Hermitian-holomorphic Deligne cohomology, Deligne pairing for singular metrics, and hyperbolic metrics

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Dedicated to µ-T

Abstract

We introduce a model for Hermitian holormorphic Deligne cohomology on a projective algebraic manifold which allows to incorporate singular hermitian structures along a normal crossing divisor. In the case of a projective curve, the cup-product in cohomology is shown to correspond to a generalization of the Deligne pairing to line bundles with "good" hermitian metrics in the sense of Mumford and others. A particular case is that of the tangent bundle of the curve twisted by the negative of the singularity divisor of a hyperbolic metric: its cup square (corrected by the total area) is shown to be a functional whose extrema are the metrics of constant negative curvature.

Contents

1 Introduction

1.1 Preliminaries and statement of the results

Let X be a smooth projective curve over $\mathbf{C}$ (a Riemann surface) and let $\mathscr{L}$ and $\mathscr{M}$ be two line bundles (= invertible sheaves) over X. In [10] Deligne defines a pairing $\langle\mathscr{L},\mathscr{M}\rangle$ as a complex line (i.e. a one-dimensional vector space over $\mathbf{C}$). Moreover, if both line bundles carry smooth hermitian fiber metrics $\|\cdot\|$, then the complex line carries a hermitian scalar product, whose expression was also defined in [10] in terms of two conveniently chosen rational sections l and m. (See section 5.1 for the relevant formulas.)

A remarkable fact is that the metrized line $(\langle\mathscr{L},\mathscr{M}\rangle,\|\cdot\|)$ can be obtained by entirely cohomological means, as a cup product in Hermitian holomorphic Deligne cohomology between the classes of the metrized line bundles $\bar{\mathscr{L}}=(\mathscr{L},\|\cdot\|)$ and $\bar{\mathscr{M}}=(\mathscr{M},\|\cdot\|)$, [2, 7].

For any proper algebraic manifold X, the Hermitian holomorphic Deligne cohomology groups of weight p were originally introduced by Brylinski in [7] as the hypercohomology groups, denoted by $\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X;p)$, of certain complexes $C(p)^\bullet_X$, whose definition is reminiscent of the corresponding one for the standard Deligne complexes. In particular, if $\widehat{\mathrm{Pic}}(X)$ denotes the group of isomorphism classes of line bundles on X equipped with a smooth hermitian metric, it is shown that

$$\widehat{\mathrm{Pic}}(X)\cong\widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1)\,.$$

The groups $\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X;p)$ have cup products behaving in the standard way:

$$\widehat{\mathrm{H}}^k_{\mathcal{D}}(X;p)\otimes\widehat{\mathrm{H}}^l_{\mathcal{D}}(X;q)\xrightarrow{\cup}\widehat{\mathrm{H}}^{k+l}_{\mathcal{D}}(X;p+q)\,.$$

Thus, given the two metrized line bundles as above we can multiply their classes in $\widehat{\mathrm{Pic}}(X)$ using this product to obtain a class $[\bar{\mathscr{L}}]\cup[\bar{\mathscr{M}}]\in\widehat{\mathrm{H}}^4_{\mathcal{D}}(X;2)$.

If X is a curve, the result follows since the latter group is isomorphic to $\mathrm{H}^2(X,\mathbf{R})$. (See ref. [2] and sect. 4, below. We have neglected the Tate twists by $2\pi\sqrt{-1}$, here.) The number so obtained is the length of a generator of $\langle\mathscr{L},\mathscr{M}\rangle$ associated to a specific choice of l and m, see sect. 5.1 below.

The case $\mathscr{L}=\mathscr{M}=T_X$ turns out to be quite interesting on its own and relevant to the hyperbolic geometry of X considered as a Riemann surface. A hermitian fiber metric is now just a conformal metric on X. If the genus is ≥ 2, we have shown in [2, Thm. 5.1] that the cup square $[T_X]\cup[T_X]$, supplemented by the area form, determines a functional whose extremum is precisely the hyperbolic metric of constant curvature equal to -1. In fact, we have shown it coincides with the Liouville functional studied in refs. [18, 19, 16], in relation to Weil-Petersson geometry.

Returning to general case of the pairing $\langle\mathscr{L},\mathscr{M}\rangle$ where $\mathscr{L}$ and $\mathscr{M}$ are not necessary equal, it has been observed—also by Deligne, [10]—that the metric on the pairing could be defined also when the fiber metrics on both $\mathscr{L}$ and $\mathscr{M}$ are allowed to be singular, provided the corresponding loci are disjoint. In a different direction, if $\mathscr{X}/_{\mathbf{Z}}$ is an arithmetic surface such that X is the corresponding fiber at infinity, U. Kühn ([14]) has formulated a pairing at arithmetic infinity (generalizing Deligne's one) for line bundles where the metric has logarithmic singularities at a finite set of points. (This is shown in the same work to be compatible with an earlier version by Bost, cf. [6], based on Green's currents.)

This leads to the interesting question of whether pairings with singular metrics can be expressed in terms of a natural cup product, and if the results outlined above extend to singular metrics. We show this is indeed the case.

In this work we first present an extension of the framework of Hermitian holormorphic Deligne cohomology to pairs (X,D), where X is a complete complex algebraic manifold, and D is a normal crossing divisor. Then, by working on a regular projective curve over $\mathbf{C}$, we use the cup product in this cohomology to obtain the pairing for two hermitian line bundles with hermitian metrics singular along D.

In slightly more detail, we define a new model $\mathfrak{D}_{h.h.}(p)^\bullet_X$ for the Hermitian Deligne complex (namely, we replace the complex $C(p)^\bullet_X$ above by a quasi-isomorphic one) and obtain the corresponding cup product structure in Theorem 2.6. In fact, this is first defined for the case where D is empty, a case which is already of independent interest. In this case $\mathfrak{D}_{h.h.}(p)^\bullet_X$ is simply the cone of a morphism between two complexes,

$$\mathfrak{D}_{h.h.}(p)^\bullet_X=\mathrm{Cone}\big(\mathbf{Z}(p)^\bullet_{\mathcal{D}}\longrightarrow\sigma^{<2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X,p)\big)[-1]\,,$$

where $\mathbf{Z}(p)^\bullet_{\mathcal{D}}$ is the $\mathbf{Z}$-valued Deligne complex, and $\sigma^{<2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X,p)$ is a suitable truncation of a Deligne algebra based on the Dolbeault complex of X. When $D\neq\varnothing$, the Hermitian Deligne complex and the corresponding (relative) cohomologies are immediately defined for the pair $(X, U = X \setminus D)$ in terms of a sheaf $\mathfrak{D}_{h.h.}(p)^\bullet_{X,U}$ on a pair of spaces in the sense of ref. [4].

A further refinement is obtained by imposing growth conditions by considering appropriate subsheaves of $\jmath_*\sigma^{<2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_U, p)$. We are ultimately interested in good metrics in the sense of Mumford ([15]), whose characteristic forms have singularities akin to those of the standard Poincaré metric on a punctured disk. While we are able to directly use good forms in complex dimension 1, it is not possible to do so to set the homological algebra machinery in general. In order to proceed, we exploit the complex of *pre-log-log* forms recently introduced in ref. [9]. It is possible to consider slightly different kinds of growth conditions: in particular, the complex of *log-log* forms, also mentioned in loc. cit., seems to have several advantages, including a Poincaré lemma.[1] The issue is briefly touched upon in sect. 4.1, but we have not dwelt on it, since we were able to work with good forms to a greater extent than the considerably more difficult ref. [9].

Having set up the general framework of Hermitian holomorphic Deligne cohomology with growth condition in general in sect. 3, we state our main applications and results in the case of a complete regular complex curve X, where D reduces to a finite set of points. If now $\mathscr{L}$ and $\mathscr{M}$ are line bundles on X carrying a good metric along D, we first show in Theorem. 5.2 that the cup product in Hermitian Deligne cohomology extends the Deligne pairing to this case, thus generalizing the corresponding result obtained in ref. [2] for the empty divisor. As a consequence, we can obtain a generalization of the earlier result in ref.[2] concerning hyperbolic metrics. Namely, good metrics $\mathrm{d}s^2$ on T_U correspond to the extension $\mathscr{L} = T_X(-D)$. We obtain in Theorem 6.1 that the cup square $[\mathscr{L}] \cup [\mathscr{L}]$ supplemented by the area integral (which is well defined, since good forms are locally integrable) is still the generating functional for the uniformizing metric. We do not need the correction terms required in earlier definitions (cf. e.g. [18]).

1.2 Organization

Let us briefly illustrate the organization of this work. The definition of the Hermitian holomorphic Deligne complex $\mathfrak{D}_{h.h.}(p)^\bullet_X$ and the corresponding cohomology groups, as well as the calculation of the multiplicative structure, are carried out in section 2, where we state Theorem 2.6. We only sketch the proof, since many of the arguments are either already available in the literature or straightforward, but long to reproduce in detail.

Definitions are initially given in the proper case, without reference to the divisor D. This is remedied in section 3, where definitions are extended to pairs (X, D). We also recall the relevant notions necessary to consider specific growth conditions: Poincaré forms, good forms, good metrics, pre-log-log forms.

Section 4 collects some intermediate results to be used later. We specialize the framework developed in sections 2 and 3 to calculate the cup product $[\bar{\mathscr{L}}] \cup [\bar{\mathscr{M}}]$ of two good hermitian line bundles $\bar{\mathscr{L}} = (\mathscr{L}, \|\ \|)$ and $\bar{\mathscr{M}} = (\mathscr{M}, \|\ \|)$, both on a general (X, D) and then further specializing to the case $\dim X = 1$. We also collect some remarks about the integration map in dimension 1 which is also needed in the subsequent sections.

Sections 5 and 6 are devoted to our main results. In the former we extend the Deligne pairing to good hermitian line bundles, and in the latter we show that, after correcting with an area term, the pairing is the generating functional for the constant negative curvature hyperbolic metric. By then, most of the necessary preliminaries will have already been obtained, therefore the corresponding proofs have been kept to a minimum.

1.3 Conventions and notations

For a subring A of $\mathbf{R}$ and an integer p, $A(p) = (2\pi\sqrt{-1})^p A$ is the Tate twist of A. We identify $\mathbf{C}/\mathbf{Z}(p) \cong \mathbf{C}^\times$ via the exponential map $z \mapsto \exp(z/(2\pi\sqrt{-1})^{p-1})$, and $\mathbf{C} \cong \mathbf{R}(p) \oplus \mathbf{R}(p-1)$, so $\mathbf{C}/\mathbf{R}(p) \cong \mathbf{R}(p-1)$.

The projection $\pi_p \colon \mathbf{C} \to \mathbf{R}(p)$ is given by $\pi_p(z) = \frac{1}{2}(z + (-1)^p \bar{z})$, for $z \in \mathbf{C}$, and similarly for any other complex quantity, e.g. complex valued differential forms.

