BUTTERFLIES I: MORPHISMS OF 2-GROUP STACKS

by

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Abstract. — Weak morphisms of non-abelian complexes of length 2, or crossed modules, are morphisms of the associated 2-group stacks, or gr-stacks. We present a full description of the weak morphisms in terms of diagrams we call butterflies. We give a complete description of the resulting bicategory of crossed modules, which we show is fibered and biequivalent to the 2-stack of 2-group stacks. As a consequence we obtain a complete characterization of the non-abelian derived category of complexes of length 2. Deligne's analogous theorem in the case of Picard stacks and abelian sheaves becomes an immediate corollary. Commutativity laws on 2-group stacks are also analyzed in terms of butterflies, yielding new characterizations of braided, symmetric, and Picard 2-group stacks. Furthermore, the description of a weak morphism in terms of the corresponding butterfly diagram allows us to obtain a long exact sequence in non-abelian cohomology, removing a preexisting fibration condition on the coefficients short exact sequence.

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1. Introduction

The notion of n-group or, loosely speaking, n-categorical group is by now quite established in mathematics. This paper is the first of a series aimed at a systematic study of the n-category of n-groups. We do it in a geometric fashion, working with n-groups over a general Grothendieck site, therefore we should rather be saying that we are studying the n-stack of n-groups. (This is a bit imprecise, though, and it will be appropriately qualified later, especially concerning laxness of the various n-categorical constructions we consider.) Torsors over n-groups are also included in our study.

The n-categorical aspect is emphasized, since what is of fundamental importance are the morphisms (1-morphisms and higher) between n-groups. In particular, while it is relatively harmless to consider n-groups as "strict," in the categorical sense, we now understand it is not so when dealing with their morphisms. In other words, *weak* morphisms are the important ones—and they cannot be made strict. Thus one of our main points is to characterize precisely and provide an explicit and—we believe—very manageable construction of these weak morphisms: the butterflies of this series' title. In doing so, we are also in position to obtain a very complete and concrete description of *torsors* over said n-groups, in particular concerning the functorial aspects, where previously only strict morphisms were considered.

To explain things in slightly more detail, it is useful to do it in the case $n = 2$, which is covered in the present paper and its sequel [**AN08a**], which deals with the torsors, whereas the situation for $n = 3$ and higher is dealt with in [**AN08b, AN08c**]. The reader shall be aware that many remarks and observations apply to higher n's as well.

What we do in this paper is present a general framework for morphisms of 2-groups on a Grothendieck site. It is roughly divided in two parts. The first, more general part deals with the 2-categorical aspects; the second, discusses what may even be called applications of the theory of morphisms of 2-groups built in the first part. We discuss exact sequences of 2-groups, the long exact sequence in non-abelian cohomology, and devote some space to 2-groups in abelian categories, that is Picard groupoids, as in [**Del73**]. This is the fundamental motivating example, and in a sense, one could characterize the present (and similar) investigations as quest for a geometric approach to the (non-abelian) derived category in the same spirit as loc. cit.

1.1. General beginning remarks. — First, a question of terminology: in this introduction, and when used in the rest of the paper, the term "2-group" without further qualification usually means "2-group stack." This is (should be?) a modern substitute for the term "gr-stack." Over a point, a gr-stack becomes a gr-category, and this is also called a 2-group (over a point). Not having been able to complete the transition ourselves, we too pervasively use the terms gr-stack and gr-category in the main body.

Whichever way it is called, a 2-group is usually by nature a lax object, in the sense that its algebraic operations involve higher coherences; for example, multiplication is only associative up to coherent isomorphism. This certainly the case for 2-groups *stacks* over a site.

Over a point, it has been known for quite some time (see [**Sí75**]) that a 2-group $\mathscr{G}$ can always be made strict. This means that $\mathscr{G}$ is equivalent, as a category, to some G such that for the latter the group operations hold with equalities, and not just up to isomorphism, and the equivalence is an additive functor. Standard arguments then imply that G is the strict gr-category determined by—and determining—a crossed module

$$G^\bullet \colon G^{-1} \xrightarrow{\delta} G^0 .$$

$G^\bullet$ is the reduced Moore complex obtained from the nerve $\underline{G}_\bullet = \mathrm{N}_\bullet \mathsf{G}$, which is a simplicial group. Over a site S something very similar can be achieved, namely it is possible to find a (sheaf of) crossed module(s) $G^\bullet$ such that $\mathscr{G}$ is now equivalent to the stack associated to the (sheaf of) groupoid(s) G—so this is a prestack—determined by $G^\bullet$. This is certainly well known, but since it plays an important rôle in our arguments throughout, we have decided to provide an explicit statement with proof (see 5.3.7).

On the other hand, when we turn our attention to morphisms between 2-groups, it is not possible to make them strict, and this is already true—and well known—in the set-theoretic case, that is, over a point. To express it in a more precise way, let $F\colon \mathscr{H} \to \mathscr{G}$ be an additive functor between 2-groups. Assuming $\mathscr{H}$ and $\mathscr{G}$ are equivalent to strict 2-groups H and G, in general F will not be isomorphic to a strict morphism $\mathsf{H} \to \mathsf{G}$ or, to put it in a different but equivalent way, cannot be realized as a morphism of crossed modules $H^\bullet \to G^\bullet$. Thus we must grapple with the problem that the groupoid $\mathscr{H}\!om(\mathscr{H},\mathscr{G})$ of additive functors is much larger than that of strict morphisms from H to G. It follows that in order to work with strict 2-groups and still retain all the features of the 2-category of 2-groups, one must find a "derived" version $\mathbf{R}\,\mathsf{Hom}(\mathsf{H},\mathsf{G})$ of the groupoid of strict morphisms from H to G. The requirement is that this new groupoid be equivalent to $\mathscr{H}\!om(\mathscr{H},\mathscr{G})$. It objects are the *weak morphisms* from H to G.

In the set-theoretic case, this problem was solved by the second-named author in refs. [**Noo07, Noo05**]. The existence of $\mathbf{R}\,\mathsf{Hom}(\mathsf{H},\mathsf{G})$ was established by methods of homotopical algebra. Then, a very concrete description of it was given in terms of group diagrams called *butterflies.*

A butterfly from $H^\bullet$ to $G^\bullet$ is a diagram of the form

$$(*)\qquad \begin{tikzcd} H^{-1} \arrow[dd] \arrow[dr] & & G^{-1} \arrow[dl] \arrow[dd] \\ & E \arrow[dl] \arrow[dr] & \\ H^0 & & G^0 \end{tikzcd}$$

where the NW-SE sequence is a complex, the NE-SW sequence is exact, i.e. a group extension, plus some other conditions which will be explained later. There is also a notion of morphism of butterflies: it is induced by a group isomorphism $E \xrightarrow{\sim} E'$ compatible with the rest of the various maps, so one has a groupoid. Strict morphisms corresponds to butterfly diagrams whose NE-SW sequence is split, with the group at the center equal to the semi-direct product $H^0 \ltimes G^{-1}$.

Incidentally, the adjective "derived" used earlier is not entirely out of place: an equivalent formulation of the butterfly is that a weak morphism can be realized as a triangle

$$
\begin{tikzcd}
 & {[H^{-1} \times G^{-1} \to E]} \arrow[dl, "\sim"'] \arrow[dr] & \\
{[H^{-1} \to H^0]} & & {[G^{-1} \to G^0]}
\end{tikzcd}
\tag{+}
$$

where each arrow is a strict morphism of crossed modules, and the left one is a quasi-isomorphism (it preserves homotopy groups).

Since butterflies can be nicely composed by juxtaposition, one obtains a nice bicategory of strict 2-groups and weak morphisms which carries the right kind of homotopical information.

The actual choice of the definition of weak morphism is discussed in section 4.2, where it is compared to the topological one given in refs. [**Noo07, Noo05**]. One may also wonder whether it is possible to further weaken the notion of weak morphism as given here. We observe (still in section 4.2) that it is related to a categorification of Morita's theory and is, however, essentially a theory of 2-stacks. Furthermore it leads to non-additive functors.

1.2. The content of the paper. — The following is a rather long discussion of the main ideas in this paper. For a quick glance at the content's description, the reader is invited to read section 1.3 first.

1.2.1. Butterflies and 2-groups. — The first part of this paper, and in part its immediate sequel [**AN08a**], center on the same question of finding a correct model for the 2-category of 2-group stacks: we want to strictify the objects, but retain an accurate information on the morphisms. At a minimum, the "model" in question is a bicategory, whose objects are the crossed modules over the site, equipped with a biequivalence with the 2-category of 2-groups. More is true when working over a general site, as we now explain.

First, let us be clear about the 2-category of 2-groups, let us denote it by $\underline{\mathsf{Gr\text{-}Stacks}}(\mathsf{S})$. This is an honest 2-category, and it is a sub-2-category of the 2-category $\underline{\mathsf{Stacks}}(\mathsf{S})$. Moreover, considering the site S/U, for each object $U \in \mathsf{S}$, and the 2-category $\underline{\mathsf{Gr\text{-}Stacks}}(\mathsf{S}/U)$ yields a fibered 2-category over the site S (in the sense of [**Hak72**]). It turns out this fibered 2-category, denoted $\textsc{Gr-Stacks}(\mathsf{S})$, actually is a 2-stack over S. This fact is certainly well known to experts, but, being unable to find a published account of it, we have included it here (cf. Appendix A).

We have already observed that given a 2-group $\mathscr{G}$ over S we can find a crossed module $G^\bullet$ such that its associated stack is equivalent to $\mathscr{G}$ (cf. Proposition 5.3.7). It is convenient to denote by $[G^{-1} \to G^0]^\sim$ the stack associated to $G^\bullet$. Note that the groupoid G determined by $G^\bullet$ is in general only a prestack: in going from the 2-group to the crossed module, the price we pay is to lose the stack condition, namely the gluing conditions on objects.

With the obvious changes, the notion of butterfly diagram (*) still makes sense over S, as well as that of morphism of butterflies. Let $\mathsf{B}(H^\bullet, G^\bullet)$ be the groupoid of butterflies from $H^\bullet$ to $G^\bullet$. In the main text we prove (cf. Theorem 4.3.1):

Theorem. — *There is an equivalence*

$$\mathsf{B}(H^\bullet, G^\bullet) \xrightarrow{\sim} \mathsf{Hom}_{\underline{\mathsf{Gr\text{-}Stacks}}(\mathsf{S})}(H^{\bullet\sim}, G^{\bullet\sim}).$$

The right-hand side is more or less by definition the groupoid of *weak morphisms* from $H^\bullet$ to $G^\bullet$. Moreover, both sides have fibered analogs

$$\mathscr{B}(H^\bullet, G^\bullet)\,, \quad \mathscr{H}om_{\textsc{Gr-Stacks}(\mathsf{S})}(H^{\bullet\sim}, G^{\bullet\sim})$$

over S which are stacks (in groupoids).

For three crossed modules $K^\bullet$, $H^\bullet$, and $G^\bullet$ there is a non-associative composition

$$\mathsf{B}(K^\bullet, H^\bullet) \times \mathsf{B}(H^\bullet, G^\bullet) \longrightarrow \mathsf{B}(K^\bullet, G^\bullet),$$

again given by juxtaposition, and similarly for $\mathscr{B}(-,-)$ replacing $\mathsf{B}(-,-)$. Combining with the fact that for every 2-group $\mathscr{G}$ we can find a crossed module $G^\bullet$ such that there is an equivalence $\mathscr{G} \simeq G^{\bullet\sim}$, we obtain the following statement (roughly corresponding to Theorem 5.3.6):

Theorem. — *There is a bicategory $\mathfrak{XMod}(\mathsf{S})$ fibered over S whose objects are crossed modules, 1-morphisms are butterflies, and 2-morphisms are morphisms of butterflies. Moreover $\mathfrak{XMod}(\mathsf{S})$ is a bistack over S and the correspondence $G^\bullet \to G^{\bullet\sim}$ induces a biequivalence*

$$\mathfrak{XMod}(\mathsf{S}) \xrightarrow{\sim} \textsc{Gr-Stacks}(\mathsf{S}).$$

The right-hand side above is a genuine 2-stack, i.e. it is fibered in 2-categories, which is considered as being fibered in bicategories in the obvious way.

This theorem has the rather striking consequence that crossed modules can be glued relative to 2-descent data formulated in terms of butterflies.

We would like to remark that the previous theorem gives us the right take on on the strict/weak question. Namely, on one hand we have strict 2-groups: they are simpler to deal with, but somewhat non geometric, in the sense that the stack condition does not hold, and therefore there is no gluing on their objects, in general. Strictness of the group law entails that they must comprise a bicategory: their morphisms compose in a non-associative way. At the same time morphisms can be described rather concretely, in terms of diagram, but observe that they necessarily cannot be functors relative to the (strict) group law. On the other hand, weak 2-groups, that is 2-group stacks, are seemingly more complicated, but they are the true geometric, in that their objects glue and can be given a description in terms of torsors. Weakness of the group law allows them to collectively form a genuine 2-category: morphisms between weak 2-groups are *functors* relative to the weak group law, and therefore they obviously compose in an associative way.

1.2.2. Exact sequences and abelian categories. — There is a number of immediate applications ensuing from the notion of butterfly diagram, which are discussed in the rest of the paper. The reason for including them in this paper is that they are closer to the general theory developed in the first part of the paper. In particular, they result from the analysis of the homotopy kernel and homotopy fiber of a butterfly, that is of a weak morphism.

The motivating example is again [**Del73**]: weak morphisms of length-two complexes of abelian sheaves correspond to additive functors of Picard stacks. This allows to geometrically describe the derived category $\mathsf{D}^{[-1,0]}(\mathsf{S})$ of complexes of abelian sheaves over S whose homology is concentrated in degrees $[-1,0]$.

Deligne's constructions become a special case of those of sections 4 and 5, when they are specialized to the abelian category of abelian sheaves over S. In this case, the objects are complexes $G^{-1}\to G^0$ of abelian sheaves without further qualifications. The associated stack $[G^{-1}\to G^0]^\sim$ is Picard. Since the butterfly diagram from $H^\bullet$ to $G^\bullet$ is a more or less canonical representation of a weak morphism, i.e. an additive functor of Picard stacks $[H^{-1}\to H^0]^\sim \to [G^{-1}\to G^0]^\sim$, its cone in the derived category $\mathsf{D}(\mathsf{S})$ (the cone is no longer in $\mathsf{D}^{[-1,0]}(\mathsf{S})$ is visible in the butterfly: it is the NW-SE diagonal of (*).

In the non-abelian setting one should rather be using the homotopy fiber construction, as it was done in [**Noo05**] in the set-theoretic context. Over a general site S, the corresponding construction is the following (cf. Theorem 6.3.10).

Theorem. — *Let $F^\bullet\colon H^{-1}\to E\to G^0$ be the complex in degrees $[-2,0]$ corresponding to the NW-SE diagonal of (*). There is a 2-stack $\mathfrak{F}$ over S associated to $F^\bullet$ whose homotopy groups in the sense of* [**Bre94**] *are*

$$\pi_i(\mathfrak{F}) = \mathrm{H}^{-i}(F^\bullet)\,,\quad i=0,1,2,$$

and fit into the expected exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \pi_2(\mathfrak{F}) \arrow[r] & \pi_1(\mathscr{H}) \arrow[r] & \pi_1(\mathscr{G}) \arrow[r] & \pi_1(\mathfrak{F}) \arrow[dllll, rounded corners] \\ \pi_0(\mathscr{H}) \arrow[r] & \pi_0(\mathscr{G}) \arrow[r] & \pi_0(\mathfrak{F}) \arrow[r] & 1 \end{tikzcd}$$

The 2-stack $\mathfrak{F}$ is part of the homotopy fiber sequence of 2-stacks over S:

$$\mathscr{H}[0] \xrightarrow{F[0]} \mathscr{G}[0] \xrightarrow{\iota} \mathfrak{F} \xrightarrow{\Delta} \textsc{Tors}(\mathscr{H}) \xrightarrow{F_*} \textsc{Tors}(\mathscr{G}).$$

This provides an imperfect analog of the exact triangle in the non-abelian setting. The fact that $\mathfrak{F}$ is a 2-stack corresponds to the fact that in the (abelian) derived category situation by taking the cone we are now dealing with complexes of length three. More serious is the fact that $\mathfrak{F}$ does not admit a group law, in general.

A special case occurs when $\pi_0(\mathfrak{F}) = *$. In this case the butterfly (*) corresponds to an essentially surjective morphism $\mathscr{H} \to \mathscr{G}$. The homotopy fiber of such a morphism is correspondingly simpler

$$\mathfrak{F} \simeq \mathrm{TORS}(\mathscr{K}),$$

where $\mathscr{K}$ is the homotopy kernel of the morphism. It is itself a 2-group stack and it has an explicit characterization in terms of the group-objects occurring in the butterfly; namely we have

***Proposition* (6.1.1)**. — *The homotopy kernel of* (*) *is equivalent to the 2-group stack associated to the crossed module*

$$H^{-1} \longrightarrow \ker(E \to G^0).$$

Thus the butterfly corresponds to the second morphism of an extension of 2-group stacks:

$$\mathscr{K} \longrightarrow \mathscr{H} \longrightarrow \mathscr{G}, \tag{**}$$

as in [**Bre92**]. Unlike loc. cit., we do *not* require that the second morphism be a fibration. Thanks to (+), $\mathscr{H}$ can be replaced with an equivalent $\mathscr{E}$. The morphism $\mathscr{E} \to \mathscr{G}$ *is* a fibration. Appealing to a former results of Breen ([**Bre90**], which uses the fibration condition) we are able to conclude that the short exact sequence (**) without the fibration condition still induces the long exact sequence in non-abelian cohomology (Proposition 6.4.1).

We discuss the relation with Deligne's constructions in [**Del73**, §1.4] in detail in section 8. The general idea is that in an abelian category a crossed module is simply a complex of length 2, therefore all constructions carry over, by simply forgetting most of the action requirement.

If $A^\bullet\colon A^{-1} \to A^0$ is such a complex, its associated stack is Picard. This is a very strong commutativity condition. It is well known that on a 2-group one can impose various degree of commutativity on the monoidal operation, so that it becomes, in order of increasing specialization: (1) Braided, (2) Symmetric, (3) Picard.

One thing we do is obtain all these conditions from a special kind of butterfly diagram, which is necessarily associated to a braided 2-group $\mathscr{G}$. We argue that a braiding on the monoidal structure of $\mathscr{G}$ is tantamount to requiring that the monoidal structure itself

$$\otimes\colon \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G}$$

be an additive functor of 2-group stacks. By our theorem this must be realized by an appropriate butterfly diagram of crossed modules

$$(++)\qquad \begin{tikzcd} G^{-1}\times G^{-1} \arrow[d] \arrow[dr] & & G^{-1} \arrow[dl] \arrow[d] \\ & P \arrow[dl] \arrow[dr] & \\ G^0\times G^0 & & G^0 \end{tikzcd}$$

which must satisfy other conditions too, most notably that the extension given by the NE-SW diagonal be split (with a fixed splitting) when restricted to either factor. We can go as far as *defining* a crossed module to braided if it admits such a structure. This is reasonable in view of the following

***Proposition* (7.1.8)**. — *A 2-group stack $\mathscr{G}$ is braided if and only if a (hence, any) corresponding crossed module $G^\bullet$ is braided in the standard sense by a braiding map $\{-,-\}\colon G^0\times G^0\to G^{-1}$, if and only if it is braided in the sense of admitting a butterfly diagram such as* (++).

The symmetry and Picard conditions on the braiding structures can be described in an entirely similar fashion. Namely, let $T\colon \mathscr{G}\times\mathscr{G}\to\mathscr{G}\times\mathscr{G}$ be the swap morphism, and let $\Delta\colon\mathscr{G}\to\mathscr{G}\times\mathscr{G}$ be the diagonal morphism. We have:

***Proposition* (7.2.3 and 7.3)**. —

- *The braiding on $\mathscr{G}$ is symmetric if and only if* (++) *is isomorphic to its own pullback by T;*
- *The braiding is furthermore Picard if in addition the induced isomorphism on the butterfly pulled back by Δ is the identity.*

There is of course a notion of *braided butterfly* between braided 2-groups which expresses the fact that the corresponding morphism is a morphism of braided objects.

We are therefore able to approach Picard stacks along two slightly different lines: as a direct byproduct of the general theory via the additional properties imposed by the Picard condition, as explained above, or as a special repetition of the general theory for an abelian category.

An interesting example of the latter arises when considering extra structures, in particular the one provided by the existence of a sheaf of commutative rings on the base site. In a ringed site we can talk about *modules,* and in particular about locally free ones. As an application we show that in such situation all butterflies among complexes of locally free modules are themselves given by locally free objects, and, more importantly, they are always *locally split,* in the sense that upon restricting to a suitable cover they split and correspond to a strict morphism. This culminates the discussion in section 8.

1.3. Organization of the paper. — This paper is organized as follows. Sections 2 and 3 collect a number of facts, notions, and propositions concerning the formalism of (hyper)covers and descent data, stackification, gr-categories, gr-stacks, and crossed modules. With it we have made an attempt at making the paper somewhat self-contained and at easing the reader's task in hunting down the various needed results from the literature. The idea is for the reader to refer back to them as needed. We have broken this rule for facts concerning 2-stacks, for which we entirely rely on the existing references, with the possible exception of some elementary facts concerning 2-descent data. Section 3, in particular, recalls several results of ref. [**Bre90**], which we have reviewed in some detail, also due to the use of different conventions.

