho\rho'] - \frac{1}{4} \omega[\log \frac{\rho}{\rho'}] \tag{4.10}$$

where we have omitted the indexes for simplicity of notation. We shall comment later on the significance of eq. (4.10).

4.2.2 Computation

Let us extend (4.8) to a degree 2 cocycle in the total simple complex associated to the double complex $\check{C}^\bullet(\mathfrak{U}_X, \underline{\mathcal{E}}^\bullet_X(1))$ of Čech cochains with values in the de Rham complex of imaginary smooth forms. This is accomplished in the usual fashion (see e.g. [8]) by finding a 1-cochain of 1-forms $\omega^1_{ij}[\log \rho]$ on U_{ij} and a 2-cochain of 0-forms $\omega^2_{ijk}[\log \rho]$ on U_{ijk} such that the relations

$$\check{\delta}\omega^0 = -d\omega^1\,, \qquad \check{\delta}\omega^1 = d\omega^2\,, \qquad \check{\delta}\omega^2 = 0$$

are satisfied. Of course, the remaining one, namely $d\omega = 0$ is automatically satisfied for dimensional reasons. It turns out that to a great extent these relations are explicitly computable without further assumptions, such as that the cover $\mathfrak{U}_X$ be good. The needed calculations are fairly standard, and they are presented in great detail in ref. [31], so we shall be brief. (The observation in [25] that on a Riemann surface for two smooth functions f and g one has $df \wedge d^c g = dg \wedge d^c f$ is useful in carrying out the calculations.) The first two steps are as follows.

First, one has:

$$\tag{4.11} \begin{gathered} \omega_j^0[\rho] - \omega_i^0[\rho] = -d\,\omega_{ij}^1[\rho] \\ \omega_{ij}^1[\rho] = \frac{1}{2}\log\rho_i\; d^c\log|g_{ij}| + d^c\log|g_{ij}|\;\frac{1}{2}\log\rho_j \end{gathered}$$

The next step yields:

$$\begin{aligned} \check{\delta}\big(\omega^1[\log\rho]\big)_{ijk} &= \omega_{jk}^1[\rho] - \omega_{ik}^1[\rho] + \omega_{ij}^1[\rho] \\ &= \log|g_{ij}|\; d^c\log|g_{jk}| - d^c\log|g_{ij}|\;\log|g_{jk}| \end{aligned}$$

and notice that $d^c\log|g_{ij}| = \pi_1 d\log g_{ij}$, and $\pi_{p+q-1}(a\wedge b) = \pi_p(a)\wedge\pi_{q-1}(b) + \pi_{p-1}(a)\wedge\pi_q(b)$, so we have

$$\tag{4.12} \check{\delta}\big(\omega^1[\log\rho]\big)_{ijk} = \pi_1\big(\log g_{ij}\; d\log g_{jk}\big) - d\big(\pi_1(\log g_{ij})\;\log|g_{jk}|\big)\,.$$

Observe that now the problem of continuing the descent becomes independent of the chosen metric ρ. The most direct way of proceeding is the following. If we assume the cover $\mathfrak{U}_X$ to be acyclic for the de Rham complex $\underline{\mathcal{E}}_X^\bullet(1)$, then there exists $\omega_{ijk}^2 \in \underline{\mathcal{E}}_X^0(1)(U_{ijk})$ such that $d\omega_{ijk}^2 = \check{\delta}(\omega^1[\log\rho])_{ijk}$. Furthermore, consistency on a quadruple intersection requires that $\check{\delta}\omega^2$ be a 3-cocycle with values in $\mathbb{R}(1)_X$. Since $H^3(X,\mathbb{R}(1)) = 0$, this cocycle must be a coboundary, therefore, up to readjusting the constants, there exists a choice of ω_{ijk}^2 such that $\check{\delta}\omega^2 = 0$, and furthermore, ω_{ijk}^2 does not depend on the metric structure.

4.2.3 Solution to the variational problem

The previous preliminary calculation is sufficient from the point of view of finding the extrema. To this effect, we set $L = T_X$, for ρ a conformal metric in $\mathscr{CM}(X)$. Notice that the space of conformal metrics on X is affine over $C^\infty(X,\mathbb{R}) \equiv \mathcal{E}^0(X)$: if ds^2 and ds'^2 are two conformal metrics with local expressions $\rho_i\,|dz_i|^2$ and $\rho'_i\,|dz_i|^2$ respectively, then there exists $\sigma\in C^\infty(X,\mathbb{R})$ such that $\log\rho'_i = \sigma|_{U_i} + \log\rho_i$. The change from $\Omega[\log\rho]$ to $\Omega[\log\rho+\sigma]$ can be exactly computed thanks to the fact that the last step in the determination of $\Omega[\log\rho]$ is independent of ρ and the quadratic character of (4.4). Indeed we have:

Proposition 4.4.

$$S[\log\rho+\sigma] - S[\log\rho] = \frac{1}{2\pi\sqrt{-1}}\int_X \Big(\frac{1}{2}\partial\sigma\wedge\bar\partial\sigma + \sigma\, c_1(\rho) - \sqrt{-1}(e^\sigma - 1)\omega_\rho\Big)$$

Proof. The change in $\omega_i^0[\log\rho]$ is computed as

$$\omega_i^0[\log\rho+\sigma] - \omega_i^0[\log\rho] = -\frac{1}{2}d\sigma_i\wedge\frac{1}{2}d^c\sigma_i + \sigma_i\,\frac{1}{2}dd^c\log\rho_i - \frac{1}{2}d(\sigma_i\, d^c\log\rho_i)\,,$$

where we set $\sigma_i \equiv \sigma|_{U_i}$. Note that the first two terms on the right hand side are globally well-defined 2-forms. On the other hand

$$\omega_{ij}^1[\log\rho+\sigma] - \omega_{ij}^1[\log\rho] = \sigma_i\, d^c\log|g_{ij}|\,.$$

Letting $\chi_i = \sigma_i\, d^c\log\rho_i$, we see that

$$\Omega[\log\rho+\sigma] - \Omega[\log\rho] = -\frac{1}{2}d\sigma\wedge\frac{1}{2}d^c\sigma + \sigma\,\frac{1}{2}dd^c\log\rho - D\chi\,,$$

and taking the area terms into account, establishes the formula. □

As a consequence, we obtain

Corollary 4.5. *The Liouville equation* (4.2) *is the Euler-Lagrange equation for the Liouville functional* (4.7) *introduced in definition 4.1. The critical point is non-degenerate.*

Proof. Replacing σ with $t\,\sigma$, $t\in\mathbb{R}$, in the previous proposition we find the infinitesimal change in $\Omega[\log\rho]$ to be

$$\frac{d}{dt}\bigg|_{t=0} S[\log\rho + t\sigma] = -\frac{1}{2\pi\sqrt{-1}}\int_X \sigma\big(c_1(\rho) - \sqrt{-1}\,\omega_\rho\big)\,,$$

and it follows that $S[\log\rho]$ has an extremum if and only if the Liouville equation is satisfied. Non degeneracy follows from the quadratic part in the exact change formula in proposition 4.4. □

Remark 4.6. The fact that the change in the cocycle is given by a pure 2-form term up to total coboundary can also be analyzed in terms of gluing properties of variational bicomplexes, cf. Theorem 1.2 in ref. [1] and the proof of Theorem 1 in ref. [3]. From this perspective, Corollary 4.5 is a direct consequence of the affine structure of the space $\mathscr{CM}(X)$ of conformal metrics.