If X is a complex manifold, $\Omega^\bullet_X$ denotes the de Rham complex of holomorphic forms, where we set $\mathscr{O}_X \equiv \Omega^0_X$ as usual. $\mathscr{E}^\bullet_X$ denotes the de Rham complex of sheaves of $\mathbf{R}$-valued smooth forms on the underlying smooth manifold. Furthermore, $\mathscr{A}^\bullet_X = \mathscr{E}^\bullet_X \otimes_{\mathbf{R}} \mathbf{C}$, and is $\mathscr{E}^\bullet_X(p)$ the twist $\mathscr{E}^\bullet_X \otimes_{\mathbf{R}} \mathbf{R}(p)$. Also, $\mathscr{A}^{p,q}_X$ will denote the sheaf of smooth (p,q)-forms, and $\mathscr{A}^n_X = \bigoplus_{p+q=n} \mathscr{A}^{p,q}_X$, where the differential decomposes in the standard fashion, $\mathrm{d} = \partial + \bar{\partial}$, according to types. We also introduce the imaginary operator $\mathrm{d}^c = \partial - \bar{\partial}$ (with a slight departure from convention, we omit the customary factor $1/(4\pi\sqrt{-1})$). We have the rules

$$\mathrm{d}\pi_p(\omega) = \pi_p(\mathrm{d}\omega)\,, \quad \mathrm{d}^c\pi_p(\omega) = \pi_{p+1}(\mathrm{d}^c\omega)$$

[1] A published definition is not available at the time of this writing. A paper by the same authors of ref. [9] is in preparation. We warmly thank Prof. Burgos for kindly informing us.

for any complex form ω. Note that we have $2\partial\bar\partial = \mathrm{d}^c\mathrm{d}$.

We denote the complexes of global sections arising from the above mentioned sheaf complexes by corresponding straight letters, as $A^\bullet(X)$, $A^{p,q}(X)$, $E^\bullet(X)$, $E^\bullet(X)(p) \overset{\text{def}}{=} E^\bullet(X)\otimes_{\mathbf{R}}\mathbf{R}(p)$, and so on.

An open cover of X will be denoted by $\mathfrak{U}_X$. If $\{U_i\}_{i\in I}$ is the corresponding collection of open sets, we write $U_{ij} = U_i\cap U_j$, $U_{ijk} = U_i\cap U_j\cap U_k$, and so on. More generally we can also have $\mathfrak{U}_X = \{U_i \to X\}_{i\in I}$, where the maps are regular coverings in an appropriate category. In this case intersections are replaced by $(n+1)$-fold fibered products $U_{i_0i_1\cdots i_n} = U_{i_0}\times_X\cdots\times_X U_{i_n}$.

If $\mathscr{F}^\bullet$ is a complex of abelian sheaves on X, with differential $d_{\mathscr{F}}$, its Čech resolution with respect to a covering $\mathfrak{U}_X \to X$ is the double complex

$$\mathsf{C}^{p,q}(\mathscr{F}) \overset{\text{def}}{=} \check{\mathrm{C}}^q(\mathfrak{U}_X,\mathscr{F}^p)\,,$$

where the q-cochains with values in $\mathscr{F}^p$ are given by $\prod\mathscr{F}^p(U_{i_0\cdots i_n})$. The Čech coboundary operator is denoted δ. The convention we use is to put the index along the Čech resolution in the *second* place, so the total differential is given by $D = d_{\mathscr{F}} + (-1)^p\delta$ on the component $\check{\mathrm{C}}^q(\mathfrak{U}_X,\mathscr{F}^p)$ of the total simple complex. Because of the Koszul sign rule we get the following convention for Čech resolutions of complexes of sheaves. If $\mathscr{G}^\bullet$ is a second complex of sheaves on X, then one defines the cup product

$$\cup : \mathsf{C}^{p,q}(\mathscr{F})\otimes\mathsf{C}^{r,s}(\mathscr{G}) \longrightarrow \check{\mathrm{C}}^{q+s}(\mathfrak{U}_X,\mathscr{F}^p\otimes\mathscr{G}^r) \subset \mathsf{C}^{p+r,q+s}(\mathscr{F}\otimes\mathscr{G})$$

of two elements $\{f_{i_0,\dots,i_q}\}\in\mathsf{C}^{p,q}(\mathscr{F})$ and $\{g_{j_0,\dots,j_s}\}\in\mathsf{C}^{r,s}(\mathscr{G})$ by

$$(-1)^{qr}\, f_{i_0,\dots,i_q}\otimes g_{i_q,i_{q+1},\dots,i_{q+s}}\,.$$

For a given complex of abelian objects, say $A^\bullet$, we have the sharp truncations $\sigma^p \equiv \sigma^{\geq p}$ and $\sigma^{<p}$, namely $\sigma^p A^n = 0$ for $n<p$, and $\sigma^{<p}A^n = 0$ if $n\geq p$. On occasion we will use the short-hand notations $A^{\bullet\geq p}$ and $A^{\bullet<p}$, respectively.

Finally, we denote a quasi-isomorphism with $\xrightarrow{\simeq}$, and a homotopy equivalence by $\simeq$.

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2 Models of Hermitian-holomorphic Deligne cohomology

2.1 Deligne cohomology

In the following, X will be a complete complex algebraic manifold, that is the analytic variety associated to a smooth proper scheme over $\mathbf{C}$. (Since in the following we will work in the analytic category there will be no need to distinguish notationally between X and X_{an}.)

First, recall that for a subring $A\subset\mathbf{R}$ and an integer p, the Deligne cohomology groups of weight p of X with values in A are the hypercohomology groups

$$\mathrm{H}^\bullet_{\mathcal{D}}(X,A(p)) \overset{\text{def}}{=} \mathbb{H}^\bullet(X,A(p)^\bullet_{\mathcal{D}})\,, \tag{2.1}$$

where $A(p)^\bullet_{\mathcal{D}}$ is the complex

$$A(p)^\bullet_{\mathcal{D}} = \mathrm{Cone}\big(A(p)_X\oplus F^p\Omega^\bullet_X \longrightarrow \Omega^\bullet_X\big)[-1] \tag{2.2}$$

(the map in the cone is the difference of the two inclusions) and $F^p\Omega^\bullet_X \equiv \sigma^p\Omega^\bullet_X$ is the Hodge ("bête") filtration. In the derived category the complex in (2.2) is isomorphic to:

$$0\longrightarrow A(p)_X \xrightarrow{\imath} \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{p-1}_X\,. \tag{2.3}$$

Deligne cohomology has a graded commutative cup product

$$\mathrm{H}^k_{\mathcal{D}}(X,A(p))\otimes\mathrm{H}^l_{\mathcal{D}}(X,A(q)) \xrightarrow{\cup} \mathrm{H}^{k+l}_{\mathcal{D}}(X,A(p+q)) \tag{2.4}$$

which is induced by a family $\alpha \rightsquigarrow \cup_\alpha$ of cup products at the level of complexes

$$A(p)^\bullet_{\mathcal{D}} \otimes A(q)^\bullet_{\mathcal{D}} \xrightarrow{\cup_\alpha} A(p+q)^\bullet_{\mathcal{D}}\,,$$

where $\alpha \in [0,1]$. This results from a general prescription to obtain cup products on cones of certain diagrams of complexes devised by Beĭlinson [5], and to be recalled below. In particular the product $\cup_0$ induces the following cup product on the (simpler) complex (2.3). If $a \in A(p)^\bullet_{\mathcal{D}}$ and $b \in A(q)^\bullet_{\mathcal{D}}$, then from ref. [4, 11] we have:

$$a \cup b = \begin{cases} a \cdot b & \deg a = 0\,, \\ a \wedge \mathrm{d}b & \deg a > 0 \text{ and } \deg b = q\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

2.2 Hermitian holomorphic variants

2.2.1

Let $\widehat{\mathrm{Pic}}(X)$ be the group of isomorphism classes of line bundles on X with smooth hermitian fiber metric, that is of pairs $(\mathscr{L},\rho)$, where $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module equipped with a map $\rho\colon \mathscr{L} \longrightarrow \mathscr{E}^+_X$, into (the sheaf of) positive real smooth functions. A basic observation by Deligne is that $\widehat{\mathrm{Pic}}(X)$ can be identified with the hypercohomology group:

$$\mathbb{H}^2\big(X, \mathbf{Z}(1)_X \xrightarrow{\imath} \mathscr{O}_X \xrightarrow{\pi_0} \mathscr{E}^0_X\big)\,, \tag{2.6}$$

which is easy to see in Čech cohomology. Indeed, for a cover $\mathfrak{U}_X$ of X, and a pair $(\mathscr{L},\rho)$, let s_i be a trivialization of $\mathscr{L}|_{U_i}$, with transition functions $g_{ij} \in \mathscr{O}^\times_X(U_{ij})$ determined by $s_j = s_i g_{ij}$. Let ρ_i be the value of the quadratic form associated to ρ on s_i, namely $\rho_i = \rho(s_i)$. Then we have $\rho_j = \rho_i \left|g_{ij}\right|^2$. Taking logarithms, we see that

$$\big(2\pi\sqrt{-1}c_{ijk}, \log g_{ij}, -\tfrac{1}{2}\log\rho_i\big)\,, \tag{2.7}$$

where $2\pi\sqrt{-1}c_{ijk} = \log g_{jk} - \log g_{ik} + \log g_{ij} \in \mathbf{Z}(1)$, is a cocycle representing the class of the pair $(\mathscr{L},\rho)$.

2.2.2

Hermitian-holomorphic Deligne cohomology groups are a generalization of the above fact to any weight and degree. Namely, consider the complexes

$$C(p)^\bullet = \mathrm{Cone}\big(\mathbf{Z}(p)_X \oplus (F^p\mathscr{A}^\bullet_X \cap \sigma^{2p}\mathscr{E}^\bullet_X(p)) \longrightarrow \mathscr{E}^\bullet_X(p)\big)[-1]\,, \tag{2.8}$$

where $F^p\mathscr{A}^\bullet_X = \bigoplus_{p' \geq p} \mathscr{A}^{p',\bullet-p'}_X$ is the Hodge filtration.

Definition 2.1. The *Hermitian-holomorphic Deligne* cohomology groups are the hypercohomology groups

$$\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X;p) \stackrel{\mathrm{def}}{=} \mathbb{H}^\bullet(X, C(p)^\bullet) \tag{2.9}$$

The complexes (2.8) were introduced in ref. [7] by Brylinski. It is easy to see that $C(1)^\bullet$ is quasi-isomorphic to the complex appearing in (2.6), so that $\widehat{\mathrm{Pic}}(X)$ is identified with $\widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1)$. More generally, the elements of the higher degree groups $\widehat{\mathrm{H}}^k_{\mathcal{D}}(X;p)$ for $k=3$ (resp. $k=4$) and $p=1,2$ correspond to classes of analytic (or algebraic) gerbes (resp. 2-gerbes) on X with an appropriate notion of hermitian and canonical connective structures (cf. ref. [1]).

It is advantageous to have alternative ways of calculating the groups $\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X;p)$ by means of complexes quasi-isomorphic to $C(p)^\bullet$. In ref. [2] we introduced the complexes

$$D_{h.h.}(p)^\bullet = \mathrm{Cone}\big(\mathbf{Z}(p)^\bullet_{\mathcal{D}} \oplus (F^p\mathscr{A}^\bullet_X \cap \sigma^{2p}\mathscr{E}^\bullet_X(p)) \longrightarrow \widetilde{\mathbf{R}(p)}^\bullet_{\mathcal{D}}\big)[-1]\,, \tag{2.10}$$

where the complex

$$\widetilde{\mathbf{R}(p)}^\bullet_{\mathcal{D}} = \mathrm{Cone}\left(F^p\mathscr{A}^\bullet_X \to \mathscr{E}^\bullet_X(p-1)\right)[-1]$$

also computes the real Deligne cohomology, cf. [4, 11]. The complex $D_{h.h.}(1)^\bullet$ provides a characterization of the *canonical connection* associated to the hermitian fiber metric structure. In fact the quasi-isomorphism between $D_{h.h.}(1)^\bullet$ and the complex in (2.6) allows to conclude that the canonical connection is unique. There are other advantages in using the complexes $D_{h.h.}(p)^\bullet$, discussed in detail in [2], related to the fact that analytic or algebraic structures can be described directly without resorting to the underlying smooth objects. This, however, happens at the cost of quite an amount of algebraic intricacy, especially in the product structure.