In section 4 we define weak morphisms, butterflies, and prove the main equivalence theorem. Then in section 5 we describe the bicategory of crossed modules, showing it is bi-equivalent to the 2-stack of 2-group stacks.

Sections 6, 7, and 8 are devoted to applications. In section 6 we reexamine the notion of exact sequence of 2-groups and obtain the long exact sequence in non-abelian cohomology without the fibration assumption. We also describe the homotopy fiber. This section requires more background (especially on 2-stacks) than the rest of the present work. Section 7 is devoted to the various commutativity laws we can impose on a 2-group stack. In particular, for a braided 2-group stack, we obtain the braiding bracket directly from the butterfly expressing the fact that the multiplication law is a morphism of 2-groups. We believe this is new even in the set-theoretic case. In section 8, we discuss the connection with Deligne's results, in particular theorem in ref. [**Del73**] becomes a corollary of our main theorem in sect. 4. We terminate the discussion with an exposition of the theory for modules in a ring site, devoting special attention to locally split butterflies, in section 8.6.

1.4. Conventions and notations. — We will work in the context of sheaves, stacks, etc. defined on a site S. It will be convenient to introduce the associated topos T of sheaves on S, and to say that F is an object (respectively, group object) of T, rather than specifying that F is a sheaf of sets (respectively, groups) on S. In a similar vein, we will usually adopt an "element" style notation by silently employing the device of identifying objects of S with the (pre)sheaves they represent, thereby identifying them with objects of T, as per more or less standard practice. Apart from this, we will not use the properties of T as an abstract topos in any significant manner.

We have tried not not make the paper dependent on specific hypotheses on the nature of the underlying site S. We have also tried to refrain from making cocycle-type arguments too prominent. When we did have to run these type of arguments, we used generalized covers and hypercovers. Using hypercovers does not lead to a complication of the formalism, provided the right simplicial one is used from the start.

As a general rule, objects of the underlying site S are denoted by capital letters: U, V, etc. Same for generalized covers, and the various sheaves on S, i.e. objects of T. For categorical objects we use:

– C, D,... "generic" categories;
– $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{G}$,... fibered categories, stacks, gr-stacks;
– $\underline{\mathsf{C}}$, $\underline{\mathsf{D}}$,... "generic" 2- (or bi-)categories;
– $\mathfrak{C}$, $\mathfrak{F}$,... 2-categories, 2-stacks, fibered bicategories.

Special items, such as the stack of G-torsors, for a group G, are denoted by $\textsc{Tors}(G)$. Same if $\mathscr{G}$ is a 2-group stack: its 2-stack of torsors is denoted by $\textsc{Tors}(\mathscr{G})$. $\mathsf{S}^\wedge$ denotes the category of presheaves of sets on S.

Complexes, and in particular simplicial objects, carry a bullet for additional emphasis, so that, for example, hypercovers are usually denoted by $U_\bullet, V_\bullet, \dots$ and so on. Complexes always are cohomological, and usually placed in *negative* degrees. Except for the last section (section 8), this is not reflected in the notation: for convenience, throughout most of the paper we denote a crossed module by $G_\bullet\colon [G_1 \to G_0]$.

2. Background notions

2.1. Topology. — We will work on a fixed site S, not assumed to necessarily have fibered products. It will be assumed that the topology on S is subcanonical.

Recall that for an object U of S a *sieve* R over U is a collection of morphisms $i\colon V \to U$ of S which is best described by saying that R is a subfunctor of $U = \mathrm{Hom}_{\mathsf{S}}(-, U)$. A morphism $u\colon Y \to X$ in $\mathsf{S}^\wedge$ is a *local epimorphism,* or a *generalized cover,* if for every morphism $U \to X$ in $\mathsf{S}^\wedge$ with $U \in \mathrm{Ob}\,\mathsf{S}$ there exists a covering sieve R of U such that for each $(i\colon V \to U) \in R$ the composition $V \to U \to X$ lifts to Y. A generalized cover $u\colon Y \to X$ factors as

$$Y \longrightarrow \mathrm{Im}(u) \longrightarrow X$$

where the first map is an epimorphism (hence a generalized cover) and the second a local isomorphism. In particular, if $u\colon Y \to U$, with $U \in \mathrm{Ob}\,\mathsf{S}$, is a generalized cover, then $R = \mathrm{Im}(u)$ is a sieve which is covering by definition: it is precisely the sieve comprising morphisms $V \to U$ which lift to Y (hence the name *local* epimorphims for u). This correspondence allows to recast the axioms characterizing a Grothendieck topology by reformulating them in terms of generalized covers instead of sieves (see [**KS06**] for more details).

If $u\colon Y_\bullet \to X_\bullet$ is a simplicial morphism between simplicial objects in $\mathsf{S}^\wedge$, the modern point of view is to say that u is a *hypercover* if all the maps

$$Y_n \longrightarrow (\operatorname{cosk}_{n-1} Y)_n \times_{(\operatorname{cosk}_{n-1} X)_n} X_n$$

are generalized covers ([**Jar07**]). It is shown in ref. [**DHI04**] that this is equivalent to u being a local acyclic fibration. More classically, following the formally stated definition in loc. cit. and refs. [**AM69**] and [**MR072**, Exp. V. 7] one has:

2.1.1 Definition. — The augmented simplicial object $u\colon Y_\bullet \to U$, with $U \in \operatorname{Ob}\mathsf{S}$ is a *hypercover* if:

1. u is a local acyclic fibration in $\mathsf{S}^\wedge$ (U is regarded as a constant simplicial object), and
2. each Y_n is a coproduct of representable objects.

One sees immediately that all maps $Y_n \to (\operatorname{cosk}_{n-1} Y)_n$ and $Y_0 \to U$ are local epimorphisms.

A hypercover $Y_\bullet \to U$ is *bounded* or more precisely, p-bounded, or a p-hypercover, for an integer $p \geq 0$, if these maps are actually isomorphisms for $n \geq p$. The *Čech covers* are the hypercovers in the sense of the previous definition for which $p = 0$, that is all maps as above are isomorphisms. In general, for a morphism $u\colon Y \to X$ in $\mathsf{S}^\wedge$ we define the associated *Čech complex* to be the simplicial object $\check{\mathrm{C}}\, u$ (or $\check{\mathrm{C}}_X\, Y$) defined by:

$$(\check{\mathrm{C}}\, u)_n = \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n+1}$$

Indeed, regarding Y as a constant simplicial object in $s\mathsf{S}^\wedge$, we see that ([**AM69**])

$$\check{\mathrm{C}}\, u = \operatorname{cosk}_0 Y$$

Thus, upon considering a local epimorphism $u\colon Y \to U$ with $U \in \operatorname{Ob}\mathsf{S}$, we see that a 0-hypercover is precisely the old-fashioned Čech complex.

2.2. Descent data. — We collect here a few reminders about the formalism of descent data. We choose to formulate descent data using Čech resolutions of generalized covers and hypercovers. For this, one needs to define $\mathscr{F}(X)$ when $\mathscr{F}$ is a fibered category over S and X is an object of $\mathsf{S}^\wedge$.

2.2.1. — Let $\mathscr{F}$ be a fibered category over S. For $X \in \operatorname{Ob}\mathsf{S}^\wedge$ set

$$\mathscr{F}(X) \stackrel{\text{def}}{=} \varprojlim_{(i\colon V \to X) \in (\mathsf{S}\downarrow X)} \mathscr{F}(V) \stackrel{\text{def}}{=} \mathsf{Hom}_{\mathsf{S}}(X, \mathscr{F})$$

where $(\mathsf{S} \downarrow X)$ is the overcategory of objects of S over X via the Yoneda embedding $\mathsf{S} \to \mathsf{S}^\wedge$, and the right hand side is the category of morphisms of fibered categories. The functor $X \in \operatorname{Ob}\mathsf{S}^\wedge$ is interpreted as a fibered category over S in the standard way (see e.g. [**Gir71**]).

For completeness let us recall the explicit form for objects and morphisms in $\mathscr{F}(X)$, see, e.g. [**KS06**]. We will not need to use the formulas in the sequel.

2.2.2. — An object of $\mathscr{F}(X)$ is a family $\{x_i, \varphi_j\}$ of objects $x_i \in \operatorname{Ob}\mathscr{F}(V)$ parametrized by $(i\colon V \to X) \in \operatorname{Ob}(\mathsf{S} \downarrow X)$, and isomorphisms $\varphi_j\colon j^*x_i \xrightarrow{\sim} x_{ij}$ for each $W \xrightarrow{j} V$, such that there is a commutative diagram

$$\begin{tikzcd} k^*j^*x_i \arrow[r, "k^*(\varphi_j)"] \arrow[d, "c_{j,k}"'] & k^*x_{ij} \arrow[d, "\varphi_k"] \\ (jk)^*x_i \arrow[r, "\varphi_{jk}"'] & x_{ijk} \end{tikzcd}$$

for any composable triple $Z \to W \to V \to X$. In the above diagram the vertical arrow to the left is the "cleavage" of the fibered category $\mathscr{F}$. A morphism

$$\{x_i, \varphi_j\} \longrightarrow \{x'_i, \varphi'_j\}$$

is a family $f_i\colon x_i \to x'_i$ for each $(i\colon V \to X) \in (\mathsf{S} \downarrow X)$ such that for any $j\colon W \to V$ the diagram

$$\begin{tikzcd} j^*x_i \arrow[r, "\varphi_j"] \arrow[d, "j^*(f_i)"'] & x_{ij} \arrow[d, "f_{ij}"] \\ j^*x'_i \arrow[r, "\varphi'_j"'] & x'_{ij} \end{tikzcd}$$

commutes.

For a hypercover $u\colon Y_\bullet \to U$, we define:

2.2.3 Definition (**[AM69**, §10]). — A *descent datum* for $\mathscr{F}$ over U relative to u is given by an object $x \in \mathscr{F}(Y_0)$ and an isomorphism $\varphi\colon d_0^*x \xrightarrow{\sim} d_1^*x$ in $\mathscr{F}(Y_1)$ satisfying the cocycle condition:

$$d_1^*\varphi = d_2^*\varphi \circ d_0^*\varphi\,.$$

It turns out that for a *pre-stack* $p\colon \mathscr{X} \to \mathsf{S}$ the categories of descent data with respect to hypercovers are equivalent to those determined by their 0-coskeleta, that is, Čech covers. This is explicitly proved in [**AM69**, Proposition 10.3] when S has fiber products, but the argument goes through for generalized covers as well, or it follows more generally from the results of [**DHI04**]. On the other hand, if working simplicially there really is no additional complication in working with hypercovers—even the notation would be the same.

2.2.4. — It is more appropriate to talk about the *category* of descent data—the notion of *morphism* between descent data (x, φ) and (x', φ') being defined by a morphism $\psi\colon x \to x'$ in $\mathscr{F}(Y_0)$ such that

$$d_1^*\psi\,\varphi = \varphi'\,d_0^*\psi \tag{2.2.4.1}$$

in $\mathscr{F}(Y_1)$. Let us denote by $\operatorname{Desc}(u, \mathscr{F})$ the category of descent data for $\mathscr{F}$ relative to the hypercover $u\colon Y_\bullet \to U$.

2.3. 2-Descent data. — We will need to discuss the analog of the descent condition for fibered 2-categories (or even bicategories), a concept for which we refer to ref. [**Hak72**] (see also [**Bre94**, Chapter 1]).

2.3.1. — Let $\mathfrak{C}$ be a fibered 2-category, and let $X \in \mathsf{S}^\wedge$. Define, analogously to 2.2.1

$$\mathfrak{C}(X) \stackrel{\text{def}}{=} \underline{\mathsf{Hom}}_{\mathsf{S}}(X, \mathfrak{C}).$$

where X is interpreted as a fibered 2-category. This also equals $\varprojlim_{(i\colon V \to X) \in (\mathsf{S} \downarrow X)} \mathfrak{C}(V)$. (See ref. [**Hak72**] for details.)

2.3.2 Definition. — Let $\mathfrak{C}$ be a fibered 2-category. Let $U \in \operatorname{Ob} \mathsf{S}$, and $u\colon Y_\bullet \to U$ a hypercover. A *2-descent datum* over U relative to u is given by an object $x \in \operatorname{Ob} \mathfrak{C}(Y_0)$, an isomorphism $\varphi\colon d_0^* x \xrightarrow{\sim} d_1^* x$ in $\mathfrak{C}(Y_1)$, and a 2-morphism

$$\alpha\colon d_1^* \varphi \longrightarrow d_2^* \varphi \circ d_0^* \varphi,$$

over Y_2, satisfying the cocycle condition visualized by the diagram:

$$\big((d_2 d_3)^* \varphi * d_0^* \alpha\big) \circ d_2^* \alpha = \big(d_3^* \alpha * (d_0 d_1)^* \varphi\big) \circ d_1^* \alpha$$

over Y_3.

2.4. Stack associated to a prestack. — Recall that for any prestack $\mathscr{X}$ there is canonically associated a stack $\mathscr{X}^\sim$ and a morphism (the "stackification") $a\colon \mathscr{X} \to \mathscr{X}^\sim$, such that for every morphism (of prestacks) $F\colon \mathscr{X} \to \mathscr{Y}$ to a stack $\mathscr{Y}$ there is a factorization

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "a"] \arrow[dr, bend right, "F"'] \arrow[dr, phantom, shift left=2ex, "\Swarrow"] & \mathscr{X}^\sim \arrow[d, dotted, "F^a"] \\
& \mathscr{Y}
\end{tikzcd}$$

$\mathscr{X}^\sim$ is determined up to equivalence. The previous diagram expresses the universal property of the associated stack.

There are explicit constructions of $\mathscr{X}^\sim$, which involve "adding descent data." Given $\mathscr{X}$, one defines

$$\mathscr{X}^+(U) = \varinjlim_{Y \to U} \operatorname{Desc}(\check{\mathrm{C}}\, Y \to U, \mathscr{X}),$$

where U is an object of S. This leads to the explicit description as found e.g. in [**LMB00**]: an object $(Y \to U, x, \varphi)$ of $\mathscr{X}^+$ over U comprises a generalized cover and a descent datum relative to it. A morphism

$$(Y \to U, x, \varphi) \longrightarrow (Y' \to U, x', \varphi')$$

is a morphism of descent data over $Y \times_U Y' \to U$. Equivalently, one could use homotopy classes of hypercovers. Our main example will be the gr-stack associated to a crossed module, for which there exists an explicit model (see below).

2.4.1 Theorem (**[Gir71, LMB00]**). — *If $\mathscr{X}$ is a prestack over S, then $\mathscr{X}^+$ is a stack.*

There is an obvious morphism $\mathscr{X} \to \mathscr{X}^+$ which consists in sending the object x over U to $(\mathrm{id}\colon U \to U, x, \mathrm{id})$. This allows us to take $\mathscr{X}^\sim$ to be $\mathscr{X}^+$ and the morphism $a\colon \mathscr{X} \to \mathscr{X}^+$ to be the one just described.

3. Recollections on gr-categories, gr-stacks, and crossed modules

3.1. Gr-categories and gr-stacks. — The reference for gr-categories is the not easily accessible thesis [**Sí75**] (see also [**SR72**]). The basic facts are recalled in ref. [**Bre92**], which we follow for terminology and conventions (see also [**Rou03**] and [**Vit02**]).

3.1.1. — A 2-group, or gr-category, is a monoidal, unital, compact groupoid, that is a groupoid C equipped with a composition law, a unit object I, and for each object $X \in \mathsf{C}$ a choice of (right) inverse X^*, respectively. The composition law is a functor

$$\otimes\colon \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$$

obeying an associativity constraint: for each triple $X, Y, Z \in \operatorname{Ob} \mathsf{C}$ there is a functorial isomorphism (the associator)

$$a_{X,Y,Z}\colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

required to satisfy a coherence condition expressed by the well-know Mac Lane's pentagon diagram. Furthermore, for each object $X \in \operatorname{Ob} \mathsf{C}$ there are functorial isomorphisms

$$l_X\colon X \xrightarrow{\sim} I \otimes X\,, \quad r_X\colon X \xrightarrow{\sim} X \otimes I \tag{3.1.1.1}$$

required to satisfy the compatibility diagram

$$\begin{tikzcd} (X \otimes I) \otimes Y \arrow[rr] & & X \otimes (I \otimes Y) \\ & X \otimes Y \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{3.1.1.2}$$

The inverse map $X \mapsto X^*$ is a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{C}$, and one has $(X \otimes Y)^* \xrightarrow{\sim} Y^* \otimes X^*$. Furthermore, there is an isomorphism

$$X \otimes X^* \xrightarrow{\sim} I\,.$$

The choice of the latter determines the arrow $I \xrightarrow{\sim} X^* \otimes X$. For all the remaining properties, as well as the compatibility diagrams not displayed here, we refer to the above mentioned works.

3.1.2. — To a gr-category C are associated its group of isomorphism classes of objects, $\pi_0(\mathsf{C}) = \operatorname{Ob}\mathsf{C}/\sim$, and the group of automorphisms of the identity object, $\pi_1(\mathsf{C}) = \operatorname{Aut}(I)$. The latter is an abelian group owing to the fact that for any object $x \in \operatorname{Ob}\mathsf{C}$, left (say) multiplication by x is an equivalence of C, hence it allows to coherently identify $\operatorname{Aut}(x)$ with $\operatorname{Aut}(I)$. This implies abelianness (cf. [**Sí75, Bre99**]). One also has that $\pi_1(\mathsf{C})$ carries a (right) $\pi_0(\mathsf{C})$-action, induced by right multiplication by objects of C.

Let now C and D be two gr-categories.

3.1.3. — An additive functor from C to D is a pair (F, λ), where

$$F\colon \mathsf{C} \longrightarrow \mathsf{D}$$

is a functor between the underlying groupoids, and for each pair of objects $X, Y \in \operatorname{Ob}\mathsf{C}$ there is a functorial (iso)morphism

$$\lambda_{X,Y}\colon F(X) \otimes F(Y) \xrightarrow{\sim} F(XY)\,.$$

(Since in a gr-category the multiplication functor by every object is an equivalence, the condition $I \xrightarrow{\sim} F(I)$ follows from the existence of λ, cf. the above quoted references.) The isomorphisms λ must be compatible with the associativity morphism, in the sense that the following diagram must commute:

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r] \arrow[d] & F(X \otimes Y) \otimes F(Z) \arrow[r] & F((X \otimes Y) \otimes Z) \arrow[d] \\
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r] & F(X) \otimes F(Y \otimes Z) \arrow[r] & F(X \otimes (Y \otimes Z))
\end{tikzcd} \tag{3.1.3.1}$$

The diagrams resulting from the compatibility between the isomorphism $I \xrightarrow{\sim} F(I)$ and l_X and r_X (for any object X) must commute as well.

3.1.4. — A natural transformation of additive functors (F, λ) and (G, μ) consists of a natural transformation of the underlying functors $\theta\colon F \Rightarrow G$ in the standard sense, such that the diagram

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\lambda_{X,Y}"] \arrow[d, "\theta_X \otimes \theta_Y"'] & F(XY) \arrow[d, "\theta_{XY}"] \\
G(X) \otimes G(Y) \arrow[r, "\mu_{X,Y}"'] & G(XY)
\end{tikzcd} \tag{3.1.4.1}$$

commutes. (Note that the diagram expressing the compatibility between θ and the associators obtained by combining the previous two diagrams is automatically commutative.)

3.1.5. — There is a canonical way of composing additive functors. The composition of $(F_1, \lambda^1)\colon \mathsf{C}_0 \to \mathsf{C}_1$ and $(F_2, \lambda^2)\colon \mathsf{C}_1 \to \mathsf{C}_2$ is $F_2 \circ F_1$ equipped with $\lambda^2 * \lambda^1$ given by

$$\lambda^2 * \lambda^1{}_{X,Y} = F_2(\lambda^1{}_{X,Y}) \circ \lambda^2{}_{F_1(X),F_1(Y)}.$$

Note that this composition *is* associative.

3.1.6. — The preceding constructions carry over to the sheaf-theoretic context. Suppose we are given a stack $\mathscr{G}$ in groupoids on the site S. Following ref. [**Bre92**], we will say that $\mathscr{G}$ is a 2-group stack, or a gr-stack, if, again, it is equipped with a composition law embodied by morphisms of stacks

$$\otimes\colon \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G}$$

and

$$(\cdot)^*\colon \mathscr{G} \longrightarrow \mathscr{G}\,, \quad x \longmapsto x^*$$

plus a global identity object I. These data will be required to satisfy the same formal properties as for a gr-category. A morphism $F\colon \mathscr{G} \to \mathscr{H}$ of gr-stack is actually an additive functor, that is, a pair (F, λ), where the underlying functor F is a morphism of stacks. Again, F and λ are required to satisfy the same properties listed for gr-categories. The same definitions hold with the word "stack" replaced by "pre-stack." Our main examples of gr-(pre)stacks will arise from crossed-modules, whose main definitions and properties we are going to recall below.

3.1.7. — For $\mathscr{G}$ a gr-stack we define $\pi_0(\mathscr{G})$ (or simply π_0, for short, when no danger of confusion can arise) to be the sheaf associated to the presheaf

$$U \rightsquigarrow \pi_0(\mathscr{G}_U),$$

where U is an object of the underlying site, so that $\pi_0(\mathscr{G}_U)$ is the group of isomorphism classes of objects of the gr-(fiber)-category $\mathscr{G}_U$ over U. It is known (and easy to see, cf. refs. [**LMB00, Bre94**]) that the projection

$$\mathscr{G} \longrightarrow \pi_0$$

makes $\mathscr{G}$ a gerbe over π_0. We also set $\pi_1(\mathscr{G}) = \underline{\mathrm{Aut}}(I)$ (or simply π_1 when possible), the sheaf of automorphisms of the identity object. The coherence argument mentioned above remains valid in this case, implying that π_1 is a sheaf of abelian groups. Moreover, it is the band of the gerbe $\mathscr{G}|_{\pi_0}$ ([**Bre94**]).