4.3 A cup product

Formula (4.12) can be handled in a more geometric fashion as follows. From sec. 2.4 we can rewrite (4.12) as

$$\check\delta\big(\omega^1[\log\rho]\big)_{ijk} = -r_2(g_{ij}, g_{jk})\,, \tag{4.13}$$

and we have the collection of tame symbols $\big(g_{ij}, g_{jk}\big]$ associated any triple intersection U_{ijk} in the cover $\mathfrak{U}_X$, [9]. These symbols glue to form a global symbol $\big(L, L\big]$. As a cohomology class on a curve X, however, $\big(L, L\big]$ will be zero (that is, there is a global object in the associated 2-stack, cf. [11]), so that it will be possible to choose local functions L_{ijk} such that

$$dL_{ijk} = -\log\, g_{ij}\; d\,\log\, g_{jk}\,,$$

as explained in detail in [3]. Note that we still need the cover $\mathfrak{U}_X$ to be fine enough. Moreover, cf. loc. cit., the collection L_{ijk} can be chosen in a way that

$$\check\delta L_{ijkl} = -c_{ijk}\log\, g_{kl} + n_{ijkl}\,,$$

where $n_{ijkl}\in\mathbb{Z}(2)$. Therefore from (2.21), (4.12) and (4.13), we can set

$$\omega^2_{ijk} = -\big(\pi_1(\log\, g_{ij})\;\log|g_{jk}|\big) - \pi_1 L_{ijk}\,, \tag{4.14}$$

namely according to sec. 2.4 we have set $\omega^2_{ijk} = -\mathscr{L}_2(g_{ij}, g_{jk})$. Now, the last compatibility condition is satisfied, indeed we have:

$$\begin{aligned}\check\delta\omega^2_{ijkl} &= -c_{ijk}\;\log|g_{kl}| - \pi_1(\check\delta L_{ijkl})\\ &= -c_{ijk}\;\log|g_{kl}| + \pi_1(c_{ijk}\log g_{kl})\\ &= 0\,.\end{aligned}$$

As a result, we obtain a 2-cocycle in the the total complex associated to $\check C^\bullet(\mathfrak{U}_X, \underline{\mathcal{E}}^\bullet_X(1))$ as before, with a more geometric interpretation of ω^2_{ijk} in terms of the trivialization of the symbol $\big(L, L\big]$. Moreover, notice that if X is a curve then $\underline{\mathcal{E}}^\bullet_X(1)[1] \xrightarrow{\simeq} \widetilde{\mathbb{R}(2)}^\bullet_{\mathcal{D}}$ thus we may interpret the class so determined by $\Omega[\log\rho] = \omega^0[\log\rho] + \omega^1[\log\rho] + \omega^2$ as a (degree 3) class in $H^3_{\mathcal{D}}(X, \mathbb{R}(2))$.

4.4 Two line bundles

For a pair (L,ρ), (L',ρ') of metrized line bundles we can complete (4.9) to a cocycle $\Omega[\log\rho, \log\rho']$ via an analogous procedure to the one presented in sects. 4.2 and 4.3. The relevant calculations being entirely similar, we limit ourselves to quoting the relevant expressions. Starting from (4.9), which we rewrite in the form

$$\omega^0_i[\log\rho_i, \log\rho'_i] = \frac{1}{2}d^c\log\rho_i \wedge \frac{1}{2}d\log\rho'_i\,, \tag{4.15}$$

the corresponding expression for the degree $(1,1)$ term is:

$$\omega^1_{ij}[\log\rho, \log\rho'] = \frac{1}{2}\log\rho_i\; d^c\log|g'_{ij}| + \frac{1}{2}d^c\log|g_{ij}|\;\log\rho'_j\,. \tag{4.16}$$

Computing the Čech coboundary we find:

$$\begin{aligned}\big(\check\delta\omega^1[\log\rho,\log\rho']\big)_{ijk} &= \log|g_{ij}|\; d^c\log|g'_{jk}| - d^c\log|g_{ij}|\;\log|g'_{jk}| \\ &= -r_2(g_{ij},g'_{jk})\,,\end{aligned} \tag{4.17}$$

from which ω^2_{ijk} (now independent of ρ and ρ') can be obtained as $-\mathscr{L}_2(g_{ij},g'_{jk})$ by looking at a collection L_{ijk} such that

$$dL_{ijk} = -\log\, g_{ij}\; d\,\log\, g'_{jk}\,,$$

from the triviality of the symbol $\big(L,L'\big]$.

5 The Liouville functional and the Bloch-Wigner dilogarithm

In this section we specialize some of the constructions in sect. 4, and in particular those in sect. 4.3 to the case of the cover $\mathfrak{U}_X$ of X given by a Kleinian uniformization. The reader should consult ref. [31] for a complete treatment. Our task here is to point out that the analog of the dilogarithm used in section 4.3 becomes *literally* the Bloch-Wigner function (2.20), which appears as the obstruction to closing the descent equations for the cocycle defining the Liouville functional. This clarifies the vanishing of the cohomological obstructions in the calculations in loc. cit. The methods of sect. 6 however, sidestep the problem entirely.

5.1 Kleinian groups and fractional linear transformations

As a reference, the reader can consult, among many others, the book [26]. We consider here a finitely generated purely loxodromic non-elementary Kleinian group Γ. We assume the group to be a Kleinian group of the second kind, namely to have a non empty discontinuity region $U_\Gamma \subset \mathbb{P}^1$. The limit set is $L_\Gamma = \mathbb{P}^1 \setminus U_\Gamma$. According to Ahlfors' finiteness theorem the quotient $X = U_\Gamma/\Gamma$ is a finite union of analytically finite Riemann surfaces. Thus:

$$U_\Gamma/\Gamma = U_1/\Gamma_1 \sqcup \cdots \sqcup U_n/\Gamma_n\,,$$

where $U_1,\dots,U_n$ are the inequivalent components of U_Γ and $\Gamma_1,\dots,\Gamma_n$ their stabilizers. By way of example, a Schottky group has just one component, whereas a Fuchsian or quasi-Fuchsian group will have exactly two components, and the resulting Riemann surfaces are mirror images of one another.