Both complexes (2.8) and (2.10) admit cup products according to Beĭlinson's general prescription, so that there results a corresponding graded commutative product in cohomology:

$$\widehat{\mathrm{H}}^k_{\mathcal{D}}(X;p) \otimes \widehat{\mathrm{H}}^l_{\mathcal{D}}(X;q) \xrightarrow{\cup} \widehat{\mathrm{H}}^{k+l}_{\mathcal{D}}(X;p+q)\,. \tag{2.11}$$

We refer to refs. [2, 1] for the details about the quasi-isomorphism $C(p)^\bullet \xrightarrow{\simeq} D_{h.h.}(p)^\bullet$ and more generally about the properties mentioned above.

2.2.3

We want to introduce a third—and simpler—model for hermitian holomorphic Deligne cohomology which is based on the following observation. Real Deligne cohomology can also be computed by the following complex considered in detail in refs. [8, 12] (its use had been previously suggested by Deligne):

$$\mathfrak{D}^n(\mathscr{A}^\bullet_X, p) = \begin{cases} \mathscr{E}^{n-1}_X(p-1) \bigcap \bigoplus\limits_{\substack{p'+q'=n-1\\ p'<p,q'<p}} \mathscr{A}^{p',q'}_X & \text{if } n \leq 2p-1 \\ \mathscr{E}^{n}_X(p) \bigcap \bigoplus\limits_{\substack{p'+q'=n\\ p'\geq p,q'\geq p}} \mathscr{A}^{p',q'}_X & \text{if } n \geq 2p \end{cases} \tag{2.12}$$

with differential

$$d_{\mathfrak{D}} = \begin{cases} -\pi \circ \mathrm{d} & n < 2p-1 \\ -2\bar\partial\partial & n = 2p-1 \\ \mathrm{d} & n = 2p\,, \end{cases} \tag{2.13}$$

where in the first line of (2.13) π is the projection.[2] The quasi-isomorphism is the composition

$$\mathbf{R}(p)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \mathrm{Cone}\big(\mathscr{E}^\bullet_X(p) \oplus F^p\mathscr{A}^\bullet_X \longrightarrow \mathscr{A}^\bullet_X\big)[-1] \xrightarrow{\simeq} \mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)\,.$$

The left map trivially follows from $\mathbf{R}(p) \xrightarrow{\simeq} \mathscr{E}^\bullet_X(p)$, (2.2), and the fact that the inclusion $F^p\Omega^\bullet_X \hookrightarrow F^p\mathscr{A}^\bullet_X$ is a filtered quasi-isomorphism. The right one is in fact a homotopy equivalence. It is computed in detail in ref. [8], where the homotopies are also explicitly computed.

Thus we can replace $\mathbf{R}(p)^\bullet_{\mathcal{D}}$ with $\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$ in the definition (2.10), and still obtain the same cohomology groups. From the obvious map $\mathbf{Z}(p)^\bullet_{\mathcal{D}} \to \mathbf{R}(p)^\bullet_{\mathcal{D}}$, and following the chain of quasi-isomorphisms, one has the map

$$\rho_p \colon \mathbf{Z}(p)^\bullet_{\mathcal{D}} \longrightarrow \mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$$

given by:

$$\rho^n_p = \begin{cases} 0 & \text{if } n = 0 \\ (-1)^n\pi_{p-1} & \text{if } 1 \leq n \leq p \\ 0 & \text{if } n \geq p, \end{cases} \tag{2.14}$$

where we use version (2.3) of the Deligne complex.

Note that the sharp truncation $\sigma^{<2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$ is just the first line in (2.12), which comprises forms of degree up to $2p-2$, whereas the model (2.3) of the Deligne complex is zero after degree $p-1$. Hence the map ρ_p factors through $\sigma^{<2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$, so we actually have a map

$$\rho_p \colon \mathbf{Z}(p)^\bullet_{\mathcal{D}} \longrightarrow \sigma^{<2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p) \tag{2.15}$$

which is obviously also given by (2.14).

[2] Some sign differences result from different conventions about cones. Given $f : A^\bullet \to B^\bullet$, ref. [8] uses $\mathrm{Cone}(-f)[-1]$, whereas we use $\mathrm{Cone}(f)[-1]$.

Consider the complex:

$$\mathfrak{D}_{h.h.}(p)^\bullet \stackrel{\text{def}}{=} \mathrm{Cone}\big(\mathbf{Z}(p)^\bullet_{\mathcal{D}} \xrightarrow{\rho_p} \sigma^{<2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)\big)[-1]\,. \tag{2.16}$$

We have

Lemma 2.2. *The complex $\mathfrak{D}_{h.h.}(p)^\bullet$ is quasi-isomorphic to $D_{h.h.}(p)^\bullet$.*

Proof. One need only observe that

$$\sigma^{2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p) \equiv \mathscr{E}^n_X(p) \bigcap \bigoplus_{\substack{p'+q'=n \\ p'\geq p, q'\geq p}} \mathscr{A}^{p',q'}_X = F^p\mathscr{A}^\bullet_X \cap \sigma^{2p}\mathscr{E}^\bullet_X(p)\,,$$

so that, writing:

$$D_{h.h.}(p)^\bullet = \mathrm{Cone}\big(\mathbf{Z}(p)^\bullet_{\mathcal{D}} \longrightarrow \mathrm{Cone}\big(F^p\mathscr{A}^\bullet_X \cap \sigma^{2p}\mathscr{E}^\bullet_X(p) \longrightarrow \widetilde{\mathbf{R}(p)^\bullet_{\mathcal{D}}}\big)\big)[-1]\,,$$

and replacing $\widetilde{\mathbf{R}(p)^\bullet_{\mathcal{D}}}$ with $\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$, the inner cone above is

$$\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)/(F^p\mathscr{A}^\bullet_X \cap \sigma^{2p}\mathscr{E}^\bullet_X(p)) \xrightarrow{\simeq} \sigma^{<2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)\,.$$

□

It follows from the lemma that $\widehat{\mathrm{H}}^k_{\mathcal{D}}(X;p) = \mathbb{H}^k(X, \mathfrak{D}_{h.h.}(p)^\bullet)$.

Remark 2.3. Since in the complex $\mathfrak{D}_{h.h.}(p)^\bullet$ we have the truncation of $\mathfrak{D}^\bullet(X,p)$ after degree $2p$, from the expression (2.13) of the differential $d_{\mathfrak{D}}$ we obtain a map

$$\mathfrak{D}_{h.h.}(p)^\bullet \xrightarrow{-2\bar{\partial}\partial} \mathscr{A}^{p,p}_X[-2p] \cap \mathscr{E}^{2p}_X(p)_{cl}\,. \tag{2.17}$$

This immediately gives a "characteristic class"

$$\widehat{\mathrm{H}}^{2p}_{\mathcal{D}}(X;p) \longrightarrow A^{p,p}(X)_{\mathbf{R}(p),cl}\,, \tag{2.18}$$

into $\mathbf{R}(p)$-valued *closed* forms.

Remark 2.4. The truncation occurring in the complex (2.16) is similar (but not equal) to the "truncated Deligne complex" $\mathfrak{D}^{\bullet\leq 2p}(\mathscr{A}^\bullet_X, p)$ considered by Goncharov for the Arakelov motivic complex in [12].

Also observe that the complex (2.16) is different from the complex used to define the "truncated relative homology" groups in refs. [8, 9], as in the latter the truncation occurs in the first complex in the cone.

Example 2.5. The weights $p = 1, 2$ will be of special interests later on.

The complex $\mathfrak{D}_{h.h.}(1)^\bullet$ coincides precisely with the complex appearing in (2.6):

$$\mathbf{Z}(1)_X \xrightarrow{\imath} \mathscr{O}_X \xrightarrow{\pi_0} \mathscr{E}^0_X\,.$$

The complex $\mathfrak{D}_{h.h.}(2)^\bullet$ is the cone (shifted by 1) of the map

$$\begin{tikzcd} \mathbf{Z}(2)_X \arrow[r, "\imath"] & \mathscr{O}_X \arrow[r, "\mathrm{d}"] \arrow[d, "-\pi_1"] & \Omega^1_X \arrow[d, "\pi_1"] & \\ & \mathscr{E}^0_X(1) \arrow[r, "-\mathrm{d}"] & \mathscr{E}^1_X(1) \arrow[r, "-\pi\circ\mathrm{d}"] & \mathscr{E}^2_X(1)\cap\mathscr{A}^{1,1}_X \end{tikzcd} \tag{2.19}$$

2.3 Multiplicative structure

2.3.1

The complex $\mathfrak{D}_{h.h.}(p)^\bullet$ has a relatively simple multiplicative structure. In fact there exists a one-parameter family of product structures which is inherited from that of $D_{h.h.}(p)^\bullet$, thanks to Beĭlinson's construction. We present here the (simpler) one that will be used in the sequel. We use the model (2.3) for the Deligne complex $\mathbf{Z}(p)^\bullet_{\mathcal{D}}$ with the product (2.5). Our notations are as follows: an element of degree n in $\mathfrak{D}_{h.h.}(p)^n$ is denoted by (x_p, w_p), where $x_p \in \mathbf{Z}(p)^n_{\mathcal{D}}$, and $w_p \in \sigma^{<2p}\mathfrak{D}^{n-1}(\mathscr{A}^\bullet_X, p)$. Also, $w^{r,s}_p$ denotes the (r,s)-component of w_p. Details of the proof of the following theorem will appear elsewhere.

Theorem 2.6. *There is a map of complexes*

$$\hat{\cup} : \mathfrak{D}_{h.h.}(p)^\bullet \otimes \mathfrak{D}_{h.h.}(q)^\bullet \longrightarrow \mathfrak{D}_{h.h.}(p+q)^\bullet$$

which is homotopy graded commutative. Let $(x_p, w_p) \in \mathfrak{D}_{h.h.}(p)^n$ and $(x_q, w_q) \in \mathfrak{D}_{h.h.}(q)^m$. Then:

$$(x_p, 0)\,\hat{\cup}\,(x_q, 0) = \begin{cases} (x_p \cup x_q, 0) & n = 0 \text{ or } m = 0 \\ (x_p \cup x_q, (-1)^m 2\pi_p(x_p) \wedge \pi_{q-1}(x_q)) & \text{if } n \text{ and } m \neq 0 \end{cases}$$

$$(x_p, 0)\,\hat{\cup}\,(0, w_q) = \begin{cases} 0 & n = 0 \\ (0, \pi_{p-1}(x_p) \wedge (\partial w_q^{q-1,m-q-1} - \bar{\partial} w_q^{m-q-1,q-1})) & n = 1, \dots, p-1 \\ (0, \pi_{p-1}(x_p) \wedge (\partial w_q^{q-1,m-q-1} - \bar{\partial} w_q^{m-q-1,q-1}) + (-1)^p \pi_p(\mathrm{d}x_p) \wedge w_q) & n = p \\ 0 & n \geq p+1 \end{cases}$$

$$(0, w_p)\,\hat{\cup}\,(x_q, 0) = 0$$

$$(0, w_p)\,\hat{\cup}\,(0, w_q) = \begin{cases} 0 & m \neq 2q \\ (0, w_p \wedge 2\bar{\partial}\partial w_q) & m = 2q \end{cases}$$

The map $\hat{\cup}$ induces the product structure (2.11).