3.2. Crossed modules. — The notion of crossed module is of course by now well known. We will recall the main definitions here to merely establish the necessary conventions. Following ref. [**Del73**], a crossed module will be considered as a complex

$$G^\bullet\colon [G^{-1} \xrightarrow{\partial} G^0]_{-1,0}$$

placed in (cohomological) degrees $-1, 0$. For notational convenience, we will use a homological (subscript indices) notation via the standard re-indexing $G_i = G^{-i}$. (All actions to be considered in this paper will be on the right, and crossed modules will be no exception.)

3.2.1 Definition. — A crossed module in T is a homomorphism of group objects

$$\partial \colon G_1 \longrightarrow G_0$$

together with a right action

$$G_1 \times G_0 \longrightarrow G_1$$

written as $(g, x) \mapsto g^x$ in set-theoretic terms, for $g \in G_1$ and $x \in G_0$, satisfying

$$\begin{aligned} \partial(g^x) &= x^{-1}\,\partial(g)\,x, \\ {g_0}^{\partial(g_1)} &= g_1^{-1}\,g_0\,g_1\,, \end{aligned} \tag{3.2.1.1}$$

for $x \in G_0$ and $g, g_0, g_1 \in G_1$.

3.2.2 Remark. — The use of set-theoretic element-notation in eqns. (3.2.1.1) can of course be avoided. The axioms can be written in a purely arrow-theoretic way:

$$\begin{tikzcd} G_1 \times G_0 \arrow[r] \arrow[d, "\mathrm{id}\times\partial"'] & G_1 \arrow[d, "\partial"] \\ G_0 \times G_0 \arrow[r] & G_0 \end{tikzcd} \qquad \begin{tikzcd} & G_0 \arrow[dr, bend left, "\jmath"] & \\ G_1 \arrow[ur, bend left, "\partial"] \arrow[rr, "\imath_{G_1}"'] & & \mathrm{Aut}(G_1) \end{tikzcd} \tag{3.2.2.1}$$

In the diagram to the left the top horizontal arrow is simply the action of G_0 on G_1, whereas the bottom one corresponds to the (right) action of G_0 on itself given by conjugation. In the diagram to the right $\jmath$ is the morphism corresponding to the action of G_0 on G_1, and i_{G_1} is homomorphisms given by the inner conjugation, namely $g \mapsto \imath_g \colon g' \mapsto g^{-1} g' g$.

3.2.3 Definition. — A *strict* morphism of crossed modules is a diagram

$$\begin{tikzcd} H_1 \arrow[r, "f_1"] \arrow[d, "\partial_G"'] & G_1 \arrow[d, "\partial_H"] \\ H_0 \arrow[r, "f_0"'] & G_0 \end{tikzcd} \tag{3.2.3.1}$$

of group homomorphisms, where the columns are crossed modules, and f_1 is f_0-equivariant, that is:

$$f_1(h^x) = f_1(h)^{f_0(x)} \tag{3.2.3.2}$$

for $h \in H_1$, $x \in H_0$.

As the usage of the qualifier "strict" in the previous definition suggests, there exist also *weak* morphisms, where conditions (3.2.3.1) and (3.2.3.2) are substantially relaxed. They are defined to be simply additive functors between the corresponding gr-categories, cf. refs. [**Noo07, Noo05**]. They will be treated in detail in a later section, from a rather different perspective than the one adopted in loc. cit.

As one may expect, there are also morphisms between (strict) morphisms (i.e. natural transformations), defined as follows.

3.2.4 Definition. — Given two morphisms f, f' as in (3.2.3.1), a *homotopy* $\gamma\colon f \Rightarrow f'$ between them is a map

$$\gamma\colon H_0 \longrightarrow G_1 \tag{3.2.4.1}$$

satisfying the following relations:

$$f_0(x)\,\partial_G(\gamma_x) = f'_0(x) \tag{3.2.4.2a}$$

$$\gamma_x\, f'_1(h) = f_1(h)\,\gamma_y \tag{3.2.4.2b}$$

$$\gamma_{xx'} = \gamma_x^{f_0(x')}\,\gamma_{x'} \tag{3.2.4.2c}$$

for all $h \in H_1$, $x, y \in H_0$ such that $x\,\partial_H h = y$, and $x' \in H_0$.

3.2.5. — A crossed module gives rise to a groupoid

$$\mathsf{G}\colon G_0 \times G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0$$

where the source and target maps are:

$$s(x,g) = x\,, \quad t(x,g) = x\,\partial(g)\,, \tag{3.2.5.1}$$

where $x \in G_0$, $g \in G_1$. This groupoid is in fact a strict gr-category, where the composition functor

$$\otimes\colon \mathsf{G} \times \mathsf{G} \longrightarrow \mathsf{G}$$

is given on objects (i.e. G_0) by the group law of G_0, and on morphisms by:

$$(x_0, g_0) \otimes (x_1, g_1) = (x_0 x_1, g_0^{x_1} g_1)\,, \tag{3.2.5.2}$$

with obvious meaning of the variables. It is easy to verify that this group law is strictly associative. It is also easy to check that a strict morphism in the sense of Definition 3.2.3 gives an additive functor $F\colon \mathsf{G} \to \mathsf{H}$. Note that this functor will be additive in the strictest possible sense, namely all the isomorphism $\lambda_{x,y}$ are the identity. Finally, a homotopy as in Definition 3.2.4 gives rise to a morphism of such additive functors $F \Rightarrow F'$.

3.2.6. — A crossed module $[\partial\colon G_1 \to G_0]$ gives rise to an obvious exact sequence

$$0 \longrightarrow A \longrightarrow G_1 \xrightarrow{\partial} G_0 \longrightarrow B \longrightarrow 1\,,$$

where $A = \operatorname{Ker}\partial$ and $B = \operatorname{Coker}\partial$. It is immediately verified that $B = \pi_0(\mathsf{G})$, and $A = \pi_1(\mathsf{G})$. It follows from the more general considerations about gr-categories, or from direct computations with (3.2.1.1), that A is a B-module, and it is central in G_1, hence abelian.

3.3. Cocycles. — To a crossed module $[G_1 \to G_0]^\sim$ there is a canonically associated simplicial group object of T, namely the nerve of the groupoid $\mathsf{G}\colon G_0 \times G_1 \rightrightarrows G_0$. It is well known that this simplicial group, which we denote $\underline{G}_\bullet$, is given by

$$\underline{G}_0 = G_0, \qquad \underline{G}_n = G_0 \times \underbrace{G_1 \times \cdots \times G_1}_{n}, \quad n \geq 1.$$

with face and degeneracy maps $d_i\colon \underline{G}_n \to \underline{G}_{n-1}$ and $s_i\colon \underline{G}_n \to \underline{G}_{n+1}$:

$$d_i(x, g_0, \dots, g_{n-1}) = \begin{cases} (x\,\partial g_0, g_1, \dots, g_{n-1}) & i = 0 \\ (x, g_0, \dots, g_{i-1}g_i, \dots, g_{n-1}) & 0 < i < n \\ (x, g_0, \dots, g_{n-2}) & i = n \end{cases}$$
$$s_i(x, g_0, \dots, g_{n-1}) = (x, g_0, \dots, g_{i-1}, 1, g_i, \dots, g_{n-1}), \quad i = 0, \dots, n.$$

3.3.1 Definition. — Let $Y_\bullet \to U$ be a hypercover of $\mathsf{S}^\wedge$. A 0-*cocycle over* U is a simplicial map $\xi\colon Y_\bullet \to \underline{G}_\bullet$. Two such cocycles ξ, ξ' are *equivalent* if there is a simplicial homotopy $\alpha\colon \xi \Rightarrow \xi'\colon Y_\bullet \to \underline{G}_\bullet$.

3.3.2. — Computing with simplicial maps and simplicial identities, and the above definition of $\underline{G}_\bullet$, shows that there is a one-to-one correspondence between such simplicial maps $\xi\colon Y_\bullet \to \underline{G}_\bullet$ and pairs (x, g), $x\colon Y_0 \to G_0$ and $g\colon Y_1 \to G_1$, satisfying

$$\begin{aligned} d_0^* x &= d_1^* x\, \partial g \\ d_1^* g &= d_2^* g\, d_0^* g \end{aligned} \tag{3.3.2.1}$$

and the normalization condition $s_0^* g = 1$. The simplicial map ξ itself is given by:

$$\begin{aligned} \xi_0 &= x \\ \xi_1 &= (d_1^* x, g) \\ \xi_2 &= ((d_1 d_2)^* x, d_2^* g, d_0^* g). \end{aligned}$$

(Like the nerve of any category $\underline{G}_\bullet$ is 2-coskeletal, hence ξ is completely determined by its 2-truncation.) To express the correspondence between ξ and (x, g) we will simply write $\xi = (x, g)$.

Similarly, another direct calculation using the definitions reveals that a simplicial homotopy $\alpha\colon \xi \Rightarrow \xi'$ is uniquely determined by an element $a\colon Y_0 \to G_1$ such that

$$\begin{aligned} x' &= x\, \partial a \\ g\, d_0^* a &= d_1^* a\, g'. \end{aligned} \tag{3.3.2.2}$$

According to the classical formulas found, e.g. in [**May92**] the homotopy α is concretely realized as a simplicial homotopy *from* ξ' *to* ξ in the sense of loc. cit. §5, and it consists of maps $\alpha_0^0\colon Y_0 \to G_0 \times G_1$ and $\alpha_0^1, \alpha_1^1\colon Y_1 \to G_0 \times G_1 \times G_1$ given by:

$$\begin{aligned}\alpha_0^0 &= (x, a)\\ \alpha_0^1 &= (d_1^* x, d_1^* a, g')\\ \alpha_0^1 &= (d_1^* x, g, d_0^* a).\end{aligned}$$

3.4. Gr-stacks associated to crossed modules. — A sheaf of groupoids is in an obvious way a prestack ([**LMB00**]). Given a crossed module $[G_1 \to G_0]$ and the groupoid $\mathsf{G}\colon G_0 \times G_1 \rightrightarrows G_0$, We usually indicate by $[G_1 \to G_0]^\sim$ (rather than $\mathsf{G}^\sim$) the associated stack. In general we have:

3.4.1 Proposition. — *If $\mathscr{G}$ is a gr-prestack, then the associated stack $\mathscr{G}^\sim$ acquires the structure of gr-stack, and the stackification morphism $a\colon \mathscr{G} \to \mathscr{G}^\sim$ becomes an additive functor.*

Idea of proof. — This can be seen by applying the diagram expressing the universal property of the associated stack at the beginning of 2.4 to the morphism $\mathscr{G}\times\mathscr{G} \to \mathscr{G} \to \mathscr{G}^\sim$ to obtain $\otimes\colon \mathscr{G}^\sim \times \mathscr{G}^\sim \to \mathscr{G}^\sim$; and similarly for the other diagrams expressing the associativity and inversion laws. □

It follows that $[G_1 \to G_0]^\sim$ is a gr-stack—the associated *gr*-stack to the crossed module $[G_1 \to G_0]$. Its gr-stack structure can be explicitly described in terms of descent data.

Using Definition 2.2.3, the maps in (3.2.5.1), and eq. (3.3.2.1), we see that in the present case descent data just become cocycles with values in $[G_1 \to G_0]$. (The correspondence being $(x, y) \to (x, g^{-1})$, to be precise.) Similarly, from (3.2.5.1) and (2.2.4.1) it follows that morphisms of descent data correspond to their respective cocycles being equivalent in the sense of 3.3.1.

3.4.2 Remark. — Whenever the site S admits fiber products, and the topology on S is given in terms of covers, the cocycle relations (3.3.2.1) take the more familiar form

$$x_j = x_i\, \partial g_{ij}$$

$$g_{ik} = g_{ij}\, g_{jk}\,,$$

with respect to a cover $\{U_i \to U\}_{i\in I}$, see e.g. [**Bre92**, 2.4.5.1, 2.4.5.2]. In the same way, (3.3.2.2) become:

$$x_i' = x_i\, \partial a_i$$

$$g_{ij}\, a_j = a_i\, g_{ij}'$$

expressing the familiar equivalence relation between cocycles with values in $[G_1 \to G_0]$ over U, see loc. cit. In general one must take care that $Y = \sqcup_i U_i$ in $\mathsf{S}^\wedge$ and similarly that $U_{ij} := U_i \times_U U_j$ exists a priori only in $\mathsf{S}^\wedge$ as well, so x_i and g_{ij} should properly interpreted as morphisms $x_i\colon U_i \to G_0$ and $g_{ij}\colon U_{ij} \to G_1$ in T.

3.4.3. — Now, given two cocycles $\xi, \xi' \colon Y_\bullet \to \underline{G}_\bullet$ there is an obvious definition of

$$\xi \otimes \xi' \colon Y_\bullet \longrightarrow \underline{G}_\bullet$$

by "pointwise" multiplication using the simplicial group structure of $\underline{G}_\bullet$:

$$(\xi \otimes \xi')_n := \xi_n \xi'_n. \tag{3.4.3.1}$$

Computing with (3.2.5.2), we find that if $\xi = (x, g)$ and $\xi' = (x', g')$, then

$$\xi \otimes \xi' = (xx', g^{d_1^* x} g'). \tag{3.4.3.2}$$

The unit is $(1, 1)$ and inverse maps will be the obvious one computed from (3.4.3.2).

It follows from the definitions in section 2.4 that objects of $[G_1 \to G_0]^\sim$ over $U \in \operatorname{Ob} \mathsf{S}$ are pairs $X = (Y, \xi)$ where $Y \to U$ is a generalized cover and $\xi \colon \check{\mathrm{C}}\, Y \to \underline{G}_\bullet$. A better way would probably be to visualize them as a fraction

$$X = \begin{tikzcd} & \check{\mathrm{C}}\, Y \arrow[dl] \arrow[dr, "\xi"] & \\ U & & \underline{G}_\bullet \end{tikzcd}$$

3.4.4. — Given two such objects $X = (Y, \xi)$ and $X' = (Y', \xi')$ over U we define their product as:

$$X \otimes X' = \big(Y \times_U Y', p^* \xi \otimes p'^* \xi'\big) \tag{3.4.4.1}$$

where $p^*\xi$ is the pull-back of ξ to $\check{\mathrm{C}}(Y \times_U Y')$ via $p \colon Y \times_U Y' \to Y$, and similarly for $p'^*\xi'$. The $\otimes$-product in the right-hand side of (3.4.4.1) is the one computed via (3.4.3.1). Considering that the simplicial map ξ is itself determined by the pair (x, g), we can just write the object X as $X = (Y, x, g)$, where now $x \colon Y \to G_0$ and $g \colon Y \times_U Y' \to G_1$. This is just the classical way to write descent data. Therefore given (Y, x, g) and (Y', x', g') objects of $[G_1 \to G_0]^\sim$ over U, we can simply write (3.4.4.1) more classically as:

$$(Y, x, g) \otimes (Y', x', g') = (Z, xx', g^{d_0^*(x')} g'), \tag{3.4.4.2}$$

where $Z \to U$ refines both Y, Y', e.g. $Z = Y \times_U Y'$, in $\mathsf{S}^\wedge$ and for simplicity on the right hand side we have suppressed the pullbacks to Z. Similarly, if morphisms $(Y, x, g) \to (Y_1, x_1, g_1)$ and $(Y', x', g') \to (Y'_1, x'_1, g'_1)$ are given by "elements" $a \colon Z \to G_1$ and $a' \colon Z' \to G_1$ as in (3.3.2.2), their product is given by $a^{x'} a'$ over a refinement W of Z, Z'.

The reader will be able to verify without difficulty:

3.4.5 Proposition **([Bre90])**. — *The product* (3.4.4.1) *gives* $[G_1 \to G_0]^\sim$ *the structure of a gr-stack.*

3.4.6 Remark. — Note that the group law on $[G_1 \to G_0]^\sim$ just introduced is *not* strict, even though the one on G is, due to the various pullbacks. Thus, for example, there will be an associativity morphism

$$((Y, x, g) \otimes (Y', x', g')) \otimes (Y'', x'', g'') \simeq (Y, x, g) \otimes ((Y', x', g') \otimes (Y'', x'', g''))$$

resulting from $(Y \times_U Y') \times_U Y''$ being isomorphic to $Y \times_U (Y' \times_U Y'')$.

3.4.7. — There is an equivalent but more geometric realization of the associated gr-stack of $[G_1 \to G_0]$. Let $\delta\colon G \to H$ be a group homomorphism of T. Following ref. [**Del79**], let us denote by $\mathrm{T{\small ORS}}(G, H)$ the stack of right G-torsors equipped with a trivialization of their extension to H-torsors. In other words an object of $\mathrm{T{\small ORS}}(G, H)$ is a pair (P, s) where P is a right G-torsor and s is global isomorphism $s\colon P \wedge^G H \xrightarrow{\sim} H$. An object of $\mathrm{T{\small ORS}}(G, H)$ will be called a (G, H)-torsor. The morphism s will be identified with a G-equivariant morphism $s\colon P \to H$ where all the actions are on the right, namely $s(ug) = s(u)\delta(g)$. With this convention, the precise correspondence

$$\begin{aligned} \underline{\mathrm{Hom}}_G(P, H) &\xrightarrow{\sim} P \wedge^G H \\ (P, s) &\longmapsto (u, s(u)^{-1}) \end{aligned}$$

in set-theoretic notation.

A morphism $f\colon (P, s) \to (Q, t)$ is a morphism $f\colon P \to Q$ of G-torsors compatible with the trivializations. Equivalently, the diagram

(3.4.7.1)
$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, bend right, "s"'] & & Q \arrow[dl, bend left, "t"] \\ & H & \end{tikzcd}$$

commutes.

3.4.8. — All this becomes much more interesting when it is applied to the group homomorphism underlying a crossed module $[\partial\colon G_1 \to G_0]$. It is shown in [**Bre90**] that in this situation each object (P, s) of $\mathrm{T{\small ORS}}(G_1, G_0)$ is in fact a G_1-bitorsor with the left G_1-action defined (set-theoretically) by:

$$g \star u = u\, g^{s(u)}.$$

As a consequence, posing, as in loc. cit.

(3.4.8.1)
$$(P, s) \otimes (Q, t) := (P \wedge^{G_1} Q, s \wedge t)$$

endows $\mathrm{T{\small ORS}}(G_1, G_0)$ with a gr-stack structure. Here $s \wedge t$ is the G-equivariant map from $P \wedge^{G_1} Q$ to G_0 given by $s(u)t(v)$, where (u, v) represents a point of $P \wedge^{G_1} Q$.

Moreover, we have:

3.4.9 Theorem **(**[**Bre90**, Théorème 4.6]**).** — *There is an equivalence of gr-stacks*

$$\mathrm{T{\small ORS}}(G_1, G_0) \xrightarrow{\sim} [G_1 \to G_0]^\sim.$$

Proof (Sketch). — We limit ourselves to an outline the argument leading to the equivalence of the product structures, referring to the original reference for the complete details.

By the argument of loc. cit. an object (P, s) of $\textsc{Tors}(G_1, G_0)$ determines descent data in the usual way. Let $Y \to *$ be a generalized cover of the terminal object $* \in \mathsf{T}$ with a trivialization of the underlying right-G_1-torsor P via the section $u\colon Y \to P$. These data determine an isomorphism of (G_1, G_0)-torsors, therefore the morphism $\varphi\colon d_0^* P_Y \xrightarrow{\sim} d_1^* P_Y$ must also satisfy $s(d_0^*) = s(\varphi(d_0^* u))$. This determines $g\colon Y \times Y \to G_1$ such that $\varphi(d_0^* u) = (d_1^* u) g$ and $x \equiv s(u)\colon Y \to G_0$ such that (3.3.2.1) are satisfied.

Assuming for convenience that (P, s) and (P', s') are trivialized over the same $Y \to *$ from (3.4.8.1) we have that

$$\varphi(d_0^* u) \wedge \varphi'(d_0^* u') = (d_1^* u) g \wedge (d_1^* u') g' = (d_1^* u) \wedge g \star (d_1^* u') g'$$

and using the form of the left action given above

$$g \star (d_1^* u') g' = (d_1^* u') g^{s'(d_1^* u')} g'$$

so that

$$\varphi(d_0^* u) \wedge \varphi'(d_0^* u') = (d_1^* u) \wedge (d_1^* u')(d_1^* u') g^{s'(d_1^* u')} g'$$

we conclude the morphism $\varphi \wedge \varphi'$ is represented by $g^{d_1^* x'} g'$. Since obviously the value of $s \wedge s'$ over $u \wedge u'$ is $x x'$, we finally have obtained that the cocycle corresponding to $(P, s) \otimes (P', s')$ is the product of the two cocycles in the sense of (3.4.3.2) (or, more precisely, (3.4.4.1)). $\square$

3.4.10. — We conclude this section with the following observation, which will be useful elsewhere in this paper: if $\mathscr{G}$ is the associated gr-stack to $[G_1 \to G_0]$, then there is an exact sequence:

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{\pi_G} \mathscr{G} \tag{3.4.10.1}$$

of gr-stacks over S. Here G_1 and G_0 are considered as gr-stacks in the obvious way. The map $\pi_{\mathscr{G}}$ associates to the element $x\colon U \to G_0$ the trivial (G_1, G_0)-torsor $(G_1|_U, x)$ over U, where x is identified with the equivariant map sending the global section 1 to x. Exactness is intended in the sense of stacks: there it is a pull-back square

$$\begin{tikzcd} G_1 \arrow[r, "\partial"] \arrow[d] & G_0 \arrow[d, "\pi_{\mathscr{G}}"] \\ \mathbf{1} \arrow[r] & \mathscr{G} \end{tikzcd}$$

which is 2-commutative. ($\mathbf{1}$ is the category with one object and one arrow.) This is discussed in sections 5.3, and with respect to the exactness question, 6.2.