In the language of [3] the map $U_\Gamma \to X$ is an étale cover and Čech cohomology with respect to it translates into group cohomology for the group Γ where the coefficient modules are sections over U_Γ of the relevant sheaves. The group action is by pull-back. Still according to the conventions in loc. cit. we will write the coboundary operation in the following slightly non-standard way. Assume c is an n-cochain with values in some *right* Γ-module A. Rather than turning A into a left Γ-module via the inversion $\gamma\, a\colon = a\gamma^{-1}$, we use the following expression for the coboundary:

$$(\check\delta c)_{\gamma_1,\dots,\gamma_n} = c_{\gamma_2,\dots,\gamma_n} + \sum_{i=1}^{n-1}(-1)^i c_{\gamma_1,\dots,\gamma_i\gamma_{i+1},\dots,\gamma_n} + (-1)^n (c_{\gamma_1,\dots,\gamma_{n-1}})\cdot\gamma_n\,. \tag{5.1}$$

As explained at length in ref. [3], the right action is more suited to handle the case where group acts on geometric objects by pull-back, which is a right action if the group Γ acts on U_Γ from the left, as we implicitly assume. Correspondingly, (5.1) is the Čech coboundary applied to the nerve of the cover $U_\Gamma \to X$. The convention in eq. (5.1) avoids a host of inversions in the group in order to have a more conformant left action.

We assume that Γ is normalized, namely the point ∞ belongs to the limit set L_Γ. Finally, let us set some more conventions. Let γ be an element of Γ represented as a fractional linear transformation:

$$\mathbb{P}^1 \ni z \longmapsto \gamma(z) = \frac{az+b}{cz+d}\,.$$

Then we can write the derivative with respect to z as:

$$\gamma'(z) = \frac{\det\gamma}{c^2(z-z_\gamma)^2} \tag{5.2}$$

where $z_\gamma = -\frac{d}{c} \cong \gamma^{-1}(\infty)$. We will set

$$c(\gamma) \equiv c_\gamma \overset{\text{def}}{=} \frac{\det\gamma}{c^2}\,.$$

The following properties are easily verified. If z_0 and z_∞ are the attracting and repelling fixed points, respectively, then

$$c_\gamma = \frac{(z_0 - z_\infty)^2 \lambda_\gamma}{(1-\lambda_\gamma)^2}\,,$$

where λ_γ is the dilating factor. For γ_i and γ_j two elements of Γ, denote:

$$z_i = \gamma^{-1}(\infty)\,, \quad z_{ij} = (\gamma_i\gamma_j)^{-1}(\infty) = \gamma_j^{-1}(z_i)\,, \quad c_i = c_{\gamma_i}\,, \quad c_{ij} = c_{\gamma_i\gamma_j}\,.$$

Then the following relation is easily checked:

$$c_{\gamma_1\gamma_2} = \frac{c_{\gamma_1}}{c_{\gamma_2}}\,(z_{12} - z_2)^2\,. \tag{5.3}$$

Finally, given four points $z_1, z_2, z_3, z_4 \in \mathbb{P}^1$, we define their cross-ratio by:

$$[z_1 \colon z_2 \colon z_3 \colon z_4] = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}\,. \tag{5.4}$$

5.2 Computation of the cocycle

A conformal metric ρ on $X = X_1 \sqcup \cdots \sqcup X_n$ appears as an automorphic function on U_Γ:

$$\frac{1}{2}\log\rho - \frac{1}{2}\log\rho\circ\gamma = \log|\gamma'| \tag{5.5}$$

for $\gamma \in \Gamma$. Eq. (5.5) is the direct translation of (3.8) following the principles of [3]. Accordingly, on has that the first two terms of the Liouville cocycle computed by applying the procedure explained in sect. 4 are:

$$\omega^0[\log\rho] = -\frac{1}{2}d\log\rho \wedge \frac{1}{2}d^c\log\rho \tag{5.6}$$

$$\omega^1_\gamma[\log\rho] = \big(\frac{1}{2}\log\rho + \frac{1}{2}\log\rho\circ\gamma\big)\; d^c\log|\gamma'| \tag{5.7}$$

Computing the coboundary of the term in (5.7) according to:

$$(\check{\delta}\omega^1)_{\gamma_1,\gamma_2} = \omega^1_{\gamma_2} - \omega^1_{\gamma_1\gamma_2} + (\gamma_2)^*\omega^1_{\gamma_1}\,,$$

where we have suppressed the dependency on $\log\rho$ for simplicity of notation, yields, analogously to what we have seen in sect. 4.3:

$$\check{\delta}\big(\omega^1[\log\rho]\big)_{\gamma_1,\gamma_2} = -r_2((\gamma_1\gamma_2)', \gamma_2')\,, \tag{5.8}$$

where r_2 has been introduced in eqs. (2.18) and (2.19). According to sect. 2, and in particular sect. 2.4, the coboundary of ω^1 is the cup product of the two real Deligne cohomology classes $(d\log(\gamma_1\gamma_2)', \log|(\gamma_1\gamma_2)'|)$ and $(d\log\gamma_2', \log|\gamma_2'|)$ associated to the rational functions $(\gamma_1\gamma_2)'$ and γ_2'. Thus we will have to consider a double complex of group cochains on Γ with values in Deligne complexes on the region of discontinuity U_Γ. Notice that from this point on only rational functions with singularities at certain prescribed points will appear. Hence, following ref. [21], we can consider the Deligne complex on the generic point $\eta_{\mathbb{P}^1}$ of $\mathbb{P}^1$. Now, as before, the automorphy factor describing the metric has disappeared and the problem of finding a function ω^2 such that its derivative equals $-r_2$ involves only objects related to the geometry of Γ.

As in sect. 5.1, for $\gamma_i, \gamma_j \in \Gamma$, consider the points z_j, z_{ij}—the inverse images of ∞—and define the element $T_{ij} \in \mathrm{PSL}_2(\mathbb{C})$ by

$$z \longmapsto T_{i,j}(z) = [z \colon z_{ij} \colon z_j \colon \infty]\,. \tag{5.9}$$

A fairly straightforward calculation, using (5.3) and the notation $\gamma_{12} \equiv \gamma_1\gamma_2$, shows that

$$\begin{aligned} r_2((\gamma_1\gamma_2)', \gamma_2') = 4\,T^*_{1,2}\big(r_2(1-z, z)\big) &+ 2\,\log|\gamma_{12}'(z_2)|\;\pi_1 d\log(z - z_2) \\ &- 2\,\log|\gamma_2'(z_{12})|\;\pi_1 d\log(z - z_{12})\,, \end{aligned}$$

with a slight abuse of notation in the first term on the right hand side. Therefore from eqs. (2.20) and (2.21) we have

$$r_2((\gamma_1\gamma_2)',\gamma_2') = 4\,d\big(T^*_{1,2}\mathscr{L}_2\big)(z) + 2\,\log|\gamma_{12}'(z_2)|\;\pi_1 d\log(z-z_2) - 2\,\log|\gamma_2'(z_{12})|\;\pi_1 d\log(z-z_{12})\,. \tag{5.10}$$

In analogy with ref. [31] we introduce the 1-cochain on Γ with values in $\underline{\mathcal{E}}^1_{\mathbb{P}^1}(1)(\eta_{\mathbb{P}^1})$:

$$\kappa_\gamma = \log|c_\gamma|\,\pi_1 d\log\gamma'\,. \tag{5.11}$$

Obviously $d\kappa_\gamma = 0$, and moreover we have:

Lemma 5.1. $\check{\delta}\kappa_{\gamma_1,\gamma_2} = 2\,\log|\gamma_{12}'(z_2)|\;\pi_1 d\log(z-z_2) - 2\,\log|\gamma_2'(z_{12})|\;\pi_1 d\log(z-z_{12})\,.$