Corollary 2.7. *The map $\hat{\cup}$ induces the product* (2.11).

2.3.2 Sketch of the proof of Thm. 2.6

In order to get the product structure in Thm. 2.6 one combines the following steps.

First we require a variant of the Beĭlinson product ([5]) for the cones of certain triangular diagrams of complexes introduced in [2]: namely, for $i = 1, 2, 3$ consider diagrams

$$\mathcal{D}_i \overset{\text{def}}{=} X_i^\bullet \xrightarrow{f_i} Z_i^\bullet \xleftarrow{g_i} Y_i^\bullet \tag{2.20}$$

with product structures $X_1^\bullet \otimes X_2^\bullet \xrightarrow{\cup} X_3^\bullet$, and similarly for $Y_i^\bullet$, $Z_i^\bullet$, compatible with the f_i, g_i up to homotopy, that is:

$$f_3 \circ \cup - \cup \circ (f_1 \otimes f_2) = d_Z h + h\, d_{X_1 \otimes X_2}\,,$$
$$g_3 \circ \cup - \cup \circ (g_1 \otimes g_2) = d_Z k + k\, d_{Y_1 \otimes Y_2}\,,$$

with homotopies:

$$h\colon \big(X_1 \otimes X_2\big)^\bullet \longrightarrow Z_3^{\bullet - 1}\,, \quad k\colon \big(Y_1 \otimes Y_2\big)^\bullet \longrightarrow Z_3^{\bullet-1}\,.$$

The following lemma is immediately verified ([2]):

Lemma 2.8. *Let* $\Gamma(\mathcal{D}_i) = \mathrm{Cone}(X_i^\bullet \oplus Y_i^\bullet \xrightarrow{f_i - g_i} Z_i^\bullet)[-1]$, $i = 1, 2, 3$. *For* $(x_i, y_i, z_i) \in X_i^\bullet \oplus Y_i^\bullet \oplus Z_i^{\bullet-1}$, $i = 1, 2$, *and* $\alpha \in [0, 1]$ *the formula:*

$$\begin{aligned}(x_1, y_1, z_1) \cup_\alpha (x_2, y_2, z_2) = \Big(&x_1 \cup x_2, y_1 \cup y_2, \\ &(-1)^{\deg(x_1)}\big((1-\alpha) f_1(x_1) + \alpha g_1(y_1)\big) \cup z_2 \\ &\quad + z_1 \cup \big(\alpha f_2(x_2) + (1-\alpha) g_2(y_2)\big) \\ &\quad\quad - h(x_1 \otimes x_2) + k(y_1 \otimes y_2)\Big)\,.\end{aligned} \tag{2.21}$$

defines a family of products

$$\Gamma(\mathcal{D}_1) \otimes \Gamma(\mathcal{D}_2) \xrightarrow{\cup_\alpha} \Gamma(\mathcal{D}_3)\,.$$

These products are homotopic to one another, and graded commutative up to homotopy. The homotopy formula is the same as that found in ref. [5].

As a further technical ingredient, the following proposition is used to transfer product structures among homotopically equivalent complexes. (It is probably well-known: a proof is included in ref. [8].)

Proposition 2.9. *Let $X^\bullet$ and $Y^\bullet$ be two homotopically equivalent complexes, and let $\phi : X^\bullet \to Y^\bullet$ and $\psi : Y^\bullet \to X^\bullet$ be the homotopy equivalences. Assume $Y^\bullet$ has a product structure $\cup$. Then the position*

$$\psi(\phi(x_1) \cup \phi(x_2))$$

provides a product structure on $X^\bullet$ which is homotopy associative and graded commutative (up to homotopy) if so is the one on $Y^\bullet$.

Secondly, starting from the complex $D_{h.h.}(p)^\bullet$ and using the quasi-isomorphism

$$\mathbf{R}(p)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$$

we apply the first part and lemma 2.8 to the diagram

$$\mathcal{D}_p \stackrel{\text{def}}{=} \mathbf{Z}(p)^\bullet_{\mathcal{D}} \xrightarrow{\rho_p} \mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p) \xleftarrow{incl} F^p\mathscr{A}^\bullet_X \cap \sigma^{2p}\mathscr{E}^\bullet_X(p)\,. \tag{2.22}$$

The product structure on $\mathbf{Z}(p)^\bullet_{\mathcal{D}}$ is still (2.5), and that on $\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$ can be found in ref. [8]. In particular, it coincides with the standard wedge product in degrees bigger or equal $2p$, which is the product structure for the included subcomplex $F^p\mathscr{A}^\bullet_X \cap \sigma^{2p}\mathscr{E}^\bullet_X(p) = \sigma^{2p}\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$. The inclusion in the right part of (2.22) is obviously strictly compatible with the products, whereas the map ρ_p is only compatible up to homotopy since $\mathfrak{D}^\bullet(\mathscr{A}^\bullet_X, p)$ is only isomorphic to $\mathrm{Cone}\big(\mathscr{E}^\bullet_X(p) \oplus F^p\mathscr{A}^\bullet_X \longrightarrow \mathscr{A}^\bullet_X\big)[-1]$ up to homotopy. (The latter complex carries the strict Beĭlinson product.) For $x_p \in \mathbf{Z}(p)^n_{\mathcal{D}}$ and $x_q \in \mathbf{Z}(q)^m_{\mathcal{D}}$ the homotopy h is:

$$h(x_p \otimes x_q) = \begin{cases} 0 & n = 0 \text{ or } m = 0 \\ (-1)^{m-1} 2\pi_p(x_p) \wedge \pi_{q-1}(x_q) & \text{otherwise.} \end{cases}$$

Finally, if in a diagram $\mathcal{D}$ of the form (2.20) the map $g : Y^\bullet \to Z^\bullet$ is injective, there is a homotopy equivalence

$$\Gamma(\mathcal{D}) \simeq \mathrm{Cone}\big(X \xrightarrow{f} \mathrm{Coker}\, g\big)[-1]$$

we can use to transfer the product structure (2.21) from the cone of the diagram $\mathcal{D}$ to the complex on the right by making use of proposition 2.9. By applying this procedure to $\Gamma(\mathcal{D}_p)$ in (2.22) we obtain the desired product structure on the complex $\mathfrak{D}_{h.h.}(p)^\bullet$.

3 Singular hermitian structures

3.1 Relative cohomology

Recall that X is a projective algebraic manifold. Let now D be a normal crossing divisor in X and let U be the complement of D in X. Let $\jmath : U \to X$ be the inclusion. As usual, an open coordinate subset V with coordinates $(z_1, \ldots, z_n)$ is *adapted* to D if the divisor D is locally given by the equation $z_1 \cdots z_k = 0$.

By viewing X as a compactification of U, following ref. [4], we consider the following sheaf complex on the triple $(U, X, \jmath)$:

$$\mathfrak{D}_{h.h.}(p)^\bullet_{X,U} \stackrel{\text{def}}{=} \mathrm{Cone}\big(\mathbf{Z}(p)^\bullet_{\mathcal{D},X} \xrightarrow{\rho_p} \jmath_*\mathfrak{D}^{\bullet<2p}(\mathscr{A}^\bullet_U, p)\big)[-1]\,. \tag{3.1}$$

Let us emphasize the $\mathbf{Z}$-valued Deligne complex is the one for X, whereas the smooth part is the (direct image of the) truncation of the p-th Deligne algebra of the open complement U of D.

Definition 3.1. The hypercohomology groups

$$\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X, U; p) \stackrel{\text{def}}{=} \mathbb{H}^\bullet(X, \mathfrak{D}_{h.h.}(p)^\bullet_{X,U}) \tag{3.2}$$

are the Hermitian-holomorphic Deligne cohomology groups of the triple $(U, X, \jmath)$.

Remark 3.2. The elements of the group $\widehat{\mathrm{H}}^2_{\mathcal{D}}(X, U; 1)$, correspond to pairs $(\mathscr{L}, \rho)$, where $\mathscr{L}$ is a line bundle over X, and ρ is a smooth hermitian metric on the restriction $\mathscr{L}|_U$.

3.1.1

The characteristic morphisms (2.17) and (2.18) generalize in this case to maps:

$$\mathfrak{D}_{h.h.}(p)^\bullet_{X,U} \xrightarrow{-2\bar\partial\partial} \jmath_*\big(\mathscr{A}_U^{p,p}[-2p] \cap \mathscr{E}_U^{2p}(p)\big)_{cl}$$

and

$$\widehat{\mathrm{H}}^{2p}_{\mathcal{D}}(X,U;p) \longrightarrow A^{p,p}(U)_{\mathbf{R}(p),cl}\,.$$

From the cone in (3.1) and the fact that $\jmath$ is affine we get the standard long exact cohomology sequence:

$$\cdots \longrightarrow \mathbb{H}^{k-1}(U,\mathfrak{D}^{\bullet<2p}(\mathscr{A}_U^\bullet,p)) \longrightarrow \widehat{\mathrm{H}}^k_{\mathcal{D}}(X,U;p) \longrightarrow \mathrm{H}^k_{\mathcal{D}}(X,\mathbf{Z}(p)) \longrightarrow \mathbb{H}^k(U,\mathfrak{D}^{\bullet<2p}(\mathscr{A}_U^\bullet,p)) \longrightarrow \cdots$$

Moreover, the arguments in refs. [4], and especially [11, §4], show that there is a product map

$$\hat\cup\,:\mathfrak{D}_{h.h.}(p)^\bullet_{X,U}\otimes\mathfrak{D}_{h.h.}(q)^\bullet_{X,U} \longrightarrow \mathfrak{D}_{h.h.}(p+q)^\bullet_{X,U}$$

whose expression is still given by the formula in Thm 2.6. As a consequence we obtain a cup product for the relative cohomology groups:

$$\widehat{\mathrm{H}}^k_{\mathcal{D}}(X,U;p)\otimes\widehat{\mathrm{H}}^l_{\mathcal{D}}(X,U;q) \xrightarrow{\ \cup\ } \widehat{\mathrm{H}}^{k+l}_{\mathcal{D}}(X,U;p+q)\,.$$

3.1.2

A variation on the previous theme is to consider *two* divisors with normal crossings D and D'. Let us assume that $D\cup D'$, and $D\cap D'$ are also divisors with normal crossings. Let U and U' be their complements in X, with inclusion maps $\jmath$ and $\jmath'$. Then $\mathfrak{D}_{h.h.}(p)^\bullet_{X,U_1}$ by restriction yields $\mathfrak{D}_{h.h.}(p)^\bullet_{X,U\cap U'}$, and similarly for $\mathfrak{D}_{h.h.}(q)^\bullet_{X,U'}$. Thus we get a product

$$\widehat{\mathrm{H}}^k_{\mathcal{D}}(X,U;p)\otimes\widehat{\mathrm{H}}^l_{\mathcal{D}}(X,U';q) \xrightarrow{\ \cup\ } \widehat{\mathrm{H}}^{k+l}_{\mathcal{D}}(X,U\cap U';p+q)\,.$$

3.1.3

In order to consider pairs $(\mathscr{L},\rho)$ where the hermitian metric ρ is required to have a prescribed behavior along D, we will need to consider appropriate subcomplexes of $\jmath_*\mathfrak{D}^{\bullet<2p}(\mathscr{A}_U^\bullet,p)$ with specified growth conditions.