In terms of the corresponding simplicial group objects, the above sequence corresponds to the highlighted portion of the following homotopy exact sequence [**Bre90**, eq. (3.11.2)]:

$$* \longrightarrow \Omega \underline{G}_\bullet \longrightarrow G_1 \longrightarrow G_0 \longrightarrow \underline{G}_\bullet \longrightarrow \mathrm{B}\, G_1 \longrightarrow \mathrm{B}\, G_0 \longrightarrow \overline{\mathrm{W}}\, \underline{G}_\bullet$$

where G_0 and G_1 are considered as constant simplicial groups. We will say more about (3.4.10.1) further down in the paper.

4. Butterflies and weak morphisms of crossed modules

Morphisms of crossed modules as defined in 3.2.3 can be generalized quite a bit, and the resulting theory has a more geometric flavor. Over the punctual topos, that is, when we are dealing with groups and crossed modules in Set, the notion of weak morphism has been developed by the second author in refs. [**Noo07, Noo05**]. As mentioned above, the framework of weak morphisms of crossed modules translates into the calculus of diagrams called "butterflies," owing to their shape.

In this section we recast this discussion in the sheaf theoretic context of gr-stacks. As this is more than a mere translation, our treatment is going to be quite different from that in the above mentioned references. From this more geometric point of view we posit that weak morphisms of crossed modules are additive functors between the associated gr-stacks. It is equivalent, in a sense made precise below, to considering butterfly diagrams as morphisms between crossed modules.

In a later section (sect. 5) we will show how crossed modules equipped with butterflies as their morphisms (i.e. weak morphisms) form a fibered bicategory which is biequivalent to the fibered 2-category of gr-stacks.

4.1. General definitions. — Let $[H_1 \to H_0]$ and $[G_1 \to G_0]$ be two crossed modules of T.

4.1.1 Definition. — A *weak morphism* $F\colon H_\bullet \to G_\bullet$ is an additive functor

$$F\colon [H_1 \to H_0]^\sim \longrightarrow [G_1 \to G_0]^\sim$$

between the corresponding gr-stacks. A (weak) 2-morphism is a morphism of such additive functors (as in sect. 3.1).

4.1.2 Remark. — Strict morphisms from $H_\bullet$ to $G_\bullet$, as defined in Definition 3.2.3, give rise to weak morphisms in the obvious way, since they give rise to strict additive functors

$$\Big[H_0 \times H_1 \overset{s}{\underset{t}{\rightrightarrows}} H_0 \Big] \longrightarrow \Big[G_0 \times G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0 \Big]$$

and therefore to morphisms between the associated stacks.

4.1.3 Definition. — A *butterfly* from $H_\bullet$ to $G_\bullet$ is a commutative diagram of group homomorphisms of the form

$$
\begin{tikzcd}
H_1 \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "\pi"'] \arrow[dr, "\jmath"] & \\
H_0 & & G_0
\end{tikzcd}
\tag{4.1.3.1}
$$

where E is a group object of T, the NW-SW sequence is a complex, and the NE-SW sequence is a group extension. The various maps satisfy the equivariance conditions written set-theoretically as:

$$
\imath(g^{\jmath(e)}) = e^{-1}\imath(g)e, \quad \kappa(h^{\pi(e)}) = e^{-1}\kappa(h)e
\tag{4.1.3.2}
$$

where $g \in G_1, h \in H_1, e \in E$.

Let us use the short-hand notation $[H_\bullet, E, G_\bullet]$ for the butterfly diagram (4.1.3.1), or even just E when there is no danger of confusion. As in [**Noo05**], we have:

4.1.4 Proposition. — *The images of κ and $\imath$ commute in E.*

Proof. — Easy consequence of (4.1.3.2). □

4.1.5 Definition. — A butterfly (4.1.3.1) is *flippable,* or *reversible,* if both diagonals are extensions.

A slightly stronger version of the definition of a butterfly plays a non-trivial role in some examples, most notably those related to braidings.

4.1.6 Definition. — A *strong butterfly* is a butterfly (4.1.3.1) equipped with a global section $s\colon H_0 \to E$ of $\pi\colon E \to H_0$ of underlying Set-valued sheaves, namely such that $\pi \circ s = \mathrm{id}_{H_0}$.

Morphism of butterflies are defined as follows:

4.1.7 Definition. — A morphism of butterflies $\varphi\colon [H_\bullet, E, G_\bullet] \to [H_\bullet, E', G_\bullet]$ is given by a group isomorphism $\varphi\colon E \xrightarrow{\sim} E'$ such that

$$
\begin{tikzcd}
H_1 \arrow[r] \arrow[dd] \arrow[dr] & E' \arrow[ddl, bend right] \arrow[ddr, bend left] & G_1 \arrow[l] \arrow[dd] \arrow[dl] \\
& E \arrow[u] \arrow[dl] \arrow[dr] & \\
H_0 & & G_0
\end{tikzcd}
$$

commutes and is compatible with all the conditions in 4.1.3. Two morphisms

$$\varphi\colon [H_\bullet, E, G_\bullet] \to [H_\bullet, E', G_\bullet]\,,\quad \varphi'\colon [H_\bullet, E', G_\bullet] \to [H_\bullet, E'', G_\bullet]$$

are composed in the obvious way.

4.1.8. — It is clear from Definitions 4.1.3 and 4.1.7 that butterflies from $H_\bullet$ to $G_\bullet$ and their morphisms form a groupoid. Let us denote it by $\mathsf{B}(H_\bullet, G_\bullet)$.

Another groupoid naturally associated with two crossed modules $H_\bullet$ and $G_\bullet$ is the groupoid of weak morphisms as defined in sect. 4.1. This groupoid will be denoted by $\mathsf{WM}(H_\bullet, G_\bullet)$.

4.2. Remarks on the definition of weak morphism. — In the set-theoretic case, the definition of weak morphism is seemingly different. In [**Noo07, Noo05**] weak morphisms from $H_\bullet$ to $G_\bullet$ are defined as (pointed) lax functors from $\mathsf{H}[1]$ to $\mathsf{G}[1]$. Recall that G is the groupoid determined by $G_\bullet$, and that $\mathsf{G}[1]$ is the "suspension" of G, namely the 2-category with only one object and 1-morphisms given by the objects of G, with composition law given by the monoidal law of G.

4.2.1. — In the context of sheaves over a site, given a crossed module $G_\bullet$ we have different notions of suspension: $\mathsf{G}[1]$, the suspension of the groupoid G itself; $\mathscr{G}[1]$, the suspension of the gr-stack $\mathscr{G}$ associated to $G_\bullet$; and finally $\textsc{Tors}(\mathscr{G})$, the 2-gerbe of $\mathscr{G}$-torsors (see [**Bre94**]). As remarked in loc. cit. and later on in section 6.3, the latter is the correct one from a geometric point of view, as it is associated to $\mathscr{G}[1]$ by a process of 2-stackification. ($\mathscr{G}[1]$ is a fibered bicategory over S which deserves to be called a pre-bistack, since $\mathscr{G}$ itself is a stack, but for which the 2-descent condition on objects does not hold; $\mathsf{G}[1]$ is even less geometric: as G itself is only a prestack, in the suspension only the 2-morphisms form a sheaf over S.)

4.2.2. — Given crossed modules $H_\bullet$ and $G_\bullet$, one can consider the following groupoids:

1. $\mathrm{LaxFnct}_*(\mathsf{H}[1], \mathsf{G}[1])$: lax pointed 2-functors;
2. $\underline{\mathsf{Hom}}_*(\mathscr{H}[1], \mathscr{G}[1])$: pointed Cartesian functors of fibered bicategories;
3. $\underline{\mathsf{Hom}}_*(\textsc{Tors}(\mathscr{H}), \textsc{Tors}(\mathscr{G}))$: pointed Cartesian 2-functors of fibered 2-categories.

A priori these are 2-groupoids, but since we are in the pointed case, they actually are equivalent to 1-groupoids. In the latter case, $\textsc{Tors}(\mathscr{G})$ is naturally pointed by the trivial torsor. Thus pointed morphisms send the trivial $\mathscr{H}$-torsor to the trivial $\mathscr{G}$-torsor up to equivalence.

4.2.3. — There are equivalences:

$$\mathsf{WM}(H_\bullet, G_\bullet) \xrightarrow{\sim} \underline{\mathsf{Hom}}_*(\mathscr{H}[1], \mathscr{G}[1]) \xrightarrow{\sim} \underline{\mathsf{Hom}}_*(\textsc{Tors}(\mathscr{H}), \textsc{Tors}(\mathscr{G})).$$

By Morita theory (see [**BM05**], or more precisely a categorification of it) the *un*-pointed 2-groupoid $\underline{\mathsf{Hom}}(\textsc{Tors}(\mathscr{H}), \textsc{Tors}(\mathscr{G}))$ consists of $(\mathscr{H}, \mathscr{G})$-bimodules, namely stacks with simultaneous left $\mathscr{H}$ and right $\mathscr{G}$-actions, that are actually torsors for the right $\mathscr{G}$-action. The *pointed* ones are the ones for which the corresponding bimodule is actually equivalent to the trivial torsor. Hence they correspond to actual additive functors $\mathscr{H} \to \mathscr{G}$.

The first equivalence between $\mathsf{WM}(H_\bullet, G_\bullet)$ and the groupoid 2 is an application of the definitions.

The groupoid 1 in the list is *strictly* smaller, however. It is rather easy to see that it only leads to additive functors of the form $F(U)\colon \mathsf{H}(U) \to \mathsf{G}(U)$ for each object U of the site S. In other words, it gives rise to additive functors between the corresponding prestacks. In light of sections 4.3 and 4.4, that choice only corresponds to *strong* butterflies in the sense of Definition 4.1.6.

4.3. Weak morphisms and butterflies. — One of our main results is the theorem stating that the groupoid of butterflies $\mathsf{B}(H_\bullet, G_\bullet)$ from $H_\bullet$ to $G_\bullet$ is equivalent to that of weak morphisms. More precisely:

4.3.1 Theorem. — *There exists a pair of quasi-inverse functors*

$$\Phi\colon \mathsf{B}(H_\bullet, G_\bullet) \longrightarrow \mathsf{WM}(H_\bullet, G_\bullet)$$

and

$$\Psi\colon \mathsf{WM}(H_\bullet, G_\bullet) \longrightarrow \mathsf{B}(H_\bullet, G_\bullet).$$

defining an equivalence between $\mathsf{B}(H_\bullet, G_\bullet)$ *and* $\mathsf{WM}(H_\bullet, G_\bullet)$.

We will give a proof in 4.4. It will be simpler to directly show that Φ is fully faithful and essentially surjective. However it still is worthwhile to have the explicit definition of both functors at hand.

Definition of Φ. — To define Φ, we need to construct an additive functor

$$\Phi(E)\colon \mathscr{H} \longrightarrow \mathscr{G}$$

for each object of $\mathsf{B}(H_\bullet, G_\bullet)$, i.e. a butterfly $[H_\bullet, E, G_\bullet]$. Given an H_1-torsor with an equivariant map $t\colon Q \to H_0$, consider the obvious map

$$\pi_*\colon\ \underline{\mathrm{Hom}}_{H_1}(Q, E) \longrightarrow \underline{\mathrm{Hom}}_{H_1}(Q, H_0)$$

induced by $\pi\colon E \to H_0$ in the butterfly. t is a global section of $\underline{\mathrm{Hom}}_{H_1}(Q, H_0)$, and we consider its local lifts to E, that is the fiber over t:

$$\underline{\mathrm{Hom}}_{H_1}(Q, E)_t = \left\{ e \in \mathrm{Hom}_{H_1}(Q|_U, E|_U) \,\middle|\, \pi \circ e = t|_U \right\}$$

4.3.2 Claim. — $\underline{\mathrm{Hom}}_{H_1}(Q, E)_t$ *is a* G_1*-torsor.*

Proof. — Given two lifts e, e' of t there exists a unique $g\colon U \to G_1$ such that $e' = e\,\imath(g)$. That g is not a map from Q to G_1, and only depends on U, follows from Proposition 4.1.4.

$\underline{\mathrm{Hom}}_{H_1}(Q,E)_t$ is locally non-empty since lifts exist, $\pi\colon E \to H_0$ being a sheaf epimorphism. □

Set $P = \underline{\mathrm{Hom}}_{H_1}(Q,E)_t$. Now define $s\colon P \to G_0$ as

$$\begin{aligned} s\colon \underline{\mathrm{Hom}}_{H_1}(Q,E)_t &\longrightarrow G_0 \\ e &\longmapsto \jmath \circ e \end{aligned}$$

It follows from the equivariance of e that s is well-defined map, that is, that it only depends on U, rather than the full $Q|_U$: indeed, one has, with set-theoretic notation:

$$\jmath(e(vh)) = \jmath(e(v)\kappa(h)) = \jmath(e(v)).$$

Moreover, if $e' = e\imath(g)$, for $g \in G_1$, then it immediately follows that $s(e') = s(e)\partial g$.

4.3.3. — In sum, declare (P,s) so constructed to be the object corresponding to (Q,t). If $\varphi\colon (Q_1,t_1) \to (Q_0,t_0)$ is a morphism of (H_1,H_0)-torsors as in (3.4.7.1), then the pull-back

$$\begin{aligned} (\varphi^{-1})^*\colon \underline{\mathrm{Hom}}_{H_1}(Q_1,E)_{t_1} &\longrightarrow \underline{\mathrm{Hom}}_{H_1}(Q_0,E)_{t_0} \\ e &\longmapsto e \circ \varphi^{-1} \end{aligned} \tag{4.3.3.1}$$

is clearly a morphism of (G_1,G_0)-torsors. Indeed eq. (3.4.7.1) is trivially satisfied simply because $\jmath \circ (e_0 \circ \varphi^{-1}) = (\jmath \circ e_0) \circ \varphi^{-1}$, where $e_0 \in \underline{\mathrm{Hom}}_{H_1}(Q_0,E)$. Clearly this respects composition and identity objects.

It is also clear that given two (H_1,H_0)-torsors (Q_0,t_0) and (Q_1,t_1) there is an isomorphism of (G_1,G_0)-torsors

$$\Phi(E)(Q_0,t_0) \wedge^{G_1} \Phi(E)(Q_1,t_1) \xrightarrow{\sim} \Phi(E)\big(Q_0 \wedge^{H_1} Q_1, t_0t_1\big), \tag{4.3.3.2}$$

in that given two lifts e_0, e_1 of t_0, t_1 to E the product e_0e_1 is a lift of t_0t_1. It is verified at once that this isomorphism satisfies the required properties in the definition of additive functors (cf. sect. 3.1).

4.3.4. — If $\alpha\colon E \to E'$ gives a morphism of butterflies, then there is an induced isomorphism

$$\alpha_*\colon \underline{\mathrm{Hom}}_{H_1}(Q,E)_t \xrightarrow{\sim} \underline{\mathrm{Hom}}_{H_1}(Q,E')_t$$

for each (H_1,H_0)-torsor (Q,t), obtained by pushing along α. Since this is clearly natural with respect to the pull-backs (4.3.3.1), it provides a natural transformation $\Phi(E) \Rightarrow \Phi(E')$, which is immediately verified to be compatible with (4.3.3.2), hence with the additive structure, as required.

Definition of Ψ. — Given a $F\colon \mathscr{H} \to \mathscr{G}$ be a morphism of gr-stacks over S, consider the stack fibered product:

$$\begin{tikzcd} H_0 \times_{\mathscr{G}} G_0 \arrow[rr] \arrow[d] & & G_0 \arrow[d, "\pi_{\mathscr{G}}"] \\ H_0 \arrow[r, "\pi_{\mathscr{H}}"'] & \mathscr{H} \arrow[r, "F"'] & \mathscr{G} \end{tikzcd}$$

By definition of stack fibered product ([**LMB00**]), $E = H_0 \times_{\mathscr{G}} G_0$ consists of elements (y, f, x), where $y\colon U \to H_0$ and $x\colon U \to G_0$, and f is an isomorphism $f\colon F(\pi_{\mathscr{H}}(y)) \xrightarrow{\sim} \pi_{\mathscr{G}}(x)$. Both G_0 and H_0 are group objects, hence in particular spaces, therefore so is the stack fiber product. Moreover, putting $(y_0, f_0, x_0)(y_1, f_1, x_1) = (y_0y_1, f_0f_1, x_0x_1)$, if $y_0, \ldots$ etc. are points of H_0 and G_0, obviously endows it with a group structure. Here f_0f_1 stands for the composition

$$F(\pi_{\mathscr{H}}(y_0y_1)) \simeq F(\pi_{\mathscr{H}}(y_0)) \otimes F(\pi_{\mathscr{H}}(y_1)) \xrightarrow{f_0\otimes f_1} \pi_{\mathscr{G}}(x_0) \otimes \pi_{\mathscr{G}}(x_1) = \pi_{\mathscr{G}}(x_0x_1).$$

4.3.5. — We have the following diagram over $\mathscr{G}$:

(4.3.5.1)
$$\begin{tikzcd} H_1 \arrow[dr, "\kappa"] \arrow[dd, "\partial_H"'] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial_G"] \\ & H_0 \times_{\mathscr{G}} G_0 \arrow[dl, "\pi"'] \arrow[dr, "\jmath"] & \\ H_0 \arrow[dr, "F\circ\pi_{\mathscr{H}}"'] & & G_0 \arrow[dl, "\pi_{\mathscr{G}}"] \\ & \mathscr{G} & \end{tikzcd}$$

In (4.3.5.1) the maps π and $\jmath$ are defined to be the canonical projections to the respective factors.

The precise definitions of the maps κ and $\imath$ are slightly more involved. Let

$$\lambda_F\colon F(\pi_{\mathscr{H}}(1)) \to \pi_{\mathscr{G}}(1)$$

be the isomorphism between the image of the unit object of $\mathscr{H}$ and the unit object of $\mathscr{G}$ (cf. sect. 3.1). The homomorphism κ is given by

$$\kappa(h) = (\partial_H h, f_h, 1),$$

where $f_h\colon F(\pi_{\mathscr{H}}(\partial_H h)) \xrightarrow{\sim} \pi_{\mathscr{G}}(1)$ is defined by:

(4.3.5.2)
$$\begin{tikzcd} F(\pi_{\mathscr{H}}(\partial h)) \arrow[dr, bend left, "f_h"] \arrow[dd, "F(h)"'] & \\ & \pi_{\mathscr{G}}(1) \\ F(\pi_{\mathscr{H}}(1)) \arrow[ur, bend right, "\lambda_F"'] & \end{tikzcd}$$

where h is regarded as a morphism

$$(H_1|_U, \partial h) = \pi_{\mathscr{H}}(\partial h) \longrightarrow \pi_{\mathscr{H}}(1) = (H_1|_U, 1)$$

between objects of $\mathscr{H}$ via the identification $\underline{\mathrm{Aut}}_{H_1}(H_1) \simeq H_1$ ([**Gir71**, III.1.2.7 (ii)]). Similarly, the definition of $\imath$:

$$\imath(g) = (1, f_g, \partial g)$$

where $f_g \colon F(\pi_{\mathscr{H}}(1)) \xrightarrow{\sim} \pi_{\mathscr{G}}(\partial g)$ is defined by:

$$\begin{tikzcd} & \pi_{\mathscr{G}}(\partial g) \arrow[dd, "g"] \\ F(\pi_{\mathscr{H}}(1)) \arrow[ur, "f_g"] \arrow[dr, "\lambda_F"'] & \\ & \pi_{\mathscr{G}}(1) \end{tikzcd}$$

4.3.6. — The NE-SW diagonal in (4.3.5.1) is exact, since it is the pull-back of the exact sequence

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{\pi_G} \mathscr{G}$$

(recall (3.4.10.1) at the end of sect. 3.4) to H_0. The NW-SW diagonal is only a complex since it is the pull-back to G_0 of the composite

$$H_1 \xrightarrow{\partial} H_0 \xrightarrow{\pi_H} \mathscr{H} \xrightarrow{F} \mathscr{G}$$

which is itself only a complex. It is immediately verified that the various maps satisfy the conditions in def. 4.1.3, so the diagram (4.3.5.1) is a butterfly from $H_\bullet$ to $G_\bullet$.

Therefore we define:

$$\Psi(F) = [H_\bullet, H_0 \times_{\mathscr{G}} G_0, G_\bullet]$$

Moreover, if $\theta \colon F \Rightarrow F' \colon \mathscr{H} \to \mathscr{G}$ is a morphism of additive functors, and E' is the fibered product constructed as in (4.3.5.1), with F' in place of F, then there is an induced isomorphism $E \xrightarrow{\sim} E'$, obtained by sending the triple (y, f, x) of E to (y, f', x), where f' is the composite of f with the inverse of:

$$\theta_{\pi_{\mathscr{H}}(y)} \colon F(\pi_{\mathscr{H}}(y)) \xrightarrow{\sim} F'(\pi_{\mathscr{H}}(y)).$$

One can easily check that with these definitions Ψ is indeed a functor.