Proof. A straightforward calculation exploiting relation (5.3). □

Using the lemma, from eq. (5.8) we have:

$$\check{\delta}\big(\omega^1[\log\rho]\big)_{\gamma_1,\gamma_2} = -4\,d\big(T^*_{1,2}\mathscr{L}_2\big)(z) - \check{\delta}\kappa_{\gamma_1,\gamma_2}\,,$$

so that, upon redefining $\omega^1_\gamma \to \omega^1_\gamma + \kappa_\gamma$, we have the last relation $\check{\delta}\omega^1 = d\omega^2$ with:

$$\omega^2_{\gamma_1,\gamma_2} = -4\left(T^*_{1,2}\mathscr{L}_2\right). \tag{5.12}$$

5.3 Cross ratio, tetrahedra and the Block-Wigner function

Relation eq. (5.12) is suggestive, in that the true Bloch-Wigner function, as presented in sect. 2.4 appears explicitly. However, we need to check the closure of the descent condition, or in other words, to compute the coboundary of ω^2 to ensure that we have obtained a cocycle.

For three elements $\gamma_1,\gamma_2,\gamma_3\in\Gamma$ we have the following calculation:

$$\begin{aligned}\check{\delta}\omega^2_{\gamma_1,\gamma_2,\gamma_3} &= \omega^2_{\gamma_2,\gamma_3} - \omega^2_{\gamma_1\gamma_2,\gamma_3} + \omega^2_{\gamma_1,\gamma_2\gamma_3} - \gamma_3^*\omega^2_{\gamma_1,\gamma_2}\\ &= -4\Big(\mathscr{L}_2([z\colon z_{23}\colon z_3\colon\infty]) - \mathscr{L}_2([z\colon z_{123}\colon z_3\colon\infty])\\ &\qquad + \mathscr{L}_2([z\colon z_{123}\colon z_{23}\colon\infty]) - \mathscr{L}_2([\gamma_3(z)\colon z_{12}\colon z_2\colon\infty])\Big)\end{aligned}$$

By the $\mathrm{PSL}_2(\mathbb{C})$ invariance property of the cross-ratio, in the last term in the previous relation we have

$$[\gamma_3(z)\colon z_{12}\colon z_2\colon\infty] = [z\colon z_{123}\colon z_{23}\colon z_3]\,.$$

Recall the 5-term relation satisfied by the Bloch-Wigner dilogarithm $\mathscr{D}_2$ [16, 23]: for five points $a_0,\dots,a_4\in\mathbb{P}^1$ we have the relation

$$\sum_{i=0}^{4}(-1)^i\mathscr{D}_2([a_0:\cdots:\hat{a_i}:\cdots:a_4]) = 0\,, \tag{5.13}$$

where the hat sign denotes omission. As a consequence we have:

$$\check{\delta}\omega^2_{\gamma_1,\gamma_2,\gamma_3} = -4\,\mathscr{L}_2([z_{123}\colon z_{23}\colon z_3\colon\infty])\,,$$

and again by the invariance of the cross ratio, we finally obtain:

$$\check{\delta}\omega^2_{\gamma_1,\gamma_2,\gamma_3} = -4\,\mathscr{L}_2([\infty\colon\gamma_1\infty\colon\gamma_1\gamma_2\infty\colon\gamma_1\gamma_2\gamma_3\infty])\,. \tag{5.14}$$

The right hand side of eq. (5.14) defines an $\mathbb{R}(1)$-valued 3-cocycle on $\mathrm{PSL}_2(\mathbb{C})$, restricted to Γ, thanks to the five-term relation. We can summarize the foregoing as follows:

Theorem 5.2. *The obstruction to closing the descent equations relative to the cover $U_\Gamma\to X$ associated to a Kleinian uniformization, is given by the universal volume class. This obstruction is zero for Γ of the second kind.*

Proof. First, it is clear from the calculations above that for the total coboundary we have

$$D(\omega^0 + \omega^1 + \omega^2) = -4\,\mathscr{L}_2([\infty\colon \gamma_1\infty\colon \gamma_1\gamma_2\infty\colon \gamma_1\gamma_2\gamma_3\infty])\,.$$

The only thing left to do is to collect the necessary results from the literature. From refs. [17, 15] we have that the hyperbolic volume of a totally ideal tetrahedron is computed by evaluating $\mathscr{D}_2$ at the cross ratio of its vertices. Now consider the four points in $\mathbb{P}^1$ obtained by applying three elements of $\mathrm{PSL}_2(\mathbb{C})$ to $\infty \in \mathbb{P}^1$ as in eq. (5.14). From[15], we have that computing the Bloch-Wigner dilogarithm at their cross ratio yields the imaginary part of the second Cheeger-Simons universal secondary class which generates the third cohomology group of $\mathrm{PSL}_2(\mathbb{C})$ with values in $\mathbb{R}(1)$. Pulling back this class along $\Gamma \hookrightarrow \mathrm{PSL}_2(\mathbb{C})$ we obtain zero, since the cohomology of Γ is computed by the 3-manifold $\mathbb{H}^3/\Gamma$, which, being non-compact has zero three-dimensional cohomology. □

6 Conformal metrics and hermitian holomorphic cohomology

In sect. 4 we presented a construction of a degree 2, $\mathbb{R}(1)$-valued class corresponding to a conformal metric $\rho \in \mathscr{CM}(X)$, represented by the cocycle $\Omega[\log\rho]$. Supplemented by the area of X computed with respect to ρ, it provides a global functional for the variational problem associated to the Liouville equation (4.2). We now show that it coincides with the square of the class of (T_X,ρ) in hermitian holomorphic Deligne cohomology introduced in sect. 3. Moreover, we show that this equality holds at the cocycle level. More generally, without considering the area term, we show that the cup product of (L,ρ) and (L',ρ') in hermitian holomorphic Deligne cohomology coincides with the class of $\Omega[\log\rho,\log\rho']$, and again the equality in fact holds at the cocycle level.

6.1 Comparison on a curve

Let X be a compact Riemann surface. From the results of sect. 3.3, the cup product of the classes of (L,ρ) and (L',ρ') in hermitian holomorphic Deligne cohomology yields a class in $H^4_{\mathcal{D}_{h.h.}}(X,2)$, and on a curve we only capture the 2-dimensional part of this class. Indeed, in the exact sequence (3.4), the cohomology class corresponding to the symbol $\big(L,L'\big] \in H^4_{\mathcal{D}}(X,\mathbb{Z}(2))$ is zero, and $A^{(2,2)}(X)_{\mathbb{R}(2)}$ is also zero for obvious dimensional reasons, so we have:

$$\cdots \longrightarrow H^3_{\mathcal{D}}(X,\mathbb{R}(2)) \longrightarrow H^4_{\mathcal{D}_{h.h.}}(X,2) \longrightarrow 0\,.$$

It follows that the class $[L,\rho]\cup[L',\rho']$ must come from an element in $H^3_{\mathcal{D}}(X,\mathbb{R}(2))$. As already remarked, on a curve we have $\widetilde{\mathbb{R}(2)}^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \underline{\mathcal{E}}^\bullet_X(1)[-1]$, thus $H^3_{\mathcal{D}}(X,\mathbb{R}(2)) \cong \mathbf{H}^3(X,\underline{\mathcal{E}}^\bullet_X(1)[-1]) \cong \mathbf{H}^2(X,\underline{\mathcal{E}}^\bullet_X(1)) \cong H^2(X,\mathbb{R}(1))$, in agreement with the calculations performed in sect. 4.