Definition 3.3. For $p\geq 0$ consider a subcomplex

$$\mathfrak{D}^\bullet_?(X,p)\subset \jmath_*\mathfrak{D}^\bullet(\mathscr{A}_U^\bullet,p)$$

where the "?" in the subscript denotes a growth condition to be described below, such that:

1. The morphism $\rho_p:\mathbf{Z}(p)^\bullet_{\mathcal{D},X}\to \jmath_*\mathfrak{D}^{\bullet<2p}(\mathscr{A}_U^\bullet,p)$ factors through $\mathfrak{D}^{\bullet<2p}_?(X,p)$;
2. The cup-product restricts to $\mathfrak{D}^\bullet_?(X,p)$, namely

$$\begin{array}{ccc}
\mathfrak{D}^\bullet_?(X,p)\bigotimes\mathfrak{D}^\bullet_?(X,q) & \longrightarrow & \mathfrak{D}^\bullet_?(X,p+q)\\
\big\downarrow & & \big\downarrow\\
\jmath_*\mathfrak{D}^\bullet(\mathscr{A}_U^\bullet,p)\bigotimes \jmath_*\mathfrak{D}^\bullet(\mathscr{A}_U^\bullet,q) & \longrightarrow & \jmath_*\mathfrak{D}^\bullet(\mathscr{A}_U^\bullet,p+q)\,;
\end{array}$$

3. $\mathfrak{D}^\bullet_?(X,p)=\jmath_*\mathfrak{D}^\bullet(\mathscr{A}_U^\bullet,p)\bigcap\{\text{forms with specified growth along } D\}$

The *Hermitian-holomorphic Deligne complex with growth* is the complex

$$\mathfrak{D}_{h.h.}(p)^\bullet_{X,?}\overset{\text{def}}{=}\mathrm{Cone}\big(\mathbf{Z}(p)^\bullet_{\mathcal{D},X}\longrightarrow\mathfrak{D}^{\bullet<2p}_?(X,p)\big)[-1] \tag{3.3}$$

One accordingly defines cohomology groups

$$\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X;p)_?\overset{\text{def}}{=}\mathbb{H}^\bullet(X,\mathfrak{D}_{h.h.}(p)^\bullet_{X,?})\,.$$

By restriction from the sheaf (3.1) defined on (X,U) the groups $\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X;p)_?$ will satisfy the same formal properties as those spelled in sect. 3.1.1.

3.2 Good metrics

We need to recall Mumford's notion of *good hermitian metric* [15, §1] in the case of a line bundle.

Let V an adapted neighborhood of a point $x \in D$. Consider a polycylinder neighborhood P of D in V, so that

$$P \cap U = (\dot{\Delta}_a)^k \times (\Delta_a)^{n-k} ,$$

where Δ (resp. $\dot{\Delta}$) is a standard disk (resp. punctured disk) of radius $r = |z| < a$, centered at $z = 0$.

On the punctured disk $\dot{\Delta}_a$ we have the Poincaré metric

$$ds^2 = \frac{|\mathrm{d}z|^2}{\big(|z| \log |z|\big)^2} ,$$

and the disk Δ_a is equipped with the standard Euclidean metric $|\mathrm{d}z|^2$. Denote by h_P the resulting product metric on the punctured polycylinder $(\dot{\Delta}_a)^k \times (\Delta_a)^{n-k}$.

Definition 3.4. A smooth q-form ω on U is said to have *Poincaré growth along* D if there is a cover $\mathfrak{U}_X$ of X with adapted neighborhoods and polycylinders as above such that the following estimate holds:

$$|\omega(\xi_1, \dots, \xi_q)|^2 \leq C\, h_P(\xi_1, \xi_1) \cdots h_P(\xi_q, \xi_q) . \tag{3.4}$$

(C is a positive constant, and $\xi_1, \dots, \xi_q$ are tangent vectors at some point of $P \cap U$.) A form ω is a *good form (along D)* if both ω and $\mathrm{d}\omega$ have Poincaré growth.

Clearly, good forms form a differential graded algebra. It is proved in ref. [15] that they are locally integrable and the associated currents have no residue.

Let now $\mathscr{L}$ be a line bundle on X equipped with a smooth hermitian metric ρ on the restriction $\mathscr{L}|_U$.

Definition 3.5. The metric ρ is *good along* D if for any adapted neighborhood V with polycylinder P as above, there is a non-vanishing section $s \in \mathscr{L}|_{P\cap U}$ such that:

1. $\|s\|, \|s\|^{-1} \leq C \left(\sum_{i=1}^k \log |z_i|\right)^N$, for some $C > 0$, $N \in \mathbf{N}$.
2. The forms $\partial \log\|s\|$ and $\bar{\partial}\partial \log\|s\|$ are good forms along D.

We have set $\|s\|^2 = \rho(s, s)$. We will denote such a pair $(\mathscr{L}, \rho)$ by $\bar{\mathscr{L}}$ and call it a *good hermitian line bundle.*

Given the pair $(\mathscr{L}|_U, \rho)$, there is a unique extension $\mathscr{L}$ to X such that ρ is good. Moreover, if s is a section of $\mathscr{L}$ on X, the current associated to the good form $\bar{\partial}\partial \log\|s\|^2$ represents the first Chern class of $\mathscr{L}$. (For both facts, cf. ref. [15, §1].) Owing to this fact, if $\bar{\mathscr{L}}$ is a good hermitian line bundle, we sometimes abuse the language and denote the form $\bar{\partial}\partial \log\|s\|^2$ by $c_1(\mathscr{L})$.

Example 3.6. Let X be a smooth projective algebraic curve over $\mathbf{C}$ (a Riemann surface). By our standing hypotheses D will correspond to a finite set of points on X. An adapted neighborhood V of a point $p \in D$ is simply a disk Δ_a of radius a with a coordinate function z such that $z(p) = 0$. Also, $V \cap U = \dot{\Delta}_a$, the corresponding punctured disk.

Consider the hyperbolic metric $\mathrm{d}s^2_{\mathsf{hyp}}$ on T_U of constant negative curvature equal to -1. Locally near $p \in D$ we have:

$$\mathrm{d}s^2_{\mathsf{hyp}}\Big|_{V\cap U} = \rho_{\mathsf{hyp}}\, |\mathrm{d}z|^2 ,$$

where ρ_{hyp} is a smooth positive function on $V \cap U$, such that there exists a continuous function $\alpha : V \to \mathbf{R}$, smooth on $V \cap U = \dot{\Delta}_a$, and the following asymptotic behavior at p (cf. [18]):

$$\log \rho_{\mathsf{hyp}}(z) = -\log\big(|z|^2 \log^2 |z|\big) + \alpha(z) , \quad |\partial\alpha|^2 \leq \frac{|\mathrm{d}z|^2}{|z|^2 \log^4 |z|} . \tag{3.5}$$

This is a good metric. It is well-known that the good hermitian line bundle $\mathscr{L}$ on X corresponding to the pair $(T_U, \mathrm{d}s^2_{\mathsf{hyp}})$ is $\mathscr{L} = T_X(-D)$. Indeed, in the adapted neighborhood V, $s = z\frac{\partial}{\partial z}$ is a non-vanishing holomorphic section of $\mathscr{L}|_V$, and its length square with respect to the hyperbolic metric is $\|s\|^2 = \exp \alpha / \log^2 |z|$, which implies goodness (cf. [17]).

3.2.1

It would seem natural to use good forms to realize the complex of Definition 3.3. Unfortunately, this is not possible, as we now briefly explain.

Denote by $\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{good}$ the differential graded subalgebra of $\jmath_*\mathscr{A}_U^\bullet$ comprised of good forms.

If ω is a good form, it does not necessarily follow that $\mathrm{d}\mathrm{d}^c\omega = 2\bar\partial\partial\omega$ is good too, namely it does not necessarily have Poincaré growth along D. Hence the graded module $\jmath_*\mathfrak{D}^\bullet(\mathscr{A}_U^\bullet,p)\bigcap\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{good}$ is not a complex, in general. Note that the form of the differential at degree $2p$ is the sole troublesome point. Indeed, we have

Lemma 3.7. *The graded modules*

$$\jmath_*\mathfrak{D}^{\bullet<2p}(\mathscr{A}_U^\bullet,p)\bigcap\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{good}\,,\quad \jmath_*\mathfrak{D}^{\bullet\geq 2p}(\mathscr{A}_U^\bullet,p)\bigcap\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{good}$$

form well defined complexes.

Proof. For the second, we have from eq. (2.13) that $d_{\mathfrak{D}} = \mathrm{d}$, hence the notion of "good" form is compatible with the definition of the complex $\mathfrak{D}^\bullet(\mathscr{A}_U^\bullet,p)$ in degrees $\geq 2p$.

For the first, let $\eta\in\jmath_*\sigma^{<2p}\mathfrak{D}^n(\mathscr{A}_U^\bullet,p)$ be a good form. From (2.13) and (2.12) we see that to compute $d_{\mathfrak{D}}\eta = -\pi(\mathrm{d}\eta)$ we must drop the component $\partial\eta^{p-1,n-p}$ of $\mathrm{d}\eta$. It is immediate to see that if $\mathrm{d}\eta$ has Poincaré growth, so does $\pi\circ\mathrm{d}\eta$. □

Moreover, the morphism $\rho_p\colon \mathbf{Z}(p)^\bullet_{\mathcal{D},X}\to\jmath_*\mathfrak{D}^{\bullet<2p}(\mathscr{A}_U^\bullet,p)$ clearly factors through the subcomplex

$$\jmath_*\mathfrak{D}^{\bullet<2p}(\mathscr{A}_U^\bullet,p)\bigcap\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{good}\,,$$

so that it is indeed possible to define the complex:

$$\mathfrak{D}_{h.h.}(p)^\bullet_{X,good}\stackrel{\mathrm{def}}{=}\mathrm{Cone}\big(\mathbf{Z}(p)^\bullet_{\mathcal{D},X}\longrightarrow\jmath_*\mathfrak{D}^{\bullet<2p}(\mathscr{A}_U^\bullet,p)\bigcap\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{good}\big)[-1]\,.$$

However, since in the expression of the product $\hat\cup$ in Theorem 2.6 there appears the operator $2\bar\partial\partial$, we see that condition 2 in Definition 3.3 is not satisfied.

3.3 Pre-log-log forms

We consider a larger subcomplex (in fact a subalgebra) of $\jmath_*\mathscr{A}_U^\bullet$ of forms with a log-log-type growth which has been recently introduced in ref. [9]. The following two definitions are from loc. cit., §7, whose notations we retain in part:

Definition 3.8. A smooth complex-valued function f on U has *log-log-growth* along D if

$$f(z_1,\dots,z_n)\leq C\prod_{i=1}^k\log^N|\log|z_i||$$

on any adapted neighborhood V with coordinates $z_1,\dots,z_n$ and some constants $C>0$ and $N\in\mathbf{N}$.

The *sheaf of differential forms on X with log-log-growth along D* is the subsheaf of $\jmath_*\mathscr{A}_U^\bullet$ generated by the functions with log-log-growth along D and the differentials

$$\begin{aligned}&\frac{\mathrm{d}z_i}{z_i\,\log|z_i|}\,,\ \frac{\mathrm{d}\bar z_i}{\bar z_i\,\log|z_i|} && i=1,\dots,k\,,\\ &\mathrm{d}z_i\,,\ \mathrm{d}\bar z_i && i=k+1,\dots,n\,.\end{aligned}$$

Definition 3.9. A form ω with log-log-growth such that $\partial\omega$, $\bar\partial\omega$, and $\partial\bar\partial\omega$ also have log-log-growth along D will be called a *pre-log-log form.* The sheaf of pre-log-log forms is the subalgebra of $\jmath_*\mathscr{A}_U^\bullet$ generated by the pre-log-log forms. This complex is denoted by $\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{pre}$. Similarly, the subcomplex of $\mathbf{R}(p)$-valued forms will be denoted by $\mathscr{E}_X^\bullet(p)\langle\langle D\rangle\rangle_{pre}$.