4.4. Proof of theorem 4.3.1. —

4.4.1 Lemma. — $\Phi \colon \mathsf{B}(H_\bullet, G_\bullet) \to \mathsf{WM}(H_\bullet, G_\bullet)$ *is fully faithful.*

Proof. — Suppose first α, $\beta \colon E \to E'$ are morphisms of butterflies such that

$$\alpha_* = \beta_* \colon \underline{\mathrm{Hom}}_{H_1}(Q, E)_t \longrightarrow \underline{\mathrm{Hom}}_{H_1}(Q, E')_t$$

for each (H_1, H_0)-torsor (Q, t). In particular, if $(Q, t) = \pi_{\mathscr{H}}(y)$, for $y \colon U \to H_0$, we have

$$\underline{\mathrm{Hom}}_{H_1}(H_1|_U, E)_y \simeq E_y,$$

where E_y is the fiber of $\pi\colon E \to H_0$ above y, which indeed is a G_1-torsor. In fact E_y is a G_1-*bi*torsor, and, according to the account of the Schreier theory in ref. [**Gro72**] (see also [**Bre90**]), the "bitorsor cocycle"

$$E_y \wedge^{G_1} E_{y'} \xrightarrow{\sim} E_{yy'} \tag{4.4.1.1}$$

allows to recover E as well as the extension $1 \to G_1 \to E \to H_0 \to 1$.

Thus the identity $\alpha_* = \beta_*$ reduces to two identical maps

$$E_y \longrightarrow E'_y$$

for all y, compatible, by (4.3.3.2), with (4.4.1.1) and the corresponding one for E'. It follows that $\alpha = \beta$.

If, on the other hand, $[H_\bullet, E, G_\bullet]$ and $[H_\bullet, E', G_\bullet]$ are two butterflies and $\varphi\colon \Phi(E) \to \Phi(E')$ a morphism of additive functors, by definition we have a natural morphism

$$\varphi_{Q,t}\colon\ \underline{\mathrm{Hom}}_{H_1}(Q,E)_t \longrightarrow \underline{\mathrm{Hom}}_{H_1}(Q,E')_t$$

for each (H_1,H_0)-torsor (Q,t). Once again, when $(Q,t) = \pi_{\mathscr{H}}(y)$ we obtain an isomorphism

$$\varphi_{H_1,y}\colon E_y \longrightarrow E'_y,$$

for all $y\colon U \to H_1$. The same arguments as above, in particular the compatibility with (4.4.1.1), allow to conclude that the various $\varphi_{H_1,y}$ glue into a homomorphism $E \to E'$. (That it indeed is a homomorphism, in particular, follows from the compatibility with (4.4.1.1).) □

To prove essential surjectivity, we need to construct, for any additive functor $F\colon \mathscr{H} \to \mathscr{G}$, a butterfly E_F and a morphism

$$\varphi_F\colon F \longrightarrow \Phi(E_F)$$

of additive functors. This will follow from the following

4.4.2 Proposition. — *For each (H_1,H_0)-torsor (Q,t) there is a natural isomorphism*

$$\varphi_{Q,t}\colon F(Q,t) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{H_1}(Q,E)_t,$$

where

$$E \stackrel{\mathrm{def}}{=} H_0 \times_{F,\mathscr{G}} G_0.$$

Note that in the previous statement E is simply the butterfly obtained by applying the functor Ψ to F. We have explicitly marked the dependency on F in the notation for clarity.

Proof of Proposition 4.4.2. — Let us begin by assuming, as we have repeatedly done above, that $(Q,t) = \pi_{\mathscr{H}}(y)$, for $y\colon U \to H_0$, so we have

$$\underline{\mathrm{Hom}}_{H_1}(H_1, H_0 \times_{F,\mathscr{G}} G_0)_y \simeq (H_0 \times_{F,\mathscr{G}} G_0)_y.$$

The right hand side above consists of pairs (f, x), where $x\colon U \to G_0$ is a point, and

$$f\colon F(\pi_{\mathscr{H}}(y)) \xrightarrow{\sim} \pi_{\mathscr{G}}(x),$$

the point y being fixed. Incidentally, that $(H_0 \times_{F,\mathscr{G}} G_0)_y$ is a G_1-torsor directly results from the diagram of (G_1, G_0)-torsors:

$$\begin{tikzcd}
 & \pi_{\mathscr{G}}(x) \arrow[dd, "f' f^{-1}"] \\
F(\pi_{\mathscr{H}}(tu)) \arrow[ur, "f"] \arrow[dr, "f'"'] & \\
 & \pi_{\mathscr{G}}(x')
\end{tikzcd}$$

$f' f^{-1}$, as a morphism between the underlying G_1-torsors, is identified with an element $g\colon U \to G_1$. Moreover, from (3.4.7.1) it follows that $x = x' \partial g$. From this we immediately recognize that $(H_0 \times_{F,\mathscr{G}} G_0)_y$ has a structure of (G_1, G_0)-torsor, where the equivariant map to G_0 is simply the projection (Equivariance follows at once from the diagram above).

If for simplicity we simply denote $F(\pi_{\mathscr{H}}(y))$ by (P, s), then from the previous paragraphs it follows that at the level of underlying G_1-torsors we have an isomorphism

$$(H_0 \times_{F,\mathscr{G}} G_0)_y \xrightarrow{\sim} \underline{\mathrm{Hom}}_{G_1}(G_1, P) \tag{4.4.2.1}$$

obtained by sending (f, x) to f^{-1}. The right-hand side is a G_1-torsor in a trivial way (from the left-action of G_1 onto itself), and in addition we have an isomorphism

$$\underline{\mathrm{Hom}}_{G_1}(G_1, P) \xrightarrow{\sim} P, \tag{4.4.2.2}$$

obtained by evaluating a map m on the left-hand side at the unit $1 \in G_1$ (see [**Gir71**, III.1.2.7 (i)]). It is also clear that (4.4.2.1) is a morphism of (G_1, G_0)-torsors: the projection sending (f, x) to x maps to $s \circ f^{-1}$, and this goes to s itself via the latter isomorphism.

In summary, we set $\varphi_{H_1,y}$ equal to the composition of (4.4.2.2) with (4.4.2.1).

We must verify the property that given a morphism

$$h\colon \pi_{\mathscr{H}}(y) \longrightarrow \pi_{\mathscr{H}}(y'),$$

which corresponds to an element $h\colon U \to H_1$ such that $y = y' \, \partial h$, the isomorphism just defined $\varphi_{H_1,y}$ behaves naturally with respect to it, namely that the diagram

$$\begin{tikzcd}
(H_0 \times_{F,\mathscr{G}} G_0)_y \arrow[r] \arrow[d, "\varphi_{H_1,y}"'] & (H_0 \times_{F,\mathscr{G}} G_0)_{y'} \arrow[d, "\varphi_{H_1,y'}"] \\
F(\pi_{\mathscr{H}}(y)) \arrow[r, "F(h)"'] & F(\pi_{\mathscr{H}}(y'))
\end{tikzcd} \tag{4.4.2.3}$$

commutes. By H_1-equivariance, the top horizontal map is just the right H_1-action of h^{-1} via κ, namely the one sending (y, f, x) to

$$(y, f, x)\,\kappa(h^{-1}) = (y', f f_h, 1),$$

where f_h was defined before along with κ. The isomorphisms f and ff_h fit in the commutative diagram

$$\begin{tikzcd} F(\pi_{\mathscr{H}}(y)) \arrow[dr, bend left, "f"] \arrow[dd, "F(h)"'] & \\ & \pi_{\mathscr{G}}(1) \\ F(\pi_{\mathscr{H}}(y')) \arrow[ur, bend right, "ff_h"'] & \end{tikzcd}$$

for F is additive, and we can use (4.3.5.2). Recalling the definition of $\varphi_{H_1,y}$, it is clear that (4.4.2.3) commutes.

The case of a general (H_1, H_0)-torsor (Q, t) is obtained by gluing local instances of the above construction via descent. That is, if U is an object of S, and $Y \to U$ a local epimorphism such that $u\colon Y \to Q$ gives a trivialization $(H_1)_Y \xrightarrow{\sim} Q_Y$ of the underlying H_1-torsor, we obtain a morphism of (H_1, H_0)-torsors

$$(Q, t)_Y \xrightarrow{\ \sim\ } (H_1, y)_Y$$

where $y = u^*t$, and in turn

$$h\colon \pi_{\mathscr{H}}(d_0^* y) \xrightarrow{\ \sim\ } \pi_{\mathscr{H}}(d_1^* y)$$

is realized by an element h of H_1 over $Y \times_U Y$, so that the pair (h, y) is a 0-cocycle relative to $\check{\mathrm{C}}(Y \to U)$ as seen in sect. 3.3.

By applying F, we obtain descent data for the (G_1, G_0)-torsors $F(\pi_{\mathscr{H}}(y))$ via the morphisms $F(h)$. This reconstructs $F(Q, t)$, since locally we have $F(H_1, y)_Y \xrightarrow{\sim} F(Q, t)_Y$.

Similarly, $\underline{\mathrm{Hom}}_{H_1}(Q, H_0 \times_{F,\mathscr{G}} G_0)_t$ is obtained from the corresponding descent data for the various $(H_0 \times_{F,\mathscr{G}} G_0)_y$ resulting from the diagram (4.4.2.3).

This shows at once that the $\varphi_{H_1,y}$ glue into a global

$$\varphi_{Q,t}\colon F(Q, t) \xrightarrow{\ \sim\ } \underline{\mathrm{Hom}}_{H_1}(Q, H_0 \times_{F,\mathscr{G}} G_0)_t.$$

That the morphism $\varphi_{Q,t}$ is itself natural with respect to morphisms of (H_1, H_0)-torsors follows again by patching arguments. □

This concludes the proof of Theorem 4.3.1. □

4.5. Strict morphisms and butterflies. — We have observed that strict morphisms of crossed modules induce weak morphisms in an obvious way. On the other hand, Theorem 4.3.1 entails the notion that butterflies ought to be considered as (weak) morphisms. It is natural to ask what kind of butterfly diagrams correspond to the strict morphisms of Definition 3.2.3.

4.5.1. — Let $F = (f_1, f_0)$ be a strict morphism $F\colon H_\bullet \to G_\bullet$ as in Definition 3.2.3. The corresponding butterfly is

$$\begin{tikzcd}
H_1 \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& H_0 \ltimes G_1 \arrow[dl, "\pi"'] \arrow[dr, "\jmath"] & \\
H_0 & & G_0
\end{tikzcd} \tag{4.5.1.1}$$

where $\pi = \mathrm{pr}_1$, $\imath = (1, \mathrm{id})$, $\kappa(h) = (\partial(h), f_1(h^{-1}))$, and $\jmath(y, g) = f_0(y)\partial(g)$. The semi-direct product at the center of the butterfly corresponds to the trivial extension of H_0 by G_1 ([**ML95**]). Equivalently, the NE-SW diagonal of the butterfly is a split extension. Also, the product law *depends* on the actual (strict) morphism $F = (f_1, f_0)$:

$$(y_0, g_0)(y_1, g_1) = (y_0 y_1, g_0^{f_0(y_1)} g_1).$$

Therefore it would be more appropriate to record this dependency in the notation: $H_0 \overset{F}{\ltimes} G_1$.

It is also clear that given the butterfly (4.5.1.1) one can construct a unique strict morphism $(f_1, f_0)\colon H_\bullet \to G_\bullet$. This is due to the canonical splitting homomorphism $s\colon H_0 \to H_0 \ltimes G_1$ which sends y to $(y, 1)$.

4.5.2. — More generally, a butterfly (4.1.3.1) is *splittable* if there exists a homomorphism $s\colon H_0 \to E$. As a result, from standard arguments, the NE-SW diagonal is in the same isomorphism class as the one in (4.5.1.1). Moreover, a unique strict morphism (f_0, f_1) can be constructed from a splittable butterfly once it has been equipped with a specific choice of the splitting homomorphism s: $f_0 = \jmath \circ s$, and f_1 is determined by the difference between κ and $s \circ \partial_H$, to wit:

$$s(\partial h) = \kappa(h)\imath(f_1(h)), \quad h \in H_1.$$

One can easily check that the pair (f_0, f_1) so determined has all the required properties (3.2.3.1) and (3.2.3.2).

The next statement is therefore an immediate consequence of Theorem 4.3.1:

4.5.3 Proposition. — *A weak morphism $F\colon H_\bullet \to G_\bullet$ is equivalent to a strict one if and only if its butterfly is isomorphic to a split one.*

If $\gamma\colon F \Rightarrow F'$ is a strict 2-morphism as in Definition 3.2.4, it is easy to see that the group isomorphism

$$\begin{aligned} \varphi\colon H_0 \overset{F}{\ltimes} G_1 &\longrightarrow H_0 \overset{F'}{\ltimes} G_1 \\ (x, g) &\longmapsto (x, \gamma_x^{-1} g) \end{aligned}$$

is a morphism between the split butterflies corresponding to F and F'.

In summary, we have a functor from the category of strict morphisms between $H_\bullet$ and $G_\bullet$ and $\mathsf{B}(H_\bullet, G_\bullet)$.

4.6. Stacks of butterflies and weak morphisms. — For two crossed modules $H_\bullet$ and $G_\bullet$ of T we have introduced the groupoid $\mathsf{B}(H_\bullet, G_\bullet)$ of butterflies from $H_\bullet$ to $G_\bullet$. There is a sheaf-theoretic counterpart, denoted by $\mathscr{B}(H_\bullet, G_\bullet)$, which is defined as usual by assigning to $U \in \operatorname{Ob}\mathsf{S}$ the groupoid

$$\mathsf{B}(H_\bullet|_U, G_\bullet|_U),$$

and to every arrow $V \to U$ of S the functor

$$\mathsf{B}(H_\bullet|_U, G_\bullet|_U) \longrightarrow \mathsf{B}(H_\bullet|_V, G_\bullet|_V).$$

4.6.1 Proposition. — *$\mathscr{B}(H_\bullet, G_\bullet)$ is a stack over S.*

Proof. — Since we are restricting morphisms of ordinary group objects, so in particular sets, it is clear that $\mathscr{B}(H_\bullet, G_\bullet)$ is fibered over S and it is a prestack.

The same idea applies to proving that the descent condition on objects is effective. In slightly more details, let $Y \to U$ be a local epimorphism, and let us consider butterfly descent data along it. Thus, let E' be a butterfly on Y from $(H_\bullet)_Y$ to $(G_\bullet)_Y$, and φ a morphism of butterflies

$$\varphi\colon d_0^*E' \longrightarrow d_1^*E'$$

over $\check{\mathrm{C}}\,Y_1 = Y \times_U Y$ satisfying the cocycle condition $d_1^*\varphi = d_2^*\varphi \circ d_0^*\varphi$ over $\check{\mathrm{C}}\,Y_2$.

Since sheaves of groups form a stack, these data determine a group E object over U such that $\psi\colon E' \xrightarrow{\sim} E_Y$ and $d_1^*\psi \circ \varphi = d_0^*\psi$. Moreover, it is easily seen that all the structural maps in the butterfly $[(H_\bullet)_Y, E', (G_\bullet)_Y]$ glue to provide the corresponding ones for a butterfly $[H_\bullet, E, G_\bullet]$ over U: this follows at once from the fact that $\mathscr{B}(H_\bullet, G_\bullet)$ is a prestack. Consider, for instance, $\imath'\colon (G_1)_Y \to E'$ and the composite

$$(G_1)_Y \xrightarrow{\imath'} E' \xrightarrow{\psi} E_Y.$$

Since φ is a morphism of butterflies, we have $\varphi \circ d_0^*\imath' = d_1^*\imath'$, and we get immediately the equality

$$d_0^*(\psi \circ \imath') = d_1^*(\psi \circ \imath')$$

from which it follows (since $\mathscr{B}(H_\bullet, G_\bullet)$ is a prestack) that there is a morphism

$$\imath\colon G_1 \longrightarrow E$$

of group objects over U. The remaining structural maps, as well as the relations (4.1.3.2) are handled in an entirely similar manner. ☐

If we start from $\mathsf{WM}(H_\bullet, G_\bullet)$, we can define $\mathscr{WM}(H_\bullet, G_\bullet)$ in the same way as we have done for $\mathscr{B}(H_\bullet, G_\bullet)$. Then, as a consequence of Proposition 4.6.1 and the equivalence in Theorem 4.3.1, we have the following:

4.6.2 Corollary. — *$\mathscr{WM}(H_\bullet, G_\bullet)$ is a stack over S.*

4.7. Weak vs. Strict morphisms. — We have seen strict morphisms $H_\bullet \to G_\bullet$ give rise to special kinds of butterflies, namely the split ones (cf. Proposition 4.5.3).

To see how strict morphisms relate to the weak ones, and in particular how they fit into the stack structure for the weak morphisms introduced in sect. 4.6, fix the crossed modules $H_\bullet$ and $G_\bullet$ and consider the category $\mathsf{B}(H_\bullet, G_\bullet)$. Within it, consider the sub-category comprising strict morphisms and 2-morphisms from $H_\bullet$ to $G_\bullet$, which we denote by $\mathsf{B}(H_\bullet, G_\bullet)_{\mathrm{str}}$. The notation means that we view strict morphisms as split butterflies as explained in sect. 4.5.

The obvious inclusion

$$\mathsf{B}(H_\bullet, G_\bullet)_{\mathrm{str}} \hookrightarrow \mathsf{B}(H_\bullet, G_\bullet)$$

extends to the fibered situation. Namely, we have an inclusion of fibered categories.

$$\mathscr{B}(H_\bullet, G_\bullet)_{\mathrm{str}} \hookrightarrow \mathscr{B}(H_\bullet, G_\bullet),$$

where $\mathscr{B}(H_\bullet, G_\bullet)_{\mathrm{str}}$ is defined by repeating the procedure of sect. 4.6. Thus its fiber category over $U \in \operatorname{Ob}\mathsf{S}$ is simply category

$$\mathsf{B}(H_\bullet|_U, G_\bullet|_U)_{\mathrm{str}}.$$

That that this defines a fibered category is immediate, since after all morphisms of (sheaves of) groups pull-back along arrows in S.

Consider the stack completion diagram

$$\begin{tikzcd}
\mathscr{B}(H_\bullet, G_\bullet)_{\mathrm{str}} \arrow[r, "a"] \arrow[dr, bend right] & \mathscr{B}(H_\bullet, G_\bullet)^{\sim}_{\mathrm{str}} \arrow[d, dotted] \arrow[dl, phantom, "\Downarrow"] \\
 & \mathscr{B}(H_\bullet, G_\bullet)
\end{tikzcd}$$

It is clear that the objects of the stack $\mathscr{B}(H_\bullet, G_\bullet)^{\sim}_{\mathrm{str}}$ are butterflies, since the stackification process happens "inside" $\mathscr{B}(H_\bullet, G_\bullet)$. Also, by the very nature of this process, each object is locally isomorphic to one of $\mathscr{B}(H_\bullet, G_\bullet)_{\mathrm{str}}$, hence to a split one. It follows that $\mathscr{B}(H_\bullet, G_\bullet)^{\sim}_{\mathrm{str}}$ is the stack of *locally split* butterflies, in the sense of the following definition:

4.7.1 Definition. — A butterfly E from $H_\bullet$ to $G_\bullet$ is *locally split* if there is a generalized cover V such that the extension on the NE-SW diagonal splits over V.

Morphisms of locally split butterflies are butterfly morphisms $\varphi\colon E \to E'$ such that, if E splits over V and E' splits over V', then $\varphi|_{V\times V'}$ is a strict 2-morphism as expounded at the end of sect. 4.5.

5. The bicategory of crossed modules and weak morphisms

To analyze the global structure of crossed modules equipped with weak morphisms provided by butterflies we need to use a few facts regarding fibered bicategories. Our goal is to prove that crossed modules and weak morphisms over S comprise a fibered bicategory which is in fact a bistack, in the sense made explicit below.

The definitions one states in the context of 2-categories fibered over a site, and the ensuing consequent distinctions based on what actually glues—fibered, prestack, stack—, have a mirror in the realm of fibered bicategories. (We refer to [**Hak72**] for fibered 2-categories over a site, and to [**Bre94**, Chapter 1] for a discussion of the extensions of these concepts to the bicategorical situation.) In particular, by a pre-bistack, we mean a fibered bicategory where the descent conditions are satisfied at all levels, except for objects. In particular, the morphism (fibered) category between any pair of objects will form a stack. A fibered bicategory is a bistack if in addition the 2-descent condition on objects, which is formulated in much the same way as for 2-categories, is effective.

5.1. Composition of butterflies and the bicategory of crossed modules. —

Recall that $\mathsf{B}(H_\bullet, G_\bullet)$ is the groupoid of butterflies from $H_\bullet$ to $G_\bullet$. There is a composition functor

$$\mathsf{B}(K_\bullet, H_\bullet) \times \mathsf{B}(H_\bullet, G_\bullet) \longrightarrow \mathsf{B}(K_\bullet, G_\bullet)$$

which is constructed in the following way (cf. ref. [**Noo05**]).