More in detail, in complex dimension 1 the second hermitian holomorphic Deligne complex $D(1)^\bullet_{h.h.}$ simplifies considerably and diagram (3.18) reduces to

(6.1)
$$\begin{array}{ccccc}
\mathbb{Z}(2) & \xrightarrow{-\imath} & \mathcal{O}_X & \xrightarrow{-d} & \underline{\Omega}^1_X \\
 & & \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_1} \\
 & & \underline{\mathcal{E}}^0_X(1) & \xrightarrow{-d} & \underline{\mathcal{E}}^1_X(1) & \xrightarrow{-d} & \underline{\mathcal{E}}^2_X(1)
\end{array}$$

so that $D(2)^\bullet_{h.h.}$ becomes just the cone of the morphism $\mathbb{Z}(2)^\bullet_{\mathcal{D}} \xrightarrow{\pi_1} \underline{\mathcal{E}}^\bullet_X(1)[-1]$. In other words, on a curve X we have that $D(2)^\bullet_{h.h.}$ is given by the complex

(6.2)
$$\mathbb{Z}(2)_X \xrightarrow{-\imath} \mathcal{O}_X \xrightarrow{(-d,-\pi_1)} \underline{\Omega}^1_X \oplus \underline{\mathcal{E}}^0_X(1) \xrightarrow{-\pi_1+d} \underline{\mathcal{E}}^1_X(1) \xrightarrow{d} \underline{\mathcal{E}}^2_X(1)\,,$$

where the differentials have been written explicitly. We can see the complex $\underline{\mathcal{E}}^\bullet_X$ appears as a subcomplex in (6.2) and the shift of two positions to the right clearly accounts for the cohomology degree shift from 2 to 4.

Our main result is the following comparison

Theorem 6.1. *Let X be a compact Riemann surface of genus $g>1$. Let (L,ρ) and (L',ρ') be two hermitian holomorphic line bundles. The class of $[L,\rho]\cup[L',\rho']$ in $H^4_{\mathcal{D}_{h.h.}}(X,2) \cong H^2(X,\mathbb{R}(1))$ coincides with the one represented by the cocycle $\Omega[\log\rho,\log\rho']$ constructed in section 4.*

Proof. We have observed above that $H^4_{\mathcal{D}_{h.h.}}(X,2) \cong H^2(X,\mathbb{R}(1))$, and by construction the class of $\Omega[\log\rho,\log\rho']$ is in $\mathbf{H}^2(X,\underline{\mathcal{E}}^\bullet_X(1)) \cong H^2(X,\mathbb{R}(1))$. Note that for X connected they must coincide up to a proportionality factor, since $H^2(X,\mathbb{R}(1)) \cong \mathbb{R}(1)$ in this case. In general, we compute the proportionality factor from the explicit cocycles from sects. 3.3 and 4.2 to 4.4. (Since $\Omega[\log\rho, \log\rho']$ is computed under suitable acyclicity assumptions on the cover, so we will use such a cover to establish the comparison.)

Let us assume L and L' and their respective hermitian metric structures are represented by cocycles of type (3.17) with respect to the chosen cover $\mathfrak{U}_X$. Specializing the general expression in table 1 in sect. 3.3 to the case at hand we obtain, with reference to (6.2):

(6.3)

$(0,4)$	$c_{ijk}c'_{klm}$
$(1,3)$	$-c_{ijk}\log g'_{kl}$
$(2,2)$	$-\log g_{ij}\, d\log g'_{jk} \oplus -\pi_1 \log g_{ij}\, \log\lvert g'_{jk}\rvert$
$(3,1)$	$(1-\alpha)\big(d^c \log\lvert g_{ij}\rvert\, \frac{1}{2}\log\rho'_j - \log\lvert g_{ij}\rvert\, \frac{1}{2} d^c \log\rho'_j\big) + \alpha\big(-\frac{1}{2}d^c\log\rho_i\, \log\lvert g'_{ij}\rvert + \frac{1}{2}\log\rho_i\, d^c\log\lvert g'_{ij}\rvert\big)$
$(4,0)$	$\alpha\, \bar\partial\partial\log\rho_i\, \frac{1}{2}\log\rho'_i + (1-\alpha)\,\frac{1}{2}\log\rho_i\, \bar\partial\partial\log\rho'_i$

where we have followed the convention explained in the introduction for the bidegrees. Let us denote by θ^i the term of bidegree $(4-i,i)$ in (6.3) and by Θ the total cocycle. (For simplicity, we suppress ρ and ρ' from the notation.) A direct calculation shows that

$$\theta^0_i = \omega^0_i + d\lambda^0_i\,, \qquad \theta^1_{ij} = \omega^1_{ij} - \check\delta\lambda^0_{ij}\,,$$

where ω^0_i is given by eq. (4.15), ω^1 is given by eq. (4.16), and

$$\lambda^0_i = \alpha\,\frac{1}{2}d^c\log\rho_i\, \log\rho'_i + (1-\alpha)\,\frac{1}{2}\log\rho_i\, d^c\log\rho'_i\,.$$

Note that at this point we could simply *define* $\Omega = \Theta - D\lambda^0$. Furthermore, note that Ω does not explicitly depend on the parameter α from the Beĭlinson product (2.5). To finish the comparison, let us assume the cover $\mathfrak{U}_X$ allows us to find a collection $L_{ijk} \in \mathcal{O}_X(U_{ijk})$ such that

$$\begin{aligned} dL_{ijk} &= -\log g_{ij}\; d\log g'_{jk} \\ \check\delta L_{ijkl} &= -c_{ijk}\log g'_{kl} + n_{ijkl}\,, \end{aligned} \tag{6.4}$$

as in sect. 4.3. In this way we have $\Omega = (\omega^0,\omega^1,\omega^2)$ with $\omega^2_{ijk} = -\mathscr{L}_2(g_{ij},g'_{jk})$. Ω is a cocycle of total degree 2 in $\operatorname{Tot}\check C^\bullet(\mathfrak{U}_X, \underline{\mathcal{E}}^\bullet_X)$, and it injects (via the exact sequence of the cone) into $\operatorname{Tot}\check C^\bullet(\mathfrak{U}_X, D(2)^\bullet_{h.h.})$ as

$$(\omega^0,\omega^1, 0\oplus\omega^2)\,.$$

Then via equations (6.4) it is easily seen that

$$\theta^2_{ijk} = d_{D(2)^\bullet_{h.h.}}(-L_{ijk}) + 0\oplus\omega^2_{ijk}\,,$$

where $d_{D(2)^\bullet_{h.h.}}$ is the differential in $D(2)^\bullet_{h.h.}$, and therefore

$$\Theta = (\omega^0,\omega^1,0\oplus\omega^2) + D\lambda^0 + D(-L,n)\,,$$

where we have put $D = d_{D(2)^\bullet_{h.h.}} \pm \check\delta$ for the total differential. Thus the two cocycles constructed via the direct method of sect. 4.2 and the cup product of metrized bundles define the same class. By direct comparison, the proportionality factor is 1. □