We refer to op. cit. for more properties of $\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{pre}$. We only observe that while pre-log-log forms and good forms are not the same, the former have some of the salient features of the latter, notably, pre-log-log forms are locally integrable and their associated currents do not have residues ([9, Proposition 7.6]).

Since $\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{pre}$ has the same formal properties of a Dolbeault complex, it makes sense to consider the complex $\mathfrak{D}^\bullet(\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{pre}, p)$ which is a subalgebra of $\jmath_*\mathfrak{D}^\bullet(\mathscr{A}_U^\bullet, p)$ satisfying the conditions in Definition 3.3. Therefore, using the same notations, we set:

$$\mathfrak{D}^\bullet_{pre}(X,p) \stackrel{\text{def}}{=} \mathfrak{D}^\bullet(\mathscr{A}_X^\bullet\langle\langle D\rangle\rangle_{pre}, p)\,. \tag{3.6}$$

Definition 3.10. The *pre-log-log Hermitian-holomorphic Deligne complex* is the complex:

$$\mathfrak{D}_{h.h.}(p)^\bullet_{X,pre} = \operatorname{Cone}\big(\mathbf{Z}(p)^\bullet_{\mathcal{D},X} \xrightarrow{\rho_p} \mathfrak{D}^{\bullet<2p}_{pre}(X,p)\big)[-1]\,. \tag{3.7}$$

The hypercohomology groups:

$$\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X;p)_{pre} \stackrel{\text{def}}{=} \mathbb{H}^\bullet(X, \mathfrak{D}_{h.h.}(p)^\bullet_{X,pre})$$

are the *pre-log-log Hermitian-holomorphic Deligne cohomology groups.*

In particular, good hermitian line bundles define elements of pre-log-log Hermitian holomorphic Deligne groups:

Proposition 3.11. *Let $\bar{\mathscr{L}} = (\mathscr{L}, \rho)$ be a good hermitian line bundle. Then $[\bar{\mathscr{L}}] \in \widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1)_{pre}$. Moreover, $c_1(\mathscr{L}) \in \Gamma(X, \mathscr{E}_X^\bullet(1)\langle\langle D\rangle\rangle_{pre})$.*

Proof. Let $\mathfrak{U}_X = \bigcup_i U_i$ be an open cover, and let $\mathscr{L}$ be trivialized by non-vanishing sections $s_i \in \mathscr{L}|_{U_i}$ as in sect. 2.2.1 (with the same notations).

It follows from definition 3.5 that since $\bar{\mathscr{L}}$ is good, the $\log\rho_i$'s have log-log growth along $D \cap U_i$, and so do $\partial\log\rho_i$ and $\partial\bar\partial\log\rho_i$. Hence $\log\rho_i$ is pre-log-log.

Moreover, since

$$c_1(\mathscr{L})|_{U_i} = \bar\partial\partial\log\rho_i$$

is of type $(1,1)$ and closed, good implies pre-log-log (cf. [9]). □

4 Cup product of hermitian line bundles

Let us temporarily revert to the non-relative case.

We will need to have a closer look at the product

$$\widehat{\mathrm{Pic}}(X) \equiv \widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1) \otimes \widehat{\mathrm{Pic}}(X) \equiv \widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1) \longrightarrow \widehat{\mathrm{H}}^4_{\mathcal{D}}(X;2)\,.$$

induced by the cup product $\mathfrak{D}_{h.h.}(1)^\bullet \otimes \mathfrak{D}_{h.h.}(1)^\bullet \xrightarrow{\hat\cup} \mathfrak{D}_{h.h.}(2)^\bullet$, in particular we need an explicit formula in Čech cohomology for the induced product at the level of total Čech complexes.

Let $\bar{\mathscr{L}} = (\mathscr{L}, \rho)$ and $\bar{\mathscr{L}}' = (\mathscr{L}', \rho')$ be two hermitian line bundles on X, and assume they are trivialized as in sect. 2.2.1 with respect to a cover $\mathfrak{U}_X$. Consider two Čech cocycles of the form (2.7) representing the corresponding classes in $\widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1)$, where primed symbols refer to the second pair.

Using the product from Thm 2.6, and the conventions on Čech resolutions outlined in sect. 1.3, the cup-product $[\bar{\mathscr{L}}]\,\hat\cup\,[\bar{\mathscr{L}}']$ in $\widehat{\mathrm{H}}^4_{\mathcal{D}}(X;2)$ is represented with a Čech resolution by the cocycle:

(4.1)

$(0,4)$	$(2\pi\sqrt{-1})^2\, c_{ijk}\, c'_{klm}$
$(1,3)$	$2\pi\sqrt{-1}\, c_{ijk} \log\, g'_{kl}$
$(2,2)$	$(-\log\, g_{ij}\ \mathrm{dlog}\ g'_{jk}) \oplus \pi_1(\log\, g_{ij})\ \log\, \boxed{g'_{jk}}$
$(3,1)$	$-\log\lvert g_{ij}\rvert\ \frac{1}{2}\,\mathrm{d}^c \log\rho'_j + \mathrm{d}^c\log\lvert g_{ij}\rvert\ \frac{1}{2}\log\rho'_j$
$(4,0)$	$\frac{1}{2}\log\rho_i\ \bar\partial\partial\log\rho'_i$

Remark 4.1. From (4.1) and remark 2.3, the characteristic form associated to $[\bar{\mathscr{L}}]\,\hat\cup\,[\bar{\mathscr{L}}']$ is the global form locally given by:

$$\bar\partial\partial\log\rho_i \wedge \bar\partial\partial\log\rho'_i\,. \tag{4.2}$$

This form represents the product $c_1(\mathscr{L}) \cup c_1(\mathscr{L}')$ of the two Chern classes. Moreover, according to refs. [7, 1], this form is the Chern class of the hermitian $(2,2)$-curving on the 2-Gerbe $\big(\mathscr{L}, \mathscr{L}'\big]$.

Assume now D is a normal crossing divisor in X as before, with $U = X \setminus D$, and let $\bar{\mathscr{L}}$, $\bar{\mathscr{L}}'$ be good hermitian line bundles. The following is evident:

Proposition 4.2. *The cup product*

$$\widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1)_{pre} \otimes \widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1)_{pre} \longrightarrow \widehat{\mathrm{H}}^4_{\mathcal{D}}(X;2)_{pre}$$

is computed again by eq. (4.1)*, where elements in* $\check{\mathrm{C}}^{4-k}(\mathfrak{U}_X, \mathfrak{D}_{h.h.}(p)^k_{X,pre})$*, for* $k = 3, 4$*, are pre-log-log forms along* $D \cap U_J$*, where* J *is the multi-index* $J = (i_0, i_1, \dots, i_{4-k})$*.*

In particular, combining this with proposition 3.11, we immediately have that the cup product of good hermitian line bundles is a pre-log-log hermitian holomorphic Deligne class, whereas the characteristic form (4.2) is a pre-log-log form on X along D.

Remark 4.3. It is immediately verified that the pre-log-log forms in the total cocycle (4.1) representing the class $[\bar{\mathscr{L}}] \hat{\cup} [\bar{\mathscr{L}}']$ as well as the characteristic form (4.2) are in fact good forms.

In other words, given two good hermitian line bundles, their cup-product can be represented in Čech cohomology by a total cocycle with values in the complex $\mathfrak{D}_{h.h.}(2)^\bullet_{X,good}$ introduced in sect. 3.2.1.

4.1 Product on a curve

4.1.1

Let us now consider in particular the case where X is a smooth proper curve over $\mathbf{C}$ as in example 3.6. Since $\dim_{\mathbf{R}} X = 2$, the complex $\mathfrak{D}_{h.h.}(2)^\bullet_{X,?}$, where ? = *good* or *pre* (or any other growth satisfying the requirements of Definition 3.3), is the cone shifted by 1 of:

$$\begin{array}{ccccccc}
\mathbf{Z}(2)_X & \xrightarrow{\imath} & \mathscr{O}_X & \xrightarrow{\mathrm{d}} & \Omega^1_X & & \\
& & \big\downarrow{\scriptstyle -\pi_1} & & \big\downarrow{\scriptstyle \pi_1} & & \\
& & \mathscr{E}^0_X(1)\langle\langle D\rangle\rangle_? & \xrightarrow{-\mathrm{d}} & \mathscr{E}^1_X(1)\langle\langle D\rangle\rangle_? & \xrightarrow{-\mathrm{d}} & \mathscr{E}^2_X(1)\langle\langle D\rangle\rangle_?
\end{array}$$

where now we have the full complex $\mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?$. (So the choice ? = *good* does make sense in this case.)

Again by dimensional reasons, the part of the total cocycle (4.1) corresponding to the element of $\mathrm{H}^4_{\mathcal{D}}(X, \mathbf{Z}(2))$ representing $\big(\mathscr{L}, \mathscr{L}'\big]$ becomes trivial, in fact we have:

Lemma 4.4. *The cocycle* (4.1) *reduces to a total Čech cocycle of degree* 2 *in* $\operatorname{Tot}\check{\mathrm{C}}^\bullet(\mathfrak{U}_X, \mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?)$*. That is, for a curve* X *as above the cup product in Proposition 4.2 reduces to:*

$$\widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1)_? \otimes \widehat{\mathrm{H}}^2_{\mathcal{D}}(X;1)_? \xrightarrow{\cup} \mathbb{H}^2(X, \mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?)$$

Proof. This follows immediately from a calculation identical to [2, Thm. 5.1], one need only observe that no problems with singularities at D ever occur, since only sections of $\mathscr{O}_X$ and $\mathscr{E}^0_X(1)$ are involved. Namely, the cohomology class of the higher symbol $\big(\mathscr{L}, \mathscr{L}'\big]$ is zero and it can be written as the coboundary of a collection (of Bloch's type dilogarithms) $L_{ijk} \in \mathscr{O}_X(U_{ijk})$ such that:

$$\mathrm{d}L_{ijk} = -\log g_{ij}\ \mathrm{d}\log g'_{jk}\,.$$

This allows to construct a cochain $\omega^2_{ijk} \in \mathscr{E}^0_X(1)(U_{ijk}) \subset \mathscr{E}^0_X(1)\langle\langle D\rangle\rangle_?(U_{ijk})$ (in fact, a Čech cocycle) which, together with the $(4,0)$ and $(3,1)$ entries in (4.1), respectively denoted ω^0_i and ω^1_{ij}, forms a total degree 2 cocycle $\Omega = \omega^0 + \omega^1 + \omega^2 \in \operatorname{Tot}\check{\mathrm{C}}^\bullet(\mathfrak{U}_X, \mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?)$. This cocycle can be injected via the standard cone exact sequence into $\operatorname{Tot}\check{\mathrm{C}}^\bullet(\mathfrak{U}_X, \mathfrak{D}_{h.h.}(2)^\bullet_{X,?})$: then its difference with (4.1) is shown to be a coboundary. ☐

4.1.2

Depending on the chosen growth conditions, there remains the question of whether in Lemma 4.4 one has

$$\mathbb{H}^2(X, \mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?) \cong \mathrm{H}^2(X, \mathbf{R}(1))$$

(Of course, the latter is isomorphic to $\mathbf{R}(1)$ if X is irreducible). This will be the case *if* Poincaré lemma for the complexes $\mathscr{E}^\bullet_X(p)\langle\langle D\rangle\rangle_?$ and $\mathscr{A}^\bullet_X\langle\langle D\rangle\rangle_?$ holds. In general, as observed in ref. [9, remark 7.19], $\mathrm{H}^2(X,\mathbf{R}(1))$ must be at least a direct factor of $\mathbb{H}^2(X,\mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?)$, since the composite map

$$\mathscr{A}^\bullet_X \longrightarrow \mathscr{A}^\bullet_X\langle\langle D\rangle\rangle_? \longrightarrow \mathscr{D}^\bullet_X \equiv {}^\backprime\mathscr{A}^\bullet_X[-2\dim X]$$

(where the complex on the right is the complex of complex-valued currents) is a quasi-isomorphism. This remark can be generalized to any X, not necessarily of complex dimension one, implying that $\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X;p)$ is a direct factor of $\widehat{\mathrm{H}}^\bullet_{\mathcal{D}}(X;p)_?$.