5.1.1 Definition. — Given two butterflies

$$\begin{tikzcd} K_1 \arrow[dd, "\partial_K"'] \arrow[dr] & & H_1 \arrow[dl, "\imath'"'] \arrow[dd, "\partial_H"] \\ & F \arrow[dl] \arrow[dr, "\jmath'"] & \\ K_0 & & H_0 \end{tikzcd} \qquad \begin{tikzcd} H_1 \arrow[dd, "\partial_H"'] \arrow[dr, "\kappa"] & & G_1 \arrow[dl] \arrow[dd, "\partial_G"] \\ & E \arrow[dl, "\pi"'] \arrow[dr] & \\ H_0 & & G_0 \end{tikzcd}$$

their composition is the butterfly (defined set-theoretically in loc. cit.):

$$\begin{tikzcd} K_1 \arrow[dd, "\partial_K"'] \arrow[dr] & & G_1 \arrow[dl] \arrow[dd, "\partial_G"] \\ & F \times^{H_1}_{H_0} E \arrow[dl] \arrow[dr] & \\ K_0 & & G_0 \end{tikzcd} \tag{5.1.1.1}$$

where the center is given by the following pull-back/push-out construction: the pull-back of the extension $1 \to G_1 \to E \to H_0 \to 1$ along $\jmath' \colon F \to H_0$ gives the extension

$$1 \longrightarrow G_1 \longrightarrow F \times_{H_0} E \longrightarrow F \longrightarrow 1$$

Then the exact NE-SW diagonal of (5.1.1.1) arises as the cokernel of the morphism

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & H_1 \arrow[r, equal] \arrow[d, "(\imath',\kappa)"] & H_1 \arrow[d, "\imath'"] \\ G_1 \arrow[r] & F \times_{H_0} E \arrow[r] & F \end{tikzcd}$$

All vertical maps are monomorphisms, and that the image of $H_1 \xrightarrow{(\imath',\kappa)} F \times_{H_0} E$ is normal thanks to the properties (4.1.3.2) of the maps in the butterflies.

5.1.2. — It is also clear that if

$$[K_\bullet, F, H_\bullet] \to [K_\bullet, F', H_\bullet]\,, \quad [H_\bullet, E, G_\bullet] \to [H_\bullet, E', G_\bullet]$$

are morphisms of butterflies given by $\psi\colon F \xrightarrow{\sim} F'$ and $\varphi\colon E \xrightarrow{\sim} E'$, respectively, then there is a corresponding morphism

$$[K_\bullet, F \times^{H_1}_{H_0} E, G_\bullet] \longrightarrow [K_\bullet, F' \times^{H_1}_{H_0} E', G_\bullet]$$

where

$$F \times^{H_1}_{H_0} E \xrightarrow{\sim} F' \times^{H_1}_{H_0} E'$$

is induced by

$$(\varphi, \psi)\colon F \times_{H_0} E \xrightarrow{\sim} F' \times_{H_0} E'$$

after taking the quotient by the image of H_1 in both ends.

5.1.3. — By the usual arguments, the construction of $F \times^{H_1}_{H_0} E$, is such that if we consider a third butterfly $[L_\bullet, M, K_\bullet]$, then there only is an isomorphism

$$\big(M \times^{K_1}_{K_0} F\big) \times^{H_1}_{H_0} E \xrightarrow{\sim} M \times^{K_1}_{K_0} \big(F \times^{H_1}_{H_0} E\big).$$

As a result the composition law is only associative up to isomorphism. With these provisions, we immediately have the following analog of [**Noo05**, Theorem 10.1]:

5.1.4 Theorem. — *When equipped with the morphism groupoids* $\mathsf{B}(-,-)$*, crossed modules in* T *form a bicategory.*

It is convenient to recall at this point the following special cases. Their handling is unchanged from the set-theoretical situation of ref. [**Noo05**], to which we refer for more details.

5.2. Special cases. — Composition of butterflies assumes a simpler form than (5.1.1.1) when one of the morphisms is strict (cf. sect.4.5). If the morphism $(q_1, q_0)\colon K_\bullet \to H_\bullet$ is strict, then the composition is

$$\begin{tikzcd} K_1 \arrow[dd, "\partial_K"'] \arrow[dr] & & G_1 \arrow[dl] \arrow[dd, "\partial_G"] \\ & q_0^*(E) \arrow[dl] \arrow[dr] & \\ K_0 & & G_0 \end{tikzcd}$$

where the NE-SW diagonal is the pull-back of the extension

$$1 \to G_1 \longrightarrow E \longrightarrow H_0 \longrightarrow 1$$

along the homomorphism $q_0\colon K_0 \to H_0$. In particular, $q_0^*(E) = F \times_{H_0} E$ is the fiber product.

Similarly, if the second morphism $(p_1, p_0)\colon H_\bullet \to G_\bullet$ is strict instead, then the composition is

$$\begin{tikzcd} K_1 \arrow[dd, "\partial_K"'] \arrow[dr] & & G_1 \arrow[dl] \arrow[dd, "\partial_G"] \\ & {p_1}_*(F) \arrow[dl] \arrow[dr] & \\ K_0 & & G_0 \end{tikzcd}$$

where this time the NE-SW diagonal is the push-forward of the extension

$$1 \to H_1 \longrightarrow F \longrightarrow K_0 \longrightarrow 1$$

along the homomorphism $p_1\colon H_1 \to G_1$. Also, ${p_1}_*(F) = F \sqcup_{H_1} E$ is the push-out.

We close with the following simple

5.2.1 Lemma. — *The weak morphism F determined by a butterfly* (4.1.3.1) *is an* equivalence *if and only if the butterfly is* flippable, *or* reversible, *as in Definition 4.1.5.*

In this case the same butterfly, but read right-to-left:

$$\begin{tikzcd} G_1 \arrow[dd, "\partial"'] \arrow[dr, "\imath"] & & H_1 \arrow[dl, "\kappa"'] \arrow[dd, "\partial"] \\ & E \arrow[dl, "\jmath"'] \arrow[dr, "\pi"] & \\ G_0 & & H_0 \end{tikzcd}$$

is a diagram corresponding to the choice of an inverse functor F^* of F.

Proof of the lemma. — If the butterfly (4.1.3.1) is reversible, composing it with its flipped counterpart according to (5.1.1.1) yields the isomorphisms

$$E \times^{G_1}_{G_0} E \xrightarrow{\sim} H_0 \ltimes^{\mathrm{Id}} H_1 \,, \quad E \times^{H_1}_{H_0} E \xrightarrow{\sim} G_0 \ltimes^{\mathrm{Id}} G_1 .$$

so that F is an equivalence.

Conversely, if F is an equivalence, then we have seen from the proof of Theorem 4.3.1 that

$$E = H_0 \times_{\mathscr{G}} G_0$$

with respect to $F \circ \pi_{\mathscr{H}} \colon H_0 \to \mathscr{G}$. Since F is an equivalence, the sequence

$$H_1 \longrightarrow H_0 \xrightarrow{F \circ \pi_{\mathscr{H}}} \mathscr{G}$$

is still homotopy-exact. Hence, by pull-back along $G_0 \to \mathscr{G}$, the sequence

$$H_1 \longrightarrow E \longrightarrow G_0$$

is short-exact, that is, the butterfly is flippable. □

5.3. The bistack of crossed modules. — Let $\underline{\mathsf{XMod}}(\mathsf{S})$, or $\underline{\mathsf{XMod}}$ for short, the bicategory of crossed modules of T in Theorem 5.1.4. Define $\mathfrak{XMod}(\mathsf{S})$, or simply $\mathfrak{XMod}$, by assigning to every object U of S the bicategory

$$\underline{\mathsf{XMod}}(\mathsf{S}/U)$$

of crossed modules over S/U, and to every arrow $V \to U$ of S the homomorphism [(1)]

$$\underline{\mathsf{XMod}}(\mathsf{S}/U) \longrightarrow \underline{\mathsf{XMod}}(\mathsf{S}/V)$$

obtained by composition with the morphism $\mathsf{S}/V \to \mathsf{S}/U$.

5.3.1 Proposition. — *$\mathfrak{XMod}$ is fibered over S. Moreover, we have an equivalence*

$$\underline{\mathsf{XMod}} \xrightarrow{\sim} \varprojlim_{U \in \mathrm{Ob}(\mathsf{S})} \mathfrak{XMod}$$

in the sense of bicategories.

Proof. — Sheaves of groups over S form a stack, hence in particular a fibered category over S, which is in fact split [**Gir71**]. Since all morphisms and 2-morphisms in $\underline{\mathsf{XMod}}(\mathsf{S}/U)$ are in effect diagrams of morphisms of sheaves of groups, it follows that pull-backs (as bifunctors) exist in $\mathfrak{XMod}$. □

Moreover, as an immediate consequence of Proposition 4.6.1 we obtain:

5.3.2 Proposition. — *The fibered bicategory $\mathfrak{XMod}$ of crossed modules over S is a pre-bistack.*

---

(1) We use the term "homomorphism" in the sense of: 1-morphism between bicategories, such that the structural 2-morphisms are isomorphisms—see [**Bé67**]

Proof. — Given two objects, $G_\bullet$ and $H_\bullet$, the fibered category of morphisms

$$\mathscr{H}om_{\mathfrak{XMod}}(H_\bullet, G_\bullet)$$

is precisely $\mathscr{B}(H_\bullet, G_\bullet)$, which is a stack. $\square$

We now turn to the question of whether 2-descent for objects of $\mathfrak{XMod}$ is effective.

5.3.3. — For the sake of definiteness, let us provide a detailed list for a 2-descent datum for $\mathfrak{XMod}$ over $U \in \operatorname{Ob}\mathsf{S}$.

1. a hypercover $Y_\bullet \to U$ (e.g. the Čech complex $\check{C}\,Y$ associated to a generalized cover $Y \to U$);
2. a crossed module $G'_\bullet$ over Y_0;
3. a reversible butterfly $[d_0^* G'_\bullet, E, d_1^* G'_\bullet]$ over Y_1;
4. a morphism of butterflies $\alpha\colon d_1^*E \Rightarrow d_2^*E \circ d_0^*E$ over Y_2;
5. a coherence condition for the $d_i^*\alpha$ over Y_3, $i = 0,1,2,3$.

The descent datum is effective if there exists a crossed-module $G\bullet$ over U, a reversible butterfly $[G'_\bullet, F, G_\bullet|_{Y_0}]$, and a morphism of butterflies

$$\beta\colon d_1^*F \circ E \Rightarrow d_0^*F$$

over Y_1, which is coherent over Y_2.

To prove that all 2-descent data in $\mathfrak{XMod}$ are effective (so that $\mathfrak{XMod}$ is a bistack) it is best to exploit the relationship of $\mathfrak{XMod}$ with the 2-category of gr-stacks (as opposed to giving a direct proof).

For this, first consider the 2-category $\underline{\mathsf{Stacks}}(\mathsf{S})$ of stacks over the site S. It gives rise to a fibered 2-category $\textsc{Stacks}(\mathsf{S})$ over S whose fiber over $U \in \operatorname{Ob}\mathsf{S}$ is $\underline{\mathsf{Stacks}}(\mathsf{S}/U)$. To every morphism $V \to U$ in S it corresponds a Cartesian 2-functor

$$f^*\colon \underline{\mathsf{Stacks}}(\mathsf{S}/U) \longrightarrow \underline{\mathsf{Stacks}}(\mathsf{S}/V)$$

arising from the pull-back $\mathscr{F} \rightsquigarrow f^*\mathscr{F}$, where $\mathscr{F}$ is a stack over S/U. It is known that $\textsc{Stacks}(\mathsf{S})$ is a 2-stack over S (see [**Bre94**]).

Gr-stacks form a sub-2-category $\underline{\mathsf{Gr\text{-}Stacks}}(\mathsf{S})$ of $\underline{\mathsf{Stacks}}(\mathsf{S})$ in the obvious way: the 1-morphisms are the additive functors and the 2-morphisms are the natural transformations of additive functors. There is an obvious forgetful functor

$$\underline{\mathsf{Gr\text{-}Stacks}}(\mathsf{S}) \longrightarrow \underline{\mathsf{Stacks}}(\mathsf{S})$$

which simply forgets the additive structure. Once again, these considerations extend to the fibered situation to yield $\textsc{Gr-Stacks}(\mathsf{S})$, the fibered 2-category of gr-stacks over the site S, as well as the forgetful functor

$$\textsc{Gr-Stacks}(\mathsf{S}) \longrightarrow \textsc{Stacks}(\mathsf{S}).$$

We have the following important result:

5.3.4 Theorem. — $\textsc{Gr-Stacks}(\mathsf{S})$ *is a 2-stack.*

A direct proof is sketched in Appendix A. A more conceptual proof will be available in a forthcoming paper by the authors.

Note that Corollary 4.6.2 also follows directly from Thm. 5.3.4, which is logically independent of any statement about crossed modules.

5.3.5 Definition. — Let F be the homomorphism

$$F\colon \mathfrak{XMod}(\mathsf{S}) \longrightarrow \text{GR-STACKS}(\mathsf{S}) \tag{5.3.5.1}$$

defined by sending the crossed module $[G_1 \to G_0]$ to its associated gr-stack $[G_1 \to G_0]^\sim$, and, for two crossed modules $H_\bullet$ and $G_\bullet$, $\mathscr{B}(H_\bullet, G_\bullet)$ to $\mathscr{WM}(H_\bullet, G_\bullet)$.

On the right-hand side of (5.3.5.1) we consider GR-STACKS(S) as a bicategory (and in fact, as a bistack) in the obvious way.

It immediately follows from Theorem 4.3.1 that F is 2-faithful—or fully faithful—in the sense of bicategories (cf. [**Hak72**] for the definition of i-faithful—$i = 0, \dots, 3$—in the context of 2-categories). The main results is:

5.3.6 Theorem. — *The homomorphism F in* (5.3.5.1) *is a biequivalence. Therefore* $\mathfrak{XMod}(\mathsf{S})$ *is a bistack.*

Since we have already remarked that F is 2-faithful, the only thing to be proved is essential surjectivity. We state it separately in the next proposition, which is also of independent interest.

5.3.7 Proposition. — *Let $\mathscr{G}$ be a gr-stack. Then there exists a crossed module $[G_1 \to G_0]$ such that $\mathscr{G}$ is equivalent to the gr-stack $[G_1 \to G_0]^\sim$.*

Proof. — The first step is to construct an additive functor

$$\pi\colon G_0 \longrightarrow \mathscr{G}$$

where G_0 is a group-object in T. G_0 is considered, of course, as a gr-stack in the obvious way. First, we have the following:

5.3.8 Lemma. — *There exists an essentially surjective map*

$$\pi_0\colon X \longrightarrow \mathscr{G},$$

where X is a space *over* S.

Proof of the lemma. — Choose a skeleton $\operatorname{sk}\mathscr{G}$ of $\mathscr{G}$. (Recall that a skeleton is a full subcategory having one object for each isomorphism class of objects of the ambient category, cf. [**Sch72**].) The inclusion $\imath\colon \operatorname{sk}\mathscr{G} \to \mathscr{G}$ is an equivalence, and it is easy to show that $\operatorname{sk}\mathscr{G}$ is fibered, and in fact split, over S. Therefore $\operatorname{Ob}(\operatorname{sk}\mathscr{G})$ is a presheaf of sets. It is actually a *separated* presheaf, as follows.

A descent datum for $\operatorname{Ob}\operatorname{sk}\mathscr{G}$ is given by a pair $(x, V_\bullet)$, where $V_\bullet \to U$ is a (hyper)cover of $U \in \operatorname{Ob}\mathsf{S}$, and x is an object of $\operatorname{sk}\mathscr{G}$ over V_0 such that $d_0^* x = d_1^* x$ over V_1. This defines a descent datum for $\mathscr{G}$ (with identity maps as morphisms), so that there exists an object y of $\mathscr{G}$ over U such that its pullback to V_0 is isomorphic to x. The object y is defined up to isomorphism (in $\mathscr{G}_U$), which shows that if it is in $\operatorname{sk}\mathscr{G}_U$, then it must be unique. (Incidentally, this argument also shows why we ought not expect $\operatorname{Ob}\operatorname{sk}\mathscr{G}$ to be a sheaf.)

We define X to be the associated sheaf: since $\operatorname{Ob}\operatorname{sk}\mathscr{G}$ is separated to begin with, the "plus" construction will have to be done only once. Define π_0 to be the extension of $\imath$ to X. It exists, because an element of $X(U)$ is given by a pair $\xi = (x, V_\bullet)$ as above, with gluing object $y \in \operatorname{Ob}\mathscr{G}_U$, so we set $\pi_0(\xi) = y$. $\square$

For any set S, let $F\langle S\rangle$ denote the free group over S. Define G_0 as

$$G_0 = F\langle X\rangle^{\sim},$$

where the tilde denotes sheafification. In other words, the right-hand side is the sheafification of the presheaf $U \rightsquigarrow F\langle X(U)\rangle$.

5.3.9 Lemma. — *The map π_0 extends to an additive functor $\pi\colon G_0 \to \mathscr{G}$.*

Proof of the lemma. — One follows the same pattern used to show the free group $F\langle S\rangle$ has the universal property with respect to set morphisms from S to groups. Since the group law of $\mathscr{G}$ is only a weak one in general, an ordering problem arises. Locally, given a word $x_1 \cdots x_n$ where $x_1, \dots, x_n$ are element of $X(U)$ over some $U \in \operatorname{Ob}\mathsf{S}$, we define $\pi(x_1 \dots x_n)$ by choosing a specific nesting of parentheses and then using the group law of $\mathscr{G}$. Specifically, we set:

$$\pi_0(x_1 \dots x_n) \overset{\text{def}}{=} (\cdots((x_1 \otimes x_2) \otimes x_3) \otimes \cdots \otimes x_n)$$

associating from the left. That this is well defined follows from results of Laplaza about coherence in gr-categories—cf. [**Lap83**]. $\square$

G_1 is defined by the square

$$\begin{tikzcd} G_1 \arrow[r, "\partial"] \arrow[d] & G_0 \arrow[d, "\pi"] \\ \mathbf{1} \arrow[r] & \mathscr{G} \end{tikzcd}$$

of gr-stacks, where $\mathbf{1} \to \mathscr{G}$ corresponds to the unit object ($\mathbf{1}$ is the category with one object and one arrow). Thus, the elements of G_1 consist of pairs (x, α) where $x \in G_0$ and $\alpha\colon I \to \pi(x)$ in $\mathscr{G}$. The map ∂ is the projection to G_0 sending $g = (x, \alpha)$ to x.

The multiplicative structure of G_1 is given by

$$(x, \alpha)(y, \beta) = (xy, \alpha\beta),$$

where $\alpha\beta$ is to be interpreted as the composite arrow

$$I \xrightarrow{\sim} I \otimes I \xrightarrow{\alpha\otimes\beta} \pi(x) \otimes \pi(y) \xrightarrow{\sim} \pi(xy).$$

The unit of G_1 is the pair $(1, \mu)$, where μ is the morphism such that

$$\mu\colon I \xrightarrow{\sim} \pi(1),$$

and 1 is the unit element of G_0. The inverse of the pair (x, α) is given by the pair $(x^{-1}, \hat{\alpha})$, where $\hat{\alpha}$ is the composite

$$I \xrightarrow{(\alpha^*)^{-1}} \pi(x)^* \xrightarrow{\sim} \pi(x^{-1}).$$

(Recall that the choice of a quasi-inverse in a gr-category—and therefore in a gr-stack—gives a functor $*\colon \mathscr{G}^{\mathrm{op}} \to \mathscr{G}$.) With these definitions the map ∂ evidently is a group homomorphism.

Let us define an action of G_0 on G_1 by setting:

$$(y, \beta)^x = (x^{-1}yx, x^{-1}\beta x). \tag{5.3.9.1}$$

Here $x \in G_0$, $(y, \beta) \in G_1$, and $x^{-1}\beta x$ is defined to be the composite

$$I \longrightarrow \pi(x)^* \otimes \pi(x) \longrightarrow (\pi(x)^* \otimes I) \otimes \pi(x) \longrightarrow (\pi(x)^* \otimes \pi(y)) \otimes \pi(x) \xrightarrow{\sim} \pi(x^{-1}yx)$$

where the star denotes the inverse operation in $\mathscr{G}$, and the last arrow on the right is in turn the composite of

$$(\pi(x)^* \otimes \pi(y)) \otimes \pi(x) \simeq (\pi(x^{-1}) \otimes \pi(y)) \otimes \pi(x) \simeq \pi(x^{-1}y) \otimes \pi(x) \simeq \pi(x^{-1}yx).$$

The arrow in the middle in the definition of $x^{-1}\beta x$ is $(\pi(x)^* \otimes \beta) \otimes \pi(x)$. Had we chosen to use $\pi(x)^* \otimes (\beta \otimes \pi(x))$ instead, then the commutativity of (3.1.1.2) in any gr-category, and the functoriality of the associator, ensure the definition of $x^{-1}\beta x$ is unaffected.

We must verify that the two axioms (3.2.1.1) of a crossed module hold for the action (5.3.9.1). The first,

$$\partial(h^x) = x^{-1}\partial h\, x$$

with $h = (y, \beta)$, is obvious. For the second, namely

$$h^{\partial g} = g^{-1}h\, g,$$

with $g = (x, \alpha)$ and $h = (y, \beta)$, to hold, we must have

$$\hat{\alpha}\beta\alpha = x^{-1}\beta x.$$

To see why this is true, consider the following diagram:

$$\begin{tikzcd}
I \arrow[r, "\sim"] & I \otimes I \arrow[r, "\sim"] \arrow[d, "(\alpha^*)^{-1}\otimes\alpha"'] & I \otimes (I \otimes I) \arrow[r, "(\alpha^*)^{-1}\otimes(\beta\otimes\alpha)"] \arrow[d, "(\alpha^*)^{-1}\otimes(I\otimes\alpha)"'] & \pi(x)^* \otimes (\pi(y) \otimes \pi(x)) \\
& \pi(x)^* \otimes \pi(x) \arrow[r] & \pi(x)^* \otimes (I \otimes \pi(x)) \arrow[ur, bend right, "\pi(x)^*\otimes(\beta\otimes\pi(x))"'] &
\end{tikzcd}$$

The relevant part of $\hat{\alpha}\beta\alpha$ is the composition in the top horizontal line. The left square is commutative due to functoriality and the definition of α^*. The composition of arrows at the bottom gives $x^{-1}\beta x$.