In light of the previous theorem, the polarization identity in Remark 4.3 is now easily explained. Using $[L\otimes L',\rho\rho'] = [L,\rho] + [L',\rho']$ and $[L\otimes L'^\vee, \rho/\rho'] = [L,\rho] - [L',\rho']$, and the (graded) commutativity of the cup product

$$H^2_{\mathcal{D}_{h.h.}}(X,1)\otimes H^2_{\mathcal{D}_{h.h.}}(X,1) \xrightarrow{\cup} H^4_{\mathcal{D}_{h.h.}}(X,2)\,,$$

we obtain the polarization identity

$$4[L,\rho]\cup[L',\rho'] = [L\otimes L',\rho\rho']^2 - [L\otimes L'^\vee,\rho/\rho']^2\,,$$

where the squares in the right hand side refer to $\cup$. A polarization identity at the level of representative cocycles, and hence the one in Remark 4.3, follow by applying Thm. 6.1 to the latter identity.

By choosing $L = L' = T_X$, the holomorphic tangent bundle of X, we immediately obtain:

Corollary 6.2. *Let $\rho \in \mathscr{CM}(X)$ be a conformal metric. The Liouville functional without area term* (4.6) *is given by the (evaluation of) the square $[T_X, \rho] \cup [T_X, \rho]$ with respect to the cup product in hermitian holomorphic Deligne cohomology. The full-fledged Liouville functional is obtained by adding the area term $\frac{1}{2\pi}\int_X \omega_\rho$ to* (4.6).

Remark 6.3. Due to the specific form of the differential in the complex $D(2)^\bullet_{h.h.}$ the descent equations are explicit and close automatically. Therefore the cocycle resulting from the calculation of the cup product sidesteps the problem of the explicit calculation of the last term, unlike the more direct version from sect. 4. Thus, thanks to the explicit character of the calculation, specific assumptions on the nature of the cover $\mathfrak{U}_X$ are not required.

It follows from Thm 6.1, corollary 6.2 and the previous remark that definition 4.1 applies to any (étale) cover $\mathfrak{U}_X \to X$. Indeed, proposition 4.4 from sect. 4.2 can be reformulated at the cocycle level as follows:

Proposition 6.4. *Let X be a compact, genus $g > 1$ Riemann surface and let $\mathfrak{U}_X \to X$ be a cover. For a conformal metric $\rho \in \mathscr{CM}(X)$ and $\sigma \in C^\infty(X, \mathbb{R})$, there is a cocycle $\hat{\Omega}_{\mathfrak{U}_X}[\log \rho]$ solving the variational problem for the Liouville equation.*

Proof. If ρ is a conformal metric, let the pair (T_X, ρ) be represented, as an hermitian line bundle, by a cocycle $c(T_X, \rho)$ with respect to the cover $\mathfrak{U}_X$. We set

$$\Omega[\log \rho] = c(T_X, \rho) \cup c(T_X, \rho)\,,$$

and a simple calculation starting from eq. (6.3) yields

$$\Omega[\log \rho + \sigma] - \Omega[\log \rho] = \sigma\, c_1(\rho) + \frac{1}{2}\,\sigma\, \partial\bar{\partial}\sigma + D\chi\,,$$

where $\chi_i = \frac{1}{2}\sigma\, \frac{1}{2} d^c \log \rho_i - \frac{1}{2} d^c\sigma\, \frac{1}{2}\log\rho_i$. Now define

$$\hat{\Omega}_{\mathfrak{U}_X}[\log \rho] = \Omega[\log \rho] - \sqrt{-1}\omega_\rho\,.$$

We see that it yields the formula in Proposition 4.4. In particular we have that

$$\left.\frac{d}{dt}\right|_{t=0} \hat{\Omega}_{\mathfrak{U}_X}[\log \rho + t\sigma] \equiv \sigma(c_1(\rho) - \sqrt{-1}\omega_\rho)\,,$$

where $\equiv$ means "up to total coboundary." □

6.2 Determinant of cohomology

Let again L and L' be two holomorphic line bundles with hermitian metrics ρ and ρ', respectively, on the compact Riemann surface X. Brylinski proves in [11] that the cup product of L and L' in hermitian holomorphic Deligne cohomology yields the (logarithm) of the metric $\|\cdot\|$ on the Deligne pairing $\langle L\,,\, L'\rangle$ defined in [12].

It follows, via Thm 6.1 and the isomorphism 3.2 between our version of hermitian holomorphic cohomology and Brylinski's, that the class of $\Omega[\log\rho, \log\rho']$ is also equal to $\log\|\langle L\,,\, L'\rangle\|$. It is worthwhile to provide a direct proof of this fact starting from the explicit cocycle given in (6.3).

First, we need to recall a few definitions from [12]. A complex line $\langle L\,,\, L'\rangle$ is assigned to the pair (L, L') as follows. Let D and D' be divisors on X corresponding to L and L', and assume they have disjoint supports. Consider two rational sections, s and s' such that $(s) = D$ and $(s') = D'$. To this datum one assigns a copy of the complex line generated by the symbol $\langle s\,,\, s'\rangle$ subject to the relations:

$$\begin{aligned}\langle fs\,,\, s'\rangle &= f(D')\langle s\,,\, s'\rangle \\ \langle s\,,\, gs'\rangle &= g(D)\langle s\,,\, s'\rangle\end{aligned} \tag{6.5}$$

whenever f is a rational function with divisor (f) disjoint from D', and similarly for g. The Weil reciprocity relation $f(div(g)) = g(div(f))$ (cf. ref. [22]) for two rational functions f and g with disjoint divisors implies that the relations (6.5) are consistent and the complex line depends only on the pair L, L'. When the line bundles are equipped with hermitian metrics, generically denoted by $\|\cdot\|$, the assignment[4]

$$\log\|\langle s\,,\, s'\rangle\|^2 = \frac{1}{2\pi\sqrt{-1}}\int_X \partial\bar{\partial}\log\|s\|^2\, \log\|s'\|^2 + \log\|s\|^2[D'] + \log\|s'\|^2[D] \tag{6.6}$$

is compatible with the relations (6.5) and defines an hermitian metric on the complex line $\langle L\,,\, L'\rangle$. In formula (6.6) the operator $\partial\bar{\partial}$ is to be computed in the sense of distributions.

Having covered the main definitions, we can now state

[4] We write the square explicitly, whereas the symbol $\|\cdot\|$ used in ref. [12] denotes the *square* of the norm.