For pre-log-log forms, that is, if $? = pre$, one indeed cannot presently invoke Poincaré lemma. In ref. [9] it is observed this could be remedied by resorting to a complex of forms with log-log-type growth conditions imposed on all derivatives (cf. sect.1, footnote 1). We can work around this issue with the integration map to be defined below.

4.1.3

Returning to X of complex dimension one, we can compose the product map in Lemma 4.4 with the map $\mathbb{H}^2(X,\mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?) \longrightarrow \mathbf{R}(1)$ which can be defined as follows. First, observe that

$$\mathbb{H}^\bullet(X,\mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?) \cong \mathrm{H}^\bullet(E^\bullet_X(1)\langle\langle D\rangle\rangle_?)\,,$$

where on the right we have the complex of global sections. Let $D = p_1 + \cdots + p_N$. Denote by $X_\varepsilon = X \setminus B_\varepsilon(D)$, where $B_\varepsilon = \bigcup_{i=1}^N \Delta_\varepsilon(p_i)$ is the union of disks of radius ε centered at each point p_i. If $\omega \in E^2_X(1)\langle\langle D\rangle\rangle_?$ corresponds to a class in $\mathbb{H}^2(X,\mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?)$, then:

$$\int_X \omega \overset{\text{def}}{=} \lim_{\varepsilon\to 0}\int_{X_\varepsilon}\omega \in \mathbf{R}(1)\,. \tag{4.3}$$

By local integrability of good or pre-log-log forms, this is well defined.

Alternatively, let $\Sigma_{\mathbf{T}}$ be a representative of the class $[X]$ obtained from a triangulation $\mathbf{T}$ subordinated to the cover $\mathfrak{U}_X = \{V_i\}_{i\in I}$. Thus there is a map of index sets $A \ni \alpha \mapsto i(\alpha) \in I$ such that $\triangle^2_\alpha \subset V_{i(\alpha)}$, $\triangle^1_{\alpha,\beta} \subset V_{i(\alpha)i(\beta)}$, etc. Assume $\operatorname{supp} D \cap \mathbf{T}^{(1)} = \varnothing$, where $\mathbf{T}^{(1)}$ is the 1-skeleton of $\mathbf{T}$.

If Ω is the cocycle representing a class $[\Omega] \in \mathbb{H}^2(X,\mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?)$, then set:

$$\langle\Omega,\Sigma_{\mathbf{T}}\rangle \overset{\text{def}}{=} \sum_\alpha \int_{\triangle^2_\alpha}\omega^0_{i(\alpha)} + \sum_{\langle\alpha\beta\rangle}\int_{\triangle^1_{\alpha\beta}}\omega^1_{i(\alpha)i(\beta)} + \sum_{\langle\alpha\beta\gamma\rangle}\int_{\triangle^0_{\alpha\beta\gamma}}\omega^2_{i(\alpha)i(\beta)i(\gamma)}\,. \tag{4.4}$$

(Note that the sums and labels run over the abstract dual triangulation.) More details can be found, e.g. in ref. [3]. A limiting process is implicit in (4.4) as well. If $\triangle^2_\alpha$ contains $p \in D$, then

$$\int_{\triangle^2_\alpha}\omega^0_{i(\alpha)} \overset{\text{def}}{=} \lim_{\varepsilon\to 0}\int_{\triangle^2_\alpha\cap X_\varepsilon}\omega^0_{i(\alpha)}\,. \tag{4.5}$$

In summary, we have:

Lemma 4.5. *Let* $[\omega] = [\Omega]$. *Then, with the above provisions:*

$$\int_X \omega = \langle\Omega,\Sigma_{\mathbf{T}}\rangle \in \mathbf{R}(1)\,.$$

Proof. The correspondence between Ω and ω can be made explicit using, e.g. partitions of unity arguments. Then we need only observe that since $\mathbf{T}^1 \cap \operatorname{supp} D = \varnothing$, singularities develop only in the integrations on the 2-chains which hit points in D. These are handled by (4.5). $\square$

5 Deligne pairing

5.1 Reminder on Determinant of Cohomology

X is a smooth proper curve over $\mathbf{C}$ as in example 3.6. We need to collect a few formulas and definitions from ref. [10].

Consider two line bundles $\mathscr{L}$ and $\mathscr{L}'$ on X and let E and E' be the corresponding divisors, assumed to have disjoint support. The Deligne pairing $\langle \mathscr{L}, \mathscr{L}' \rangle$, is a complex line generated by the symbol $\langle s, s' \rangle$, where s (resp. s') is a rational section of $\mathscr{L}$ (resp. $\mathscr{L}'$), such that their divisors are $(s) = E$ and $(s') = E'$. Replacing s with fs, where f is a rational function, yields the relation

$$\langle fs, s' \rangle = f(E')\, \langle s, s' \rangle \,,$$

and similarly if we replace s' by gs'. Consistency is ensured by the Weil reciprocity property $f((g)) = g((f))$ [13].

If $\mathscr{L}$ (resp. $\mathscr{L}'$) is equipped with a smooth hermitian metric ρ (resp. ρ') over X, the complex line $\langle \mathscr{L}, \mathscr{L}' \rangle$ acquires a hermitian metric given by ([10]):

$$\begin{aligned} \log \|\langle s, s' \rangle\|^2 = \frac{1}{2\pi\sqrt{-1}} \int_X \partial\bar{\partial} \log\|s\|^2 \, \log\|s'\|^2 \\ + \log\|s\|^2[E'] + \log\|s'\|^2[E] \,. \end{aligned} \tag{5.1}$$

Since there is no danger of confusion, we have denoted all the square norms corresponding to the various metrics by $\| \ \|$. The operator $\partial\bar{\partial}$ is to be interpreted in the sense of distributions.

Alternatively, let f_1, f_2 be two smooth $\mathbf{R}_{\geq 0}$-valued functions such that $f_1 + f_2 = 1$ and f_1 (resp. f_2) vanishes in a neighborhood of the support of E' (resp. E). (It follows that f_1 (resp. f_2) is equal to 1 near the support of E (resp. E').) Then (5.1) can be re-expressed as:

$$\begin{aligned} \log \|\langle s, s' \rangle\|^2 &= \frac{\sqrt{-1}}{\pi} \int_X f_1 \, c_1(\rho) \, \log\|s'\| + \frac{\sqrt{-1}}{\pi} \int_X f_2 \log \|s\| \, c_1(\rho') \\ &+ \frac{\sqrt{-1}}{\pi} \int_X \log \|s\| \; \mathrm{d} f_2 \wedge \mathrm{d}^c \log \|s'\| \\ &+ \frac{\sqrt{-1}}{\pi} \int_X \log \|s'\| \; \mathrm{d} f_1 \wedge \mathrm{d}^c \log \|s\| \,, \end{aligned} \tag{5.2}$$

a variation of [10, 6.5.1], due to O. Gabber. Observe that the integrations above make sense as ordinary smooth differential forms, since, as a consequence of the assumptions on f_1, f_2, both $\mathrm{d}f_1$ and $\mathrm{d}f_2$ vanish in a neighborhood of $\operatorname{supp} E \cup \operatorname{supp} E'$.

5.2 Pairing for good line bundles

Let us now include the divisor $D = p_1 + \cdots + p_N$ in the picture. Accordingly, $\bar{\mathscr{L}}$ and $\bar{\mathscr{L}}'$ are good hermitian line bundles on X along D. Recall $X_\varepsilon = X \setminus \bigcup_{i=1}^N \Delta_\varepsilon(p_i)$. The 2-forms appearing in the integrand on the right hand side of (5.2) are good forms along D, namely they belong to $E^2_X(1)\langle\langle D \rangle\rangle_{good} \overset{\text{def}}{=} \Gamma(X, \mathscr{E}^2_X(1)\langle\langle D \rangle\rangle_{good})$.

Definition 5.1. With X, D, $\bar{\mathscr{L}}$ and $\bar{\mathscr{L}}'$ as above, set

$$\log \|\langle s, s' \rangle\|^2 = \lim_{\varepsilon \to 0} \int_{X_\varepsilon} \big(\text{Integrand 2-form in RHS of (5.2)}\big)$$

The following is a direct generalization of [2, Thm. 5.5] and [7, Thm. 7.3].

Theorem 5.2. *The cup-product in Hermitian-holomorphic Deligne cohomology with growth (? = good or pre) extends the the norm on the determinant of cohomology line to good hermitian line bundles. Namely, let ω be a 2-form corresponding to $[\bar{\mathscr{L}}] \cup [\bar{\mathscr{L}}']$ as in Lemma 4.4. Then*

$$\lim_{\varepsilon \to 0} \frac{\sqrt{-1}}{\pi} \int_{X_\varepsilon} \omega = \log \|\langle \mathbf{1}, \mathbf{1} \rangle\|^2 \tag{5.3}$$

where the right hand side of (5.3) *is interpreted in the sense of definition 5.1, and* $\mathbf{1}$ *is the rational section of* $\mathscr{L}$ *and* $\mathscr{L}'$ *induced by the function* 1.

Proof. We will proceed as in the proof of the corresponding result in ref. [2].

Let again E and E' the divisors of $\mathscr{L}$ and $\mathscr{L}'$. Recall their supports are disjoint. Hence we have an open cover $\mathfrak{U}_X = \{U_1, U_2\}$, where $U_1 = X \setminus \operatorname{supp} E'$ and $U_2 = X \setminus \operatorname{supp} E$. Moreover, choosing f_1, f_2 as in sect. 5.1 above, yields a partition of unity subordinated to the cover $\mathfrak{U}_X$.

Since $\mathfrak{U}_X$ has two elements only, the only non-zero terms of the cocycle in the proof of Lemma 4.4 representing $[\bar{\mathscr{L}}] \cup [\bar{\mathscr{L}}']$, are ω_i^0 and ω_{ij}^1, where the index i takes the values $1, 2$. Furthermore, it is immediately verified that the 2-form

$$\omega = f_1\,\omega_1^0 + f_2\,\omega_2^1 + \mathrm{d} f_2 \wedge \omega_{21}^1 \tag{5.4}$$

is a well-defined element of $E_X^2(1)\langle\langle D\rangle\rangle_?$ and represents the same cohomology class. Note that here ? = *good and pre*.