Finally, the equivalence between $\mathscr{G}$ and $[G_1 \to G_0]^{\sim}$ is obvious: full faithfulness is built-in in the definition of G_1, whereas essential surjectivity follows from the definition of π.

This concludes the proof of the proposition. □

5.3.10 Remark. — An entirely similar proof in the case of strictly Picard gr-stacks is due to Deligne, cf. ref. [**Del73**, Lemme 1.4.13 i]. Breen and Messing have sketched a proof for the general case, using a rigidification of the group law on the simplicial object determined by $\mathscr{G}$, cf. [**BM05**, Appendix B].

5.4. Derived category of crossed modules. — This section is devoted to some remarks and analogies with standard (abelian) homological algebra.

The first observation is that butterflies really *are* fractions.

Let $[H_\bullet, E, G_\bullet]$ be a butterfly. It follows immediately from its properties, and explicitly from [**Noo05**], that

$$E_\bullet\colon\ H_1 \times G_1 \xrightarrow{\kappa\cdot\imath} E \qquad (h,g) \longmapsto \kappa(g)\,\imath(g),$$

is a crossed module. The butterfly thus gives rise to a "fraction," that is, a diagram of *strict* morphisms of crossed modules

$$H_\bullet \xleftarrow[Q]{\sim} E_\bullet \xrightarrow{P} G_\bullet$$

or, explicitly,

$$\begin{tikzcd} H_1 \arrow[d, "\partial"'] & H_1 \times G_1 \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\kappa\cdot\imath"'] & G_1 \arrow[d, "\partial"] \\ H_0 & E \arrow[l, "\pi"] \arrow[r, "\jmath"'] & G_0 \end{tikzcd}$$

The strict morphism Q is a quasi-isomorphism, inducing an isomorphism of homotopy groups, as it can be readily checked.

The following is very easy, but it is worthwhile to point the statement out.

5.4.1 Lemma. — *$F\colon \mathscr{H} \to \mathscr{G}$ is a quasi-isomorphism if and only if it is an equivalence.*

Proof. — Since the homotopy groups are the same, following the ideas of [**Bre94**, §7], we can consider F as a morphism over a space (=sheaf) π_0. Both $\mathscr{H}$ and $\mathscr{G}$ are π_1-gerbes over π_0, hence they must be equivalent.

The converse is obvious. □

It follows from the lemma that if we call $\mathscr{E}$ the gr-stack associated to the crossed module $E_\bullet$, the induced morphism $\mathscr{E} \to \mathscr{H}$ is an equivalence. Thus, analogously to the abelian situation in [**Del73**] F factorizes as $F = P \circ Q^*$, where Q^* is a quasi-inverse to Q (as morphisms of gr-stacks).

Continuing the analogy with loc. cit., let us denote by $\underline{\mathsf{Gr\text{-}Stacks}}^\flat(\mathsf{S})$ the category whose objects are the gr-stacks of S and whose morphisms are isomorphism classes of additive functors. Similarly, $\underline{\mathsf{XMod}}^\flat(\mathsf{S})$ will denote the category whose objects are crossed modules and whose morphisms are isomorphism classes of butterflies. Then Theorem 5.3.6 implies that there is an equivalence of categories

$$\underline{\mathsf{XMod}}^\flat(\mathsf{S}) \xrightarrow{\sim} \underline{\mathsf{Gr\text{-}Stacks}}^\flat(\mathsf{S}).$$

6. Exact sequences of gr-stacks and homotopy groups

In the set-theoretic context, given a butterfly

$$\begin{tikzcd} H_1 \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\ & E \arrow[dl, "\pi"] \arrow[dr, "\jmath"'] & \\ H_0 & & G_0 \end{tikzcd}$$

the complex $F_\bullet$ on the NW-SE diagonal participates in the long exact sequence of homotopy groups:

$$1 \to \pi_2(\underline{\mathsf{F}}) \to \pi_1(\mathsf{H}) \to \pi_1(\mathsf{G}) \to \pi_1(\underline{\mathsf{F}}) \to \pi_0(\mathsf{H}) \to \pi_0(\mathsf{G}) \to \pi_0(\underline{\mathsf{F}}) \to 1,$$

where H, G are the groupoids corresponding to $H_\bullet$ and $G_\bullet$, respectively, and $\underline{\mathsf{F}}$ is a 2-groupoid built from the complex $F_\bullet$, cf. [**Noo05**]. In all instances these homotopy groups coincide with the naïve (non-abelian) homology groups of the complexes. It is shown in loc. cit. that there is homotopy fiber sequence of simplicial objects

$$\underline{F_\bullet} \longrightarrow \underline{H_\bullet} \longrightarrow \underline{G_\bullet}.$$

All terms are the nerves of the corresponding (2-)groupoids: for $\underline{H_\bullet}$ and $\underline{G_\bullet}$ this was briefly recalled in sect. 3.3. They also are simplicial groups. For 2-groupoids, see [**MS93**]. Note that $\underline{F_\bullet}$ is *not* a simplicial group.

We give a geometric account of this circle of ideas in the sheaf-theoretic case, based on the correspondence between weak morphisms and morphisms of gr-stacks.

6.1. Homotopy kernel. — Let $F\colon \mathscr{H} \to \mathscr{G}$ be a morphism of gr-stacks. The homotopy kernel of F, $\mathscr{K} = \mathscr{K}er(F)$, is a gr-stack defined by the (stack) fiber product:

$$\begin{tikzcd} \mathscr{K} \arrow[r, "J"] \arrow[d] \arrow[dr, phantom, "\not\Downarrow_{\varepsilon}"] & \mathscr{H} \arrow[d, "F"] \\ \mathbf{1} \arrow[r] & \mathscr{G} \end{tikzcd} .$$

We have explicitly marked the 2-morphism $\varepsilon\colon F \circ J \Rightarrow I$, where I is the null functor, sending every object I of $\mathscr{K}$ to the unit object of $\mathscr{G}$, and every morphism to the identity morphism of the unit object. Explicitly, the objects of $\mathscr{K}$ are given by pairs (Y, f), where Y is an object of $\mathscr{H}$, and $f\colon F(Y) \xrightarrow{\sim} I_{\mathscr{G}}$. A morphism $a\colon (Z,g) \to (Y,f)$ is given by a morphism $a\colon Z \to Y$ in $\mathscr{H}$ such that the diagram

$$\begin{tikzcd} F(Z) \arrow[r, "F(a)"] \arrow[dr, bend right, "g"'] & F(Y) \arrow[d, "f"] \\ & I_{\mathscr{G}} \end{tikzcd}$$

commutes. The multiplication law is given by:

$$(Y, f) \otimes_{\mathscr{K}} (Z, g) = (Y \otimes_{\mathscr{H}} Z, fg),$$

where fg is the morphism obtained by composing $f \otimes_{\mathscr{G}} g$ with the obvious structural ones. In fact the construction of the group G_1 in the proof of Lemma 5.3.9 is but an instance of homotopy kernel. There will be no difficulty in realizing that the multiplication and inverse laws of $\mathscr{K}$ are just the obvious translations of those already analyzed in detail in that case.

Let $H_\bullet$ and $G_\bullet$ be crossed modules corresponding to $\mathscr{H}$ and $\mathscr{G}$, respectively. Let $[H_\bullet, E, G_\bullet]$ be the butterfly corresponding to the weak morphism $H_\bullet \to G_\bullet$ determined by $F\colon \mathscr{H} \to \mathscr{G}$. We may take $E = H_0 \times_{\mathscr{G},F} G_0$. According to Proposition 5.3.7, there exists a crossed module such that its associated gr-stack is equivalent to $\mathscr{K}$. More precisely, consider the map $\jmath\colon E \to G_0$ for the butterfly $[H_\bullet, E, G_\bullet]$. We have:

6.1.1 Proposition. — *The kernel $\mathscr{K}$ of F is equivalent to the gr-stack associated to the crossed module $[H_1 \to \operatorname{Ker}\jmath]$. Moreover, $J\colon \mathscr{K} \to \mathscr{H}$ corresponds to the obvious* strict *morphism of crossed modules*

$$\begin{tikzcd} H_1 \arrow[r, equal] \arrow[d, "\bar{\kappa}"'] & H_1 \arrow[d, "\partial"] \\ \operatorname{Ker}\jmath \arrow[r] & H_0 \end{tikzcd}$$

Proof. — It is immediately checked that $[H_1 \to \operatorname{Ker}\jmath]$ is a crossed module.

Now, assume we are given an (H_1, H_0)-torsor (Q, t) such that its image by F is isomorphic to the unit object of $\mathscr{G}$, which we identify with $(G_1, 1)$ (the trivial G_1-torsor, equipped with the equivariant map $1\colon G_1 \to G_0$). Recall that the image of (Q, t) by F is the G_1-torsor $\underline{\mathrm{Hom}}_{H_1}(Q, E)_t$ of local lifts of t to E, equipped with the global equivariant map sending each local lift e to $\jmath(e)$. To say that this is isomorphic to the identity object $(G_1, 1)$ means that there exists a global lift $e\colon Q \to E$ such that $\jmath(e) = 1$. Thus the objects of $\mathscr{K}$ are $(H_1, \operatorname{Ker}\jmath)$-torsors. It is clear that these constructions are functorial.

The second part of the proposition is obvious. □

The butterfly corresponding to J (from the proposition) and the one corresponding to F are composed according to the prescription of 5.2. It is immediately verified that the result, which corresponds to $F \circ J$, is globally split—the splitting homomorphism $s\colon \ker \jmath \to \ker \jmath \times_{H_0} E$ is simply the diagonal—and moreover it has the two properties in Lemma 10.3 of [**Noo05**]. Hence it corresponds to the trivial morphism $1\colon [H_1 \to \ker \jmath] \to [G_1 \to G_0]$.

6.2. Exact sequences. —

6.2.1. — The sequence

$$\mathscr{K} \xrightarrow{G} \mathscr{H} \xrightarrow{F} \mathscr{G} \tag{6.2.1.1}$$

of morphisms of gr-stacks is a *complex* if there is a 2-morphism ε from $F \circ G$ to the null-functor:

$$\begin{tikzcd} \mathscr{K} \arrow[r, "G"] \arrow[rr, bend right, "I"'] & \mathscr{H} \arrow[r, "F"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \mathscr{G} \\ & {} & \end{tikzcd}$$

This idea of a complex of gr-stacks is standard. (In the set-theoretic case, compare ref. [**Vit02**].) For geometric applications, see ref. [**Ald08**]. From (6.2.1.1) there results a functor:

$$\bar{G}\colon \mathscr{G} \longrightarrow \mathscr{K}er(F)$$

defined by sending $Y \in \operatorname{Ob} \mathscr{G}_U$ to the pair $(G(Y), \varepsilon_Y)$. While obvious, this allows to formulate the notion of exactness in the middle of the sequence (6.2.1.1) as follows:

6.2.2 Definition. — The sequence (6.2.1.1) is exact at $\mathscr{H}$ if the functor $\bar{G}$ is full and essentially surjective.

This definition can be found, for instance, in ref. [**Vit02**] in the context of gr-categories. The formulation for gr-stacks is the same, while taking care that "full" and "essentially surjective" must be intended in the appropriate context.

For the definition of "short exact," still in the context of gr-categories, see refs. [**Bre92, Rou03**]. We repeat it in the context of gr-stacks

6.2.3 Definition. — The sequence (6.2.1.1) is:

- *left exact* if it is exact at $\mathscr{H}$ and $\bar{G}\colon \mathscr{K} \to \mathscr{K}er(F)$ is an equivalence;
- an extension of gr-stacks if it is both left exact and F is essentially surjective.

We leave out the fibration condition found in [**Bre90**], see section 6.4 below for more details.

6.2.4. — Recall that if $\mathscr{G}$ is a gr-stack we define $\pi_1(\mathscr{G}) = \underline{\mathrm{Aut}}_{\mathscr{G}}(I)$ and $\pi_0(\mathscr{G})$ as the sheaf associated to the presheaf $U \rightsquigarrow \mathrm{Ob}\,\mathscr{G}_U/\sim$. Given the sequence (6.2.1.1) one obtains corresponding sequences

$$\pi_i(\mathscr{K}) \longrightarrow \pi_i(\mathscr{H}) \longrightarrow \pi_i(\mathscr{G}), \qquad i = 0, 1. \tag{6.2.4.1}$$

We have the following easy lemma (cf. [**Vit02**]). We sketch its proof anyway, in view of subsequent applications and the fact it is being formulated for gr-stacks, as opposed to gr-categories, as in loc. cit.

6.2.5 Lemma. — *If* (6.2.1.1) *is exact at* $\mathscr{H}$*, the sequences* (6.2.4.1) *are exact at* $\pi_i(\mathscr{H})$*,* $i = 0, 1$.

Proof. — For $i = 1$, we have that if f is an automorphism of $I_{\mathscr{H}}$ over U, then its image $\pi_1(F)(f)$ is defined by:

$$\begin{tikzcd}
F(I_{\mathscr{H}}) \arrow[r, "F(f)"] \arrow[d, "\eta_F"'] & F(I_{\mathscr{H}}) \arrow[d, "\eta_F"] \\
I_{\mathscr{G}} \arrow[r, dotted, "\pi_1(F)(f)"'] & I_{\mathscr{G}}
\end{tikzcd}$$

where η_F is the morphism resulting from the additivity. Therefore, if $\pi_1(F)(f)$ is the identity, then by definition f is an automorphism of $(I_{\mathscr{H}}, \eta_F)$, considered as an object of $\mathscr{K}er(F)$. Since $\bar{G}$ is full, there exists a generalized cover V and an automorphism g of $I_{\mathscr{K}}$ over V such that $\pi_1(g)$ becomes equal to $f|_V$ via $G(I_{\mathscr{K}}) \xrightarrow{\sim} (I_{\mathscr{H}}, \eta_F)$, as wanted.

For $i = 0$, let ξ be a section of $\pi_0(\mathscr{H})$ over $U \in \mathrm{Ob}\,\mathsf{S}$. We can assume, up to refining U, that $\xi = [X]$, where X is an object of $\mathscr{H}_U$. Then $\pi_0(F)(\xi) = [F(X)]$. If this is equal to $1 \in \pi_0(\mathscr{G})$, then we must have that there is an isomorphism $f\colon F(X) \xrightarrow{\sim} I_{\mathscr{G}}$, or in other words, (X, f) is an object of $\mathscr{K}er(F)_U$. Since $\bar{G}$ is essentially surjective, there exists a (generalized) cover $V \to U$ and an object Y of $\mathscr{K}_V$ such that $\bar{G}(Y) \xrightarrow{\sim} (X, f)|_V$. So this means there exists $a\colon G(Y) \xrightarrow{\sim} X$ in $\mathscr{H}_V$ such that

$$\begin{tikzcd}
F(G(Y)) \arrow[r, "F(a)"] \arrow[dr, bend right, "\varepsilon_Y"'] & F(X) \arrow[d, "d"] \\
& I_{\mathscr{G}}
\end{tikzcd}$$

Thus $[X]|_V = [G(Y)] = \pi_0(G)([Y])$. $\square$

6.2.6 Proposition. — *If the sequence* (6.2.1.1) *is left-exact, then there is a connecting homomorphism* $\Delta\colon \pi_1(\mathscr{G}) \to \pi_0(\mathscr{K})$ *leading to a long exact sequence*

$$0 \longrightarrow \pi_1(\mathscr{K}) \longrightarrow \pi_1(\mathscr{H}) \longrightarrow \pi_1(\mathscr{G}) \xrightarrow{\Delta} \pi_0(\mathscr{K}) \longrightarrow \pi_0(\mathscr{H}) \longrightarrow \pi_0(\mathscr{G}).$$

Proof. — We follow the set-theoretic arguments of ref. [**Rou03**].

Observe that there exists a *functor*

$$\Delta\colon \pi_1(\mathscr{G}) \longrightarrow \mathscr{K}er(F)$$

defined as follows. Any $g \in \underline{\mathrm{Aut}}_{\mathscr{G}}(I)(U)$ is sent to $(I_{\mathscr{H}}, g \circ \eta_F)$. (See the proof of the previous lemma for the meaning of symbols.) We claim that the sequence of gr-stacks

$$0 \longrightarrow \pi_1(\mathscr{K}) \longrightarrow \pi_1(\mathscr{H}) \longrightarrow \pi_1(\mathscr{G}) \xrightarrow{\Delta} \mathscr{K} \longrightarrow \mathscr{H} \longrightarrow \mathscr{G}$$

is exact in the sense of Definition 6.2.2. (The group objects in the sequence are considered as gr-stacks in the obvious way.)

First of all, it is clear that the homomorphism $\pi_1(\mathscr{K}) \to \pi_1(\mathscr{H})$ is an injection, and we need to check exactness only at $\pi_1(\mathscr{G})$ and $\mathscr{K}$. We may assume that $\mathscr{K} = \mathscr{K}er(F)$.

Exactness at $\pi_1(\mathscr{G})$ holds because if $\Delta(g) \xrightarrow{\sim} (I_{\mathscr{H}}, \eta_F)$, this means there exists $f\colon I_{\mathscr{H}} \to I_{\mathscr{H}}$ over U such that

$$\begin{tikzcd}
F(I_{\mathscr{H}}) \arrow[r, "F(f)"] \arrow[d, "\eta_F"'] & F(I_{\mathscr{H}}) \arrow[d, "\eta_F"] \\
I_{\mathscr{G}} \arrow[r, "g"'] & I_{\mathscr{G}}
\end{tikzcd}$$

commutes, which by definition means g is the image of f by $\pi_1(F)$.

Exactness at $\mathscr{K}$ holds because if (X, f) becomes isomorphic to $I_{\mathscr{H}}$ in $\mathscr{H}$, this means that there is already an isomorphism $a\colon X \xrightarrow{\sim} I_{\mathscr{H}}$, and so the diagram

$$\begin{tikzcd}
F(I_{\mathscr{H}}) \arrow[r, "F(a)"] \arrow[d, "f"'] & F(I_{\mathscr{H}}) \arrow[d, "\eta_F"] \\
I_{\mathscr{G}} & I_{\mathscr{G}} \arrow[l, dotted, "g"]
\end{tikzcd}$$

defines the required automorphism of $I_{\mathscr{G}}$.

Having established the exactness of the above sequence, we need only apply lemma 6.2.5 to it to obtain the sequence in the statement. □

6.2.7 Remark. — The sequence in the proposition is exact at the rightmost place iff F is essentially surjective.

6.3. The homotopy fiber of a butterfly. — Let us return to the morphism $F\colon \mathscr{H} \to \mathscr{G}$ and the corresponding butterfly, which we rewrite for convenience:

$$\begin{tikzcd} H_1 \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\ & E \arrow[dl, "\pi"'] \arrow[dr, "\jmath"] & \\ H_0 & & G_0 \end{tikzcd},$$

Let us denote by $\mathscr{K}$ the homotopy kernel of F. From Proposition 6.1.1 it follows that

$$\pi_1(\mathscr{K}) = \operatorname{Ker}(\bar{\kappa}\colon H_1 \to \operatorname{Ker} \jmath)$$

and

$$\pi_0(\mathscr{K}) = \operatorname{Coker}(\bar{\kappa}\colon H_1 \to \ker \jmath)$$

respectively coincide with the non-abelian cohomology sheaves $\underline{\mathrm{H}}^{-2}(F_\bullet)$ and $\underline{\mathrm{H}}^{-1}(F_\bullet)$, where $F_\bullet$ is the complex

$$[H_1 \longrightarrow E \longrightarrow G_0]_{-2,0}$$

placed in degrees $[-2,0]$. Moreover, $\pi_i(\mathscr{G}) = \underline{\mathrm{H}}^{-i}(G_\bullet)$, $i = 0,1$, and similarly for $H_\bullet$. By the previous discussion we have the exact sequence of homology sheaves:

$$0 \longrightarrow \underline{\mathrm{H}}^{-2}(F_\bullet) \longrightarrow \underline{\mathrm{H}}^{-1}(H_\bullet) \longrightarrow \underline{\mathrm{H}}^{-1}(G_\bullet) \longrightarrow \underline{\mathrm{H}}^{-1}(F_\bullet) \longrightarrow \underline{\mathrm{H}}^{0}(H_\bullet) \longrightarrow \underline{\mathrm{H}}^{0}(G_\bullet).$$

Again, the above sequence will be exact on the right if the butterfly corresponds to an essentially surjective morphism. Otherwise, it is easy to see that the obstruction is $\underline{\mathrm{H}}^{0}(F_\bullet)$.

The complex $F_\bullet$ itself begs for an explanation akin to the one given in [**Noo05**], but more in keeping with the present geometric context. We conclude this section by providing the construction of a 2-stack which, in a somewhat imprecise sense, takes the rôle of the cone in an exact triangle. The discussion in the rest of this section requires more background than we were able to provide in the rest of this work, especially concerning 2-(gr)-stacks and their relations with complexes of length 3, for which we refer to one of the sequels of this work ([**AN08b, AN08c**]). Here we will simply gloss over several details of the construction, and refer to the quoted references for the necessary background and additional details not explicitly included here.