Theorem 6.5. *The cup product of (L, ρ) with (L', ρ') in hermitian holomorphic Deligne cohomology corresponds to the logarithm of the norm* (6.6) *on the Deligne pairing $\langle L\,,\, L'\rangle$. The proportionality factor is $-\pi\sqrt{-1}$.*

Proof. Let D and D' be divisors with disjoint support on X corresponding to L and L', respectively. Using the same technique as in refs. [11, 12], consider two C^∞ positive real functions f_1 and f_2 such that $f_1 + f_2 = 1$ and f_1 (resp. f_2) vanishes in a neighborhood of the support of D' (resp. D). Also, set $U_1 = X \setminus \mathrm{supp}(D')$ and $U_2 = X \setminus \mathrm{supp}(D)$. Thus $\{f_1, f_2\}$ is just a partition of unity subordinated to the cover $\mathfrak{U}_X = \{U_1, U_2\}$.

The only two terms different from zero in the cocycle Θ in (6.3) representing the class $[L, \rho] \cup [L', \rho']$ with respect to this cover are θ^0_i and θ^1_i, with the Čech index $i \in \{1, 2\}$. Thus the class we are after is equivalently given by the integral

$$\int_X f_1\theta^0_1 + f_2\theta^1_2 + df_2 \wedge \theta^1_{21}\,, \tag{6.7}$$

which is arrived at by applying in the standard homotopy operator based on the partition of unity $\{f_1, f_2\}$: from $\check{\delta}\theta^1_{12} = 0$ we obtain that θ^1_{12} is the coboundary of the cochain $j \to \sum_{i=1,2} f_i\theta^1_{ij}$ and then we use $\theta^0_2 - \theta^0_2 = -d\theta^1_{12}$. Observe that the 2-form in (6.7) is globally well defined over X: θ^1_{21} is defined only on $U_1 \cap U_2$, but df_2 has support on $U_1 \cap U_2$, so their wedge product is defined everywhere; similarly, $f_i\,\theta^0_i$ is everywhere defined thanks to the fact that f_i has support in U_i, $i = 1, 2$.

Consider rational sections s and s' of L and L' such that $div(s) = D$ and $div(s') = D'$ as above. With respect to the two-element cover $\mathfrak{U}_X = \{U_1, U_2\}$, the section s corresponds to the pair $\{s_1, s_2\}$, and similarly for $s' = \{s'_1, s'_2\}$. Since $\mathrm{supp}(D)$ is contained in U_1 but not in U_2, and the other way around for D', it follows that s_2 and s'_1 are actually invertible functions over their respective domains. Following [25], we can assume that s and s' are in fact the rational section $\mathbf{1}$, so that $s_2 = 1$ and $s'_1 = 1$, and therefore:

$$|s|^2_\rho = \begin{cases} \log\rho_1 + \log|s_1|^2 & \text{on } U_1\,, \\ \log\rho_2 & \text{on } U_2\,, \end{cases}$$

and

$$|s'|^2_{\rho'} = \begin{cases} \log\,\rho'_1 & \text{on } U_1\,, \\ \log\rho'_2 + \log|s'_2|^2 & \text{on } U_2\,. \end{cases}$$

Furthermore, $\log|g'_{21}| = \log|s'_2|$ on $U_1 \cap U_2$. Let us denote by $\|\cdot\| = |\cdot|_\rho = |\cdot|_{\rho'}$ for simplicity. Using the relevant entries from (6.3) we have:

$$\theta^0_1 = c_1(\rho)\,\log\|s'\|\,,$$
$$\theta^0_2 = \bar\partial\partial\log\rho_2\,\frac{1}{2}\log\rho'_2 = dd^c\,\log\|s\|\,\frac{1}{2}\log\rho'_2\,.$$

An elementary integration by parts leads to:

$$\int_X f_2\theta^0_2 = \int_X f_2\log\|s\|\,c_1(\rho') + \int_X \log\|s\|\,df_2 \wedge \frac{1}{2}d^c\log\rho'_2 - \int_X \frac{1}{2}\log\rho'_2\,df_2 \wedge d^c\log\|s\|\,.$$

On the other hand, we have

$$\int_X df_2 \wedge \theta^1_{12} = \int_X \log\|s\|\,df_2 \wedge d^c\log|s'_2| - \int_X \log|s'_2|\,df_2 \wedge d^c\log\|s\|\,,$$

and putting all terms together we obtain

$$\int_X f_1\theta^0_1 + f_2\theta^1_2 + df_2 \wedge \theta^1_{21} = \int_X f_1\,c_1(\rho)\,\log\|s'\| + \int_X f_2\log\|s\|\,c_1(\rho') + \int_X \log\|s\|\,df_2 \wedge d^c\log\|s'\| + \int_X \log\|s'\|\,df_1 \wedge d^c\log\|s\| \tag{6.8}$$

which, if expressed in terms of the squares of the norms, is (up to a factor) the logarithm of $\|\langle s\,,\,s'\rangle\|$, as it is found in [12, formula 6.5.1]. This version is due to O. Gabber. Via the Poincaré-Lelong lemma (see, e.g. [22])

$$\bar{\partial}\partial \log\|s\| = c_1(\rho) + 2\pi\sqrt{-1}\,[D]\,,$$

where $[D]$ is the delta-current supported at the divisor of s, and similarly for s', formula (6.8) can be recast into:

$$\begin{aligned}\int_X f_1\theta_1^0 + f_2\theta_2^1 + df_2\wedge\theta_{21}^1 &= \int_X dd^c\log\|s\|\ \log\|s'\| - 2\pi\sqrt{-1}\ \log\|s\|[D'] - 2\pi\sqrt{-1}\ \log\|s'\|[D] \\ &= -2\pi\sqrt{-1}\ \log\|\langle s\,,\,s'\rangle\|\end{aligned} \tag{6.9}$$

which is what we wanted to show. □

This allows us to recast the Liouville functional for conformal metrics on X in the following form.

Corollary 6.6. *The exponential of the Liouville functional defines an hermitian metric on the complex line* $\langle T_X\,,\,T_X\rangle$, *namely for a conformal metric* $\rho\in\mathscr{CM}(X)$ *we have:*

$$\exp S[\log\rho] = \|\langle T_X\,,\,T_X\rangle\|\ \exp\tfrac{1}{2\pi}A_X(\rho)\,.$$

Remark 6.7. The above corollary justifies the choice made in Definition 4.1 for the various factors $2\pi\sqrt{-1}$.

A similar result has been obtained in ref. [33] by considering the Liouville action functional defined on the Schottky space, and in fact the results in loc. cit. are formulated in terms of a Schottky *family*.

Indeed, the statement in Corollary 6.6 can be immediately reformulated for a family $\pi\colon X\to S$ with base parameter space S by considering the relative holomorphic tangent line bundle $T_{X/S}$ with an hermitian fiber metric ρ. (Thus $\rho_s\in\mathscr{CM}(X_s)$ for every fiber $X_s\,,\,s\in S$.) Notice that the fiber metric ρ needs not be critical (i.e. satisfying the fiberwise constant negative curvature condition).

A Cones

In the main body of the paper we have used iterated cones to define the hermitian holomorphic Deligne complexes. One technical problem one has to face concerns the homotopy (graded) commutativity of the modified Beĭlinson product defined in eq. (2.5). A problem arises because the factors in the cones are cones themselves and therefore they have multiplication structures which are graded commutative up to homotopy to begin with. We want to show that even in this situation the final resulting product on cones is again homotopy graded commutative. This ensures that on cohomology the product will be genuinely graded commutative, so that in particular hermitian holomorphic Deligne cohomology as defined in section 3 has the correct product structure.