Now, let s and s' be rational sections as in 5.1. With respect to the cover $\mathfrak{U}_X$ they correspond to two pairs of rational functions, $s = \{s_1, s_2\}$, and $s' = \{s'_1, s'_2\}$. In fact s'_1 and s_2 are invertible in their respective domains of definition. Let us now actually use that s and s' are the rational section $\mathbf{1}$, so that $s_2 = 1$ and $s'_1 = 1$. With these choices we have:

$$\begin{aligned}
\omega_1^0 &= \frac{1}{2}\log\rho_1\,\mathrm{d}\mathrm{d}^c\log\|s'\| \\
\omega_2^0 &= \log\|s\|\, c_1(\rho') \\
\omega_{21}^0 &= \log|s_1|\;\mathrm{d}^c\log\|s'\| - \mathrm{d}^c\log|s_1|\;\log\|s'\|
\end{aligned}$$

Then simple integrations by parts analogous to those in [2, Thm. 5.5] yield:

$$\begin{aligned}
\int_{X_\varepsilon}\omega &= \int_{X_\varepsilon}\big(\text{Integrand 2-form in RHS of (5.2)}\big) \\
&\quad + \frac{1}{4}\int_{\partial X_\varepsilon} f_1\left(\log\rho_1\;\mathrm{d}^c\log\rho'_1 - \log\rho'_1\;\mathrm{d}^c\log\rho_1\right).
\end{aligned} \tag{5.5}$$

Since the metrics are good, it easily checked that each term in the second integral in (5.5) has an estimate of the type:

$$\int_0^{2\pi}\left|f_1(\varepsilon e^{\sqrt{-1}t})\right|\left|\alpha(\varepsilon e^{\sqrt{-1}t})\right|\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\,\mathrm{d}t$$

where α is bounded (and so is f_1). Hence, the second integral in (5.5) goes to zero as $\varepsilon \to 0$, as wanted. □

6 Extremal hyperbolic metrics

6.1 Preliminaries

In this section we consider in more detail the setup of example 3.6: recall that $\mathrm{d}s^2_{\mathsf{hyp}}$ is a hyperbolic metric on $U = X \setminus D$ satisfying the condition (3.5). The good extension of T_U to X is $\mathscr{L} = T_X(-D)$. Thus we denote by $\bar{\mathscr{L}}_{\mathsf{hyp}}$ the pair $(\mathscr{L}, \mathrm{d}s^2_{\mathsf{hyp}})$.

More generally, we consider the space of good conformal metrics $\mathrm{d}s^2$ on X. Namely $\mathrm{d}s^2$ is a conformal metric on U locally written as $\mathrm{d}s^2 = \rho\,|\mathrm{d}z|^2$, where ρ satisfies the same asymptotic conditions (3.5). From those conditions it follows that $\mathrm{d}s^2$ is good. We denote $\mathscr{CM}(X)_{good}$ the space of good conformal metrics on X, and $\bar{\mathscr{L}}$ will denote the pair $(\mathscr{L}, \mathrm{d}s^2)$.

Note that $\mathscr{CM}(X)_{good}$ is an affine space over $\Gamma(X, \mathscr{E}_X^0)$. Namely, if $\mathrm{d}s^2$ and $\mathrm{d}s'^2 \in \mathscr{CM}(X)$, there exists a smooth function $\sigma : X \to \mathbf{R}$ such that $\mathrm{d}s'^2 = e^\sigma\,\mathrm{d}s^2$. The hyperbolic metric $\mathrm{d}s^2_{\mathsf{hyp}}$ belongs to $\mathscr{CM}(X)_{good}$.

Let V be an adapted neighborhood of $p \in D$, that is, a disk Δ_a of radius $a > 0$ centered at p, where we write $\mathrm{d}s^2 = \rho\,|\mathrm{d}z|^2$. Using the non vanishing section $s = z\partial/\partial z$ of $\mathscr{L}|_V$ let us also set:

$$\tilde\rho(z) \stackrel{\text{def}}{=} \|s\|^2 \equiv \mathrm{d}s^2\Big(z\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}\Big), \tag{6.1}$$

so that $\rho = \tilde\rho/|z|^2$ and $\tilde\rho$ has the asymptotic behavior:

$$\log\tilde\rho(z) = -\log\big(\log^2|z|\big) + \alpha(z) \tag{6.2}$$

at $z = 0$. If, on the other hand, (V, z) is a neighborhood of an ordinary point $p \in U$, then we simply set $\tilde\rho = \rho$ and $s = \partial/\partial z$.

Finally, let ω_ρ be the Kähler form associated to $\mathrm{d}s^2$. Locally we have

$$\omega_\rho = \frac{\sqrt{-1}}{2}\rho\,\mathrm{d}z \wedge \mathrm{d}\bar z\,,$$

and the associated *area* is simply the integral $A_\rho(X) = \int_X \omega_\rho$.

6.2 The Liouville equation

The hyperbolic metric $\mathrm{d}s^2_{\mathsf{hyp}}$ whose scalar curvature is equal to -1 is of course of particular importance from the point of view of uniformization theory. With respect to a local coordinate z the constant negative curvature condition translates into the nonlinear PDE

$$\frac{\partial^2 \log \rho_{\mathsf{hyp}}}{\partial z \partial \bar{z}} = \frac{1}{2}\rho_{\mathsf{hyp}}\,, \tag{6.3}$$

known as the *Liouville equation,* which must be supplemented by the asymptotic condition (3.5) at the points of D. The equation can be recast in the more invariant fashion:

$$c_1(\rho_{\mathsf{hyp}}) = \sqrt{-1}\,\omega_{\rho_{\mathsf{hyp}}}\,. \tag{6.4}$$

Note that $c_1(\rho_{\mathsf{hyp}}) = c_1(\tilde{\rho}_{\mathsf{hyp}})$, and the latter computes $c_1(\mathscr{L})$.

It is well-known that (6.3) is the extremum of an energy-type functional. A global construction for eq. (6.4) was given in [18] for curves of type $(0,n)$ (with $n \geq 3$) and in [19, 16] for curves of type $(g,0)$ (with $g \geq 2$), and more generally, in the latter case, in [2].

The following is a direct generalization of [2, Thm. 5.1 and Cor. 5.6]:

Theorem 6.1. *Denote by* $\log \|\langle \bar{\mathscr{L}}, \bar{\mathscr{L}}\rangle\|$ *the integral of the good* 2*-form determined by* $[\bar{\mathscr{L}}]\,\hat{\cup}\,[\bar{\mathscr{L}}]$*, as in Theorem 5.2 and consider the quantity:*

$$S(\mathrm{d}s^2) \stackrel{\text{def}}{=} \log \|\langle \bar{\mathscr{L}}, \bar{\mathscr{L}}\rangle\| + \frac{1}{2\pi} A_\rho(X)\,. \tag{6.5}$$

The metric $\mathrm{d}s^2$ *satisfies equation* (6.4) *(that is:* $\mathrm{d}s^2 = \mathrm{d}s^2_{\mathsf{hyp}}$*) if and only if:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} S(e^{t\sigma}\,\mathrm{d}s^2) = 0\,,$$

for any $\sigma \in \Gamma(X, \mathscr{E}^0_X)$.

Proof. First, fix an open cover $\mathfrak{U}_X = \{V_i\}_{i\in I}$ with adapted neighborhoods. Let z_i be the corresponding coordinate on V_i. (We will drop the Čech index i if there is no danger of confusion.) Assume the same conditions on the cover and the integration map explained in sect. 4.1.3 before Lemma 4.5.

Lemma 6.2. *If* $\mathscr{L}$ *and* $\mathrm{d}s^2$ *are trivialized by sections* s_i *on* V_i *as in sect. 6.1, the resulting cocycle belongs to* $\operatorname{Tot}^2 \check{\mathrm{C}}^\bullet(\mathfrak{U}_X, \mathfrak{D}_{h.h.}(1)^\bullet_{X,?})$*, with* ? = *good or pre.*

Proof. This follows at once from the definitions. □

Using ? = *pre*, the cup square in pre-log-log Hermitian holomorphic Deligne cohomology has the form (4.1) (with $\rho' = \rho$), and let again Ω be the cocycle representing the resulting image of the class $[\bar{\mathscr{L}}]\,\hat{\cup}\,[\bar{\mathscr{L}}]$ in $\mathbb{H}^2(X, \mathscr{E}^\bullet_X(1)\langle\langle D\rangle\rangle_?)$ as per Lemma 4.4. Using sect. 4.1.3 and Lemma 4.5, eq. (6.5) is written as:

$$S(\mathrm{d}s^2) = \frac{\sqrt{-1}}{2\pi}\langle \Omega, \Sigma_{\mathbf{T}}\rangle + \frac{1}{2\pi} A_\rho(X)$$

In order to carry out the calculation of $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} S(e^{t\sigma}\,\mathrm{d}s^2)$, we can use the arguments of [2, 16] provided (4.5) is taken into account when integrating by parts. Since the calculation is the same we will not reproduce it here. As a result, we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} S(e^{t\sigma}\,\mathrm{d}s^2) &= \frac{\sqrt{-1}}{2\pi}\int_X \sigma\Big(c_1(\mathscr{L}) - \sqrt{-1}\omega_\rho\Big) \\ &\quad - \frac{\sqrt{-1}}{8\pi}\lim_{\varepsilon\to 0}\sum_{p_i\in D}\int_{C_\varepsilon(p_i)}\big(\log\tilde{\rho}_i\,\mathrm{d}^c\sigma - \sigma\,\mathrm{d}^c\log\tilde{\rho}_i\big)\,,\end{aligned} \tag{6.6}$$

where $C_\varepsilon(p_i)$ is a circle of radius ε around the point p_i. The first integral in eq. (6.6) must be understood in the sense of eq. (4.3). The second integral is treated as the analogous one at the end of the proof of Theorem 5.2. Then its limit as $\varepsilon \to 0$ is zero. This finishes the proof. □

Remark 6.3. The use of explicit correction terms, as in ref. [18], is not needed in the present framework. Notice that on an adapted neighborhood V the section $\partial/\partial z$ ought to be considered as a *rational* section of $\mathscr{L}|_V$: we have $\partial/\partial z = z^{-1}s$. In the language of ref. [9] the corresponding forms will be *mixed,* that is they present both *log-log* and *log*-type singularities, thereby losing local integrability.

The following lemma provides a precise comparison. Let $p \in \operatorname{supp} D$ and let us identify the adapted neighborhood (V, z) with a disk Δ_a of radius a centered at $z = 0$. Let $A(\varepsilon, a) = \Delta_a \cap X_\varepsilon$ be the resulting annulus of radii a and ε. Recall that $\rho(z) = \|\partial/\partial z\|^2$ and $\tilde{\rho}(z) = \|s\|^2$. Then we have:

Lemma 6.4.

$$\int_{A(\varepsilon,a)} \frac{1}{2} \log\tilde{\rho}\, \bar{\partial}\partial \log\tilde{\rho} = \int_{A(\varepsilon,a)} \frac{1}{2} \partial\log\rho \wedge \bar{\partial}\log\rho - 2\pi\sqrt{-1}\log\varepsilon - 2\pi\sqrt{-1}\log(\log^2\varepsilon) + O(1)\,.$$

Proof. One has the identity

$$\log\tilde{\rho}\,\bar{\partial}\partial\log\tilde{\rho} = \partial\log\rho \wedge \bar{\partial}\log\rho + \frac{1}{2}\,\mathrm{d}\Big\{\log\tilde{\rho}\;\mathrm{d}^c\!\log\tilde{\rho} + \log|z|^2\,\mathrm{d}^c\!\log|z|^2 - 2\log\tilde{\rho}\;\mathrm{d}^c\!\log|z|^2\Big\}\,.$$

The estimate on the boundary term follows from a direct calculation and the fact that $\log\tilde{\rho}\;\mathrm{d}^c\!\log\tilde{\rho}$ is a good form. ∎

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