A.1 Cones and homotopies

We consider the following situation. For $i=1,2,3$ we have maps of complexes: $f_i\colon A_i^\bullet\to B_i^\bullet$, and for $i<j$ maps $a_{ji}\colon A_i^\bullet\to A_j^\bullet$ and $b_{ji}\colon B_i^\bullet\to B_j^\bullet$. Also, let $C^\bullet(f_i)=\operatorname{Cone}(f_i : A_i^\bullet\to B_i^\bullet)$, for $i=1,2,3$.

First, consider the homotopy commutative diagram:

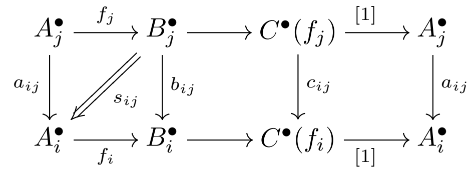
$$\begin{array}{ccc} A_j^\bullet & \xrightarrow{\ f_j\ } & B_j^\bullet \\ {\scriptstyle a_{ij}}\downarrow & \swarrow\!\!\swarrow{\scriptstyle s_{ij}} & \downarrow{\scriptstyle b_{ij}} \\ A_i^\bullet & \xrightarrow[\ f_i\ ]{} & B_i^\bullet \end{array} \tag{A.1}$$

where $s_{ij}\colon A_j^\bullet\to B_i^{\bullet-1}$ is the homotopy map of complexes:

$$f_i a_{ij} - b_{ij} f_j = d\,s_{ij} + s_{ij} d\,.$$

An immediate verification yields

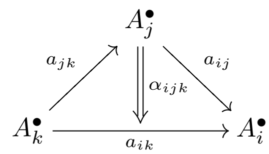
Lemma A.1. *The diagram* (A.1) *can be extended to*



where the map c_{ij} *is given by* $\begin{pmatrix} a_{ij} & 0 \\ -s_{ij} & b_{ij} \end{pmatrix}$ *and the squares containing the cones are in fact strictly commutative.*

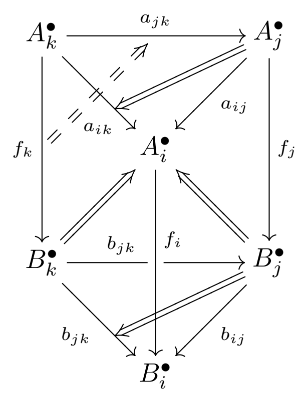
Remark A.2. This lemma is nothing other than the statement that any homotopy commutative diagram of the form (A.1) in the category of complexes in an abelian category can be extended to a (homotopy) commutative diagram of distinguished triangles, that is, one of the axioms defining a triangulated category, see, e.g. [32].

For $k < j < i$ consider the homotopy commutative triangle



where $a_{ik} - a_{ij}a_{jk} = d\alpha_{ijk} + \alpha_{ijk}d$, and similarly for the complexes $B_i^\bullet$, with a corresponding homotopy β_{ijk}. Thus $\alpha_{ijk}\colon A_k^\bullet \to A_i^{\bullet-1}$ and $\beta_{ijk}\colon B_k^\bullet \to B_i^{\bullet-1}$. Now consider the diagram:

(A.2)



The faces in (A.2) are homotopy commutative, however we assume that composing the *faces* is strictly commutative, namely the two possible homotopies $b_{ij}\, b_{jk}\, f_k \Longrightarrow f_i\, a_{ik}$ must be equal. Concretely, this corresponds to the relation

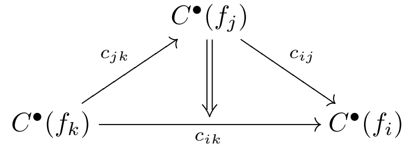
$$s_{ik} + \beta_{ijk}\, f_k = f_i\, \alpha_{ijk} + s_{ij}\, a_{jk} + b_{ij}\, s_{jk}\,. \tag{A.3}$$

We have:

Lemma A.3. *The map*

$$\begin{pmatrix} -\alpha_{ijk} & 0 \\ 0 & \beta_{ijk} \end{pmatrix} \colon C^\bullet(f_k) \longrightarrow C^{\bullet-1}(f_i)$$

realizes the homotopy



Proof. It is an elementary calculation based on writing $c_{ik} - c_{ik}\, c_{jk}$ explicitly via the matrix representation given in Lemma A.1 and using eq. (A.3). ☐

A.2 Applications

Consider the same setup as in section 2.1, with the same complexes $X_i^\bullet$, etc., and diagrams:

$$\mathcal{D}_i \stackrel{\mathrm{def}}{=} X_i^\bullet \xrightarrow{f_i} Z_i^\bullet \xleftarrow{g_i} Y_i^\bullet$$

from which we construct the cones

$$C(\mathcal{D}_i) = \mathrm{Cone}(X_i^\bullet \oplus Y_i^\bullet \xrightarrow{f_i - g_i} Z_i^\bullet)[-1]\,, \quad i = 1, 2, 3\,.$$

Moreover, following ref. [5], define $\mathcal{D}_i \otimes \mathcal{D}_j$ by taking the tensor product component-wise. Thus

$$\mathcal{D}_1 \otimes \mathcal{D}_2 = X_1^\bullet \otimes X_2^\bullet \xrightarrow{f_1 \otimes f_2} Z_1^\bullet \otimes Z_2^\bullet \xleftarrow{g_1 \times g_2} Y_1^\bullet \otimes Y_2^\bullet\,.$$

Assuming as in section 2.1 that the product maps are compatible with the f_i, etc., the diagram

$$\mathcal{D}_1 \otimes \mathcal{D}_2 \to \mathcal{D}_3$$

is of the same type as (A.1), and therefore lemma A.1 implies lemma 2.1.

Now, let the multiplication maps $X_1^\bullet \otimes X_2^\bullet \to X_3^\bullet$ be graded commutative up to homotopy and similarly for the $Y_i^\bullet$ and the $Z_i^\bullet$. We are interested in the commutativity properties of multiplication map given by the Beĭlinson product (2.5).

Proposition A.4. *The multiplication map* $\cup_\alpha \colon C(\mathcal{D}_1) \otimes C(\mathcal{D}_2) \longrightarrow C(\mathcal{D}_3)$ *given by* (2.5) *is homotopy graded commutative.*

Proof. The permutation operation on tensor products induces the diagram

$$\begin{array}{ccc} \mathcal{D}_1 \otimes \mathcal{D}_2 & \longrightarrow & \mathcal{D}_2 \otimes \mathcal{D}_1 \\ & \searrow \quad \swarrow & \\ & \mathcal{D}_3 & \end{array}$$

which is of type (A.2) and we can apply lemma A.3. ☐

It follows from the proposition that the cohomology inherits a well defined graded commutative product. This in particular applies to the definition of hermitian holomorphic Deligne cohomology that uses the cone (3.1). Therefore we conclude that the cup product (3.6) is graded commutative, as wanted.

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