6.3.1. — Following [**Noo05**], there is a (sheaf of) 2-groupoid(s), call it $\underline{\mathsf{F}}$, determined by $F_\bullet$. Its objects are given by the sections of G_0, and in general a 2-cell is

$$\begin{tikzcd} x \arrow[r, bend left=50, "e_0"] \arrow[r, bend right=50, "e_1"'] \arrow[r, phantom, "\Downarrow h"] & x' \end{tikzcd}$$

with $x, x' \in G_0$, $e_0, e_1 \in E$, and $h \in H_1$, satisfying the relations

$$e_1 = e_0\, \kappa(h), \qquad x' = x\, \jmath(e_0) = x\, \jmath(e_1).$$

The vertical composition of 2-arrows is given by multiplication in H_1. For the horizontal composition we have

$$\begin{tikzcd} x \arrow[r, bend left, "e_0"] \arrow[r, bend right, "e_1"'] \arrow[r, phantom, "\Downarrow h"] & x' \arrow[r, bend left, "e'_0"] \arrow[r, bend right, "e'_1"'] \arrow[r, phantom, "\Downarrow h'"] & x'' \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "e_0 e'_0"] \arrow[r, bend right, "e_1 e'_1"'] \arrow[r, phantom, "\Downarrow h''"] & x'' \end{tikzcd}$$

with $h'' = h^{\pi(e'_0)} h'$. This follows at once from the properties of the butterfly. Observe that this 2-groupoid is strict, in the sense that the 1-morphisms are strictly invertible, as opposed to just being equivalences.

6.3.2. — The homomorphism $\kappa\colon H_1 \to E$ defines a crossed module, where the action of E on H_1 occurs via $\pi\colon E \to H_0$. Let $\mathscr{C}$ be the associated gr-stack:

$$\mathscr{C} = \Big[H_1 \xrightarrow{\kappa} E\Big]^{\sim} \simeq \textsc{Tors}(H_1, E).$$

An object (P, e) of $\mathscr{C}$ is therefore an H_1-torsor equipped with an H_1-equivariant map $e\colon P \to E$. It follow at once that $\jmath \circ e$ is invariant under the action of H_1, hence it is an element of G_0. Moreover, for a morphism $\varphi\colon (P, e) \to (P', e')$ in $\mathscr{C}$ the corresponding element in G_0 is the same: $\jmath \circ e = (\jmath \circ e') \circ \varphi$. This can be summarized by saying that there is an additive functor

$$J\colon \mathscr{C} \longrightarrow G_0$$

sending the object (P, e) to $\jmath(e)$, where, again, G_0 is regarded as a gr-stack in the obvious way.

Incidentally, it is easy to verify that the homotopy kernel of J is again $\mathscr{K}$. There is also a morphism of gr-stacks $\pi\colon \mathscr{C} \to \mathscr{H}$ induced by the obvious corresponding strict morphism of crossed modules $[H_1 \to E] \to [H_1 \to H_0]$ contained in the butterfly. The (homotopy) kernel of the latter is G_1, as it is immediately checked. Thus we have the following proposition, whose proof will be left to the reader, giving a stacky dévissage of the butterfly:

6.3.3 Proposition. — *The butterfly E from $H_\bullet$ to $G_\bullet$ gives rise to the commutative diagram of gr-stacks*

(6.3.3.1)
$$\begin{tikzcd} & \mathscr{K} \arrow[r, equal] \arrow[d] & \mathscr{K} \arrow[d] \\ G_1 \arrow[r] \arrow[d, equal] & \mathscr{C} \arrow[r, "\pi"] \arrow[d, "J"] & \mathscr{H} \arrow[d, "F"] \\ G_1 \arrow[r, "\partial"'] & G_0 \arrow[r, "\pi_{\mathscr{G}}"'] & \mathscr{G} \end{tikzcd}$$

In order to proceed any further we need to introduce the notion of torsor over a gr-stack and that of $(\mathscr{H},\mathscr{G})$-torsor, for $F\colon \mathscr{H} \to \mathscr{G}$ is a morphism of gr-stacks.

If $\mathscr{H}$ is a gr-stack, the notion of (right) $\mathscr{H}$-torsor is expounded in detail in [**Bre90**], and essentially categorifies that of torsor over a sheaf of groups. Thus, a right $\mathscr{H}$-torsor is a stack $\mathscr{X}$ equipped with a right $\mathscr{H}$-action, and it is locally non-empty in a way that makes it locally equivalent to $\mathscr{H}$. Morphisms of $\mathscr{H}$-torsors are morphisms of the underlying stacks that weakly commute with the $\mathscr{H}$-action, that is, up to coherent natural transformation. Similarly, 2-morphisms are again 2-morphisms between the underlying stacks that are compatible with the $\mathscr{H}$-action. We denote by $\textsc{Tors}(\mathscr{H})$ the fibered 2-category of $\mathscr{H}$-torsors over S. In fact it is a (neutral) 2-gerbe over S.

6.3.4. — A $(\mathscr{H},\mathscr{G})$-torsor is the categorification of the concept of (A,B)-torsor, for a homomorphism $A \to B$ of group objects, and its definition was given in ref. [**Ald08**]—albeit in a fully abelian context. A $(\mathscr{H},\mathscr{G})$-torsor is a pair $(\mathscr{X},S)$ where $\mathscr{X}$ is a (right) $\mathscr{H}$-torsor equipped with a $\mathscr{H}$-equivariant functor

$$S\colon \mathscr{X} \longrightarrow \mathscr{G}.$$

This is equivalent to saying that $\mathscr{X}$ is equipped with a trivializing equivalence $S\colon \mathscr{X} \wedge^{\mathscr{C}} \mathscr{G} \xrightarrow{\sim} \mathscr{G}$. (The contracted product for stacks equipped with an action by a gr-stack is defined in [**Bre90**].)

The notion of morphism of $(\mathscr{H},\mathscr{G})$-torsors is the one obtained by taking the obvious generalization (=categorification) of (3.4.7.1). A morphism of $\mathscr{H}$-torsors is a pair (F,μ) such that $F\colon \mathscr{X} \to \mathscr{Y}$ is a morphisms of stacks and μ is the natural transformation expressing the weak compatibility of F with the torsor structures. We further require that the diagram

(6.3.4.1)
$$\begin{tikzcd} \mathscr{X} \arrow[r, "F"] \arrow[dr, bend right, "S"'] & \mathscr{Y} \arrow[d, "T"] \arrow[dl, phantom, "\lambda_F"] \\ & \mathscr{G} \end{tikzcd}$$

of $\mathscr{H}$-equivariant morphisms 2-commutes. Note there is no additional condition on μ. Finally, a 2-morphism $\alpha\colon (F,\mu) \Rightarrow (G,\nu)\colon \mathscr{X} \to \mathscr{Y}$ is a 2-morphism of $(\mathscr{H},\mathscr{G})$-torsors if the natural transformation α, in addition to satisfying the properties required in [**Bre90**, 6.1.7], also fits in the diagrammatic equality of 2-morphisms

(6.3.4.2)
$$\begin{tikzcd} \mathscr{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[dr, bend right, "S"'] & \mathscr{Y} \arrow[d, "T"] \arrow[dl, phantom, "\lambda_G"] \\ & \mathscr{G} \end{tikzcd} = \begin{tikzcd} \mathscr{X} \arrow[r, "F"] \arrow[dr, bend right, "S"'] & \mathscr{Y} \arrow[d, "T"] \arrow[dl, phantom, "\lambda_F"] \\ & \mathscr{G} \end{tikzcd}$$

$(\mathscr{H},\mathscr{G})$-torsors comprise a 2-stack denoted $\textsc{Tors}(\mathscr{H},\mathscr{G})$. Actually more is true: given the morphism $F\colon \mathscr{H}\to\mathscr{G}$ of gr-stacks there is an induced functor

$$F_*\colon\ \textsc{Tors}(\mathscr{H})\longrightarrow\textsc{Tors}(\mathscr{G})$$

that sends the $\mathscr{H}$-torsor $\mathscr{X}$ to $\mathscr{X}\wedge^{\mathscr{H}}\mathscr{G}$. This is the analog—or "categorified"—notion of extension of the structural group for ordinary torsors. Then $\textsc{Tors}(\mathscr{H},\mathscr{G})$ is the "homotopy fiber" of F_*. More precisely we have:

6.3.5 Lemma. — *There is a pull-back diagram*

$$\begin{tikzcd}
\textsc{Tors}(\mathscr{H},\mathscr{G}) \arrow[r] \arrow[d] & \textsc{Tors}(\mathscr{H}) \arrow[d, "F_*"] \\
\underline{\mathbf{1}} \arrow[r] & \textsc{Tors}(\mathscr{G})
\end{tikzcd}$$

in the sense of fibered product for 2-categories, as in [**Hak72**].

In the lemma we have indicated with $\underline{\mathbf{1}}$ the 2-category with only one object and identity (2-)morphisms.

Proof of the lemma. — The morphism $\underline{\mathbf{1}}\to\textsc{Tors}(\mathscr{G})$ sends the unique object of $\underline{\mathbf{1}}$ to the trivial $\mathscr{G}$-torsor. According to [**Hak72**], an object of the fibered product

$$\underline{\mathbf{1}}\times_{\textsc{Tors}(\mathscr{G})}\textsc{Tors}(\mathscr{H})$$

is by definition given by an $\mathscr{H}$-torsor $\mathscr{X}$, plus an equivalence $F_*(\mathscr{X})\simeq\mathscr{G}$. This is by definition an $(\mathscr{H},\mathscr{G})$-torsor. $\square$

6.3.6. — Returning now to $\underline{\mathsf{F}}$, note that it is a 2-category fibered in 2-groupoids over S. Moreover, this 2-category is separated in the sense that it has a *sheaf* of 2-morphisms. It is promoted to a 2-prestack by promoting the categories of morphism from simply being isomorphic copies of the groupoids associated to the crossed module $[H_1\to E]$ to be equivalent to the gr-stack $\mathscr{C}$. This 2-prestack, which should be more properly called a *bi*-prestack, since it turns out to be fibered in bicategories, is then made into a 2-stack by an additional sheafification process. This latter step rigidifies it again, so that it is an actual 2-stack.

Rather than describe the various stages in detail, let us give a convenient model for $\mathfrak{F}$, the 2-stack associated to the 2-groupoid $\underline{\mathsf{F}}$. We claim that

$$\mathfrak{F}\simeq\textsc{Tors}(\mathscr{C},G_0).$$

Thus objects are pairs $(\mathscr{X},s)$, where $\mathscr{X}$ is a $\mathscr{C}$-torsor and $s\colon\mathscr{X}\to G_0$ is a $\mathscr{C}$-equivariant map via the morphism $J\colon\mathscr{C}\to G_0$.

To have at least an intuitive idea of why $\mathfrak{F}$ should be the 2-stackification of the 2-groupoid $\underline{\mathsf{F}}$ at all, one may look into the 2-descent datum determined by an object $(\mathscr{X},s)$ of $\mathfrak{F}$ over U, resulting in a 0-cocycle over U with values in $F_\bullet$.

6.3.7. — Since a torsor is by definition locally trivial, there will exist an object X of $\mathscr{X}$ over a cover $V_0 \to U$ determining an equivalence $\mathscr{X} \simeq \mathscr{C}$ over V_0. Moreover, applying s we obtain an element $x = s(X)$ of G_0. Now, assuming we are provided with a V_1 such that $V_1 \rightrightarrows V_0$, for instance by working with a Čech resolution $\check{\mathrm{C}}(V_0 \to U)$, the fact that $\mathscr{X}$ is a $\mathscr{C}$-torsor further implies there is a morphism over V_1:

$$f\colon d_1^* X \xrightarrow{\sim} d_0^* X \cdot (P, e),$$

for an appropriate object (P, e) of $\mathscr{C}_{V_1}$. Again, applying s results in

$$d_1^*(x) = d_0^*(x)\, \jmath(e). \tag{6.3.7.1}$$

The morphism f will have to satisfy the 1-cocycle conditions given in [**Bre90**], which amount to the existence of a morphism of (H_1, E)-torsors

$$\varphi\colon d_2^*(P, e) \otimes d_0^*(P, e) \longrightarrow d_1^*(P, e)$$

over V_2 satisfying a compatibility condition over V_3, which are ultimately the conditions given in [**Bre90**, 6.2.8 and 6.2.10]. Upon choosing trivializations for the E-torsors compatible with the cover, the morphism φ above yields the relation

$$d_2^* e\, d_0^* e = d_1^* e\, \kappa(h), \tag{6.3.7.2}$$

where $h \in H_1(V_2)$ arises from expressing the morphism of the underlying torsors in terms of the trivializations. The coherence condition results in the following relation over V_3:

$$d_2^* h\, d_0^* h = d_1^* h\, (d_3^* h)^{(d_0 d_1)^* e}. \tag{6.3.7.3}$$

Equations (6.3.7.1), (6.3.7.2) and (6.3.7.3) give the required cocycle.

6.3.8. — The homotopy sheaves π_i, for $i = 0, 1, 2$, are defined for a 2-stack (cf. ref. [**Bre94**, Chap. 8]). In fact a 2-stack $\mathfrak{F}$ is a 2-gerbe over the localized site $\mathsf{S} \downarrow \pi_0(\mathfrak{F})$, as shown in loc. cit., much in the same way as a stack is a gerbe over its own π_0-sheaf. Moreover, $\mathfrak{F}$ as a 2-gerbe over $\mathsf{S} \downarrow \pi_0(\mathfrak{F})$ is neutral, as I, the trivial $(\mathscr{C}, G_0)$-torsor, provides the necessary global object. It follows that as a 2-gerbe it is equivalent to $\textsc{Tors}(\mathscr{A}ut(I))$, and therefore it is classified by the invariants of the gr-stack $\mathscr{A}ut(I)$.

6.3.9. — It can be readily verified by means of a local calculation that there is an equivalence of gr-stacks

$$\mathscr{A}ut(I) \xrightarrow{\sim} \mathscr{K} = \left[H_1 \xrightarrow{\bar{\kappa}} \ker \jmath\right]^{\sim}.$$

Hence we can use the results in [**Bre94**, Chap. 8] to conclude that

$$\underline{\mathrm{H}}^{-2}(F_\bullet) = \pi_2(\mathfrak{F}) = \pi_1(\mathscr{K}), \qquad \underline{\mathrm{H}}^{-1}(F_\bullet) = \pi_1(\mathfrak{F}) = \pi_0(\mathscr{K})$$

as wanted. Moreover, since $\mathfrak{F}$ is the 2-stack associated to the 2-groupoid $\underline{\mathsf{F}}$, it follows that $\pi_0(\mathfrak{F}) = \underline{\mathrm{H}}^0(F_\bullet)$.

The previous discussion is subsumed by part of the following statement whose proof will be sketched below.

6.3.10 Theorem. — *Let $F\colon \mathscr{H} \to \mathscr{G}$ be a morphism of gr-stacks, where $\mathscr{H} = [H_1 \to H_0]^{\sim}$ and $\mathscr{G} = [G_1 \to G_0]^{\sim}$, and let $[H_\bullet, E, G_\bullet]$ be the corresponding butterfly.*

There exists a 2-stack $\mathfrak{F}$ such that $\pi_i(\mathfrak{F}) = \underline{\mathrm{H}}^{-i}(F_\bullet)$, $i = 0, 1, 2$, where $F_\bullet$ is the complex on the NW-SE diagonal of the butterfly. Moreover, there is a homotopy fiber sequence of 2-stacks:

$$\mathscr{H}[0] \xrightarrow{F[0]} \mathscr{G}[0] \xrightarrow{\iota} \mathfrak{F} \xrightarrow{\Delta} \mathrm{T{\small ORS}}(\mathscr{H}) \xrightarrow{F_*} \mathrm{T{\small ORS}}(\mathscr{G}) \tag{6.3.10.1}$$

giving rise to the exact sequence of homotopy sheaves

$$\begin{tikzcd} 0 \arrow[r] & \pi_2(\mathfrak{F}) \arrow[r] & \pi_1(\mathscr{H}) \arrow[r] & \pi_1(\mathscr{G}) \arrow[r] & \pi_1(\mathfrak{F}) \arrow[dlll, rounded corners] \\ & \pi_0(\mathscr{H}) \arrow[r] & \pi_0(\mathscr{G}) \arrow[r] & \pi_0(\mathfrak{F}) \arrow[r] & 1 \end{tikzcd}$$

The various morphisms in the sequence are as follows. The suffix [0] is used to indicate that the affected objects are to be considered "discrete" 2-stacks with no non-trivial 2-morphisms, so $F[0]$ is just F. F_* is the "push-forward" of torsors along F, introduced before. The morphism Δ sends a $(\mathscr{C}, G_0)$ torsor $(\mathscr{X}, s)$ to the $\mathscr{H}$-torsor $\mathscr{X} \wedge^{\mathscr{C}} \mathscr{H}$ via the morphism of gr-stacks $\mathscr{C} \to \mathscr{H}$. Finally, the morphism $\iota\colon \mathscr{G}[0] \to \mathfrak{F}$ can be seen as the one induced by the 2-stackification of the strict morphism of 2-groupoids $\mathsf{G}[0] \to \underline{\mathsf{F}}$ occurring in the butterfly.

The sequence is a "homotopy fiber sequence" in the sense that each term in (6.3.10.1), starting from $\mathfrak{F}$ and moving to the left, can be understood as a pull-back (fiber product) diagram of 2-stacks in the sense of [**Hak72**]. For example:

$$\begin{tikzcd} \mathfrak{F} \arrow[r] \arrow[d] & \mathrm{T{\small ORS}}(\mathscr{H}) \arrow[d] \\ \underline{\mathbf{1}} \arrow[r] & \mathrm{T{\small ORS}}(\mathscr{G}) \end{tikzcd} \tag{6.3.10.2}$$

where $\underline{\mathbf{1}}$ is the 2-category with one object and only identity morphism and 2-morphism. A similar consideration holds for the other three-part segments. Note also that the resulting exact sequence involving the last three terms is well defined, since all the three 2-stacks appearing in the sequence are naturally pointed by the trivial torsor. (The others are 2-groups, so they are naturally pointed too.)

Proof of Theorem 6.3.10. — From Lemma 6.3.5, $\mathrm{T{\small ORS}}(\mathscr{H}, \mathscr{G})$ can be taken as the homotopy fiber of the functor F_*. Moreover, it is easy to see that it is part of the full homotopy fiber sequence:

$$\mathscr{H}[0] \xrightarrow{F[0]} \mathscr{G}[0] \xrightarrow{\iota} \textsc{Tors}(\mathscr{H},\mathscr{G}) \xrightarrow{\Delta} \textsc{Tors}(\mathscr{H}) \xrightarrow{F_*} \textsc{Tors}(\mathscr{G})$$

This time, Δ forgets the trivialization, whereas ι sends a (G_1, G_0)-torsor (P, s) to the $(\mathscr{H},\mathscr{G})$-torsor

$$(\mathscr{H}, (P, s))$$

where (P, s) is to be considered as the $\mathscr{G}$-equivariant functor sending the unit object $I_{\mathscr{H}}$ of $\mathscr{H}$ to (P, s).

The homotopy fiber sequence above can be extended to the left as follows:

$$0 \longrightarrow \pi_1(\mathscr{K}) \longrightarrow \pi_1(\mathscr{H}) \longrightarrow \pi_1(\mathscr{G}) \longrightarrow \mathscr{K} \longrightarrow \mathscr{H} \longrightarrow \cdots$$

The (abelian) groups to the left are considered as 2-stacks in the obvious way: they are totally discrete 2-stacks with only identity arrows and 2-arrows.

By applying π_0 to the combined sequence, and observing that $\pi_0(\textsc{Tors}(\mathscr{H}))$ and $\pi_0(\textsc{Tors}(\mathscr{G}))$ are trivial, we get the homotopy sequence in the statement, with $\textsc{Tors}(\mathscr{H},\mathscr{G})$ as the relevant 2-stack instead of $\mathfrak{F}$.

The statement then follows from the following proposition:

6.3.11 Proposition. — *There is a 2-equivalence (in the sense of ref.* [**Hak72**]*)* $G\colon \mathfrak{F} \xrightarrow{\sim} \textsc{Tors}(\mathscr{H},\mathscr{G})$.

Proof. — The 2-functor G is defined as follows. Let $(\mathscr{X}, s)$ be a $(\mathscr{C}, G_0)$-torsor. Then $\mathscr{X} \wedge^{\mathscr{C}} \mathscr{H}$ is an $\mathscr{H}$-torsor. We claim there is a global, $\mathscr{G}$-equivariant, functor

$$S\colon \mathscr{X} \wedge^{\mathscr{C}} \mathscr{H} \longrightarrow \mathscr{G}$$

given by the global map

$$(\pi_{\mathscr{G}} \circ s) \wedge F\colon \mathscr{X} \wedge^{\mathscr{C}} \mathscr{H} \longrightarrow \mathscr{G},$$

so that the pair $(\mathscr{X} \wedge^{\mathscr{C}} \mathscr{H}, S)$ is an $(\mathscr{H},\mathscr{G})$-torsor. Since an object of $\mathscr{X} \wedge^{\mathscr{C}} \mathscr{H}$ is a pair (X, Y), where X and Y are objects of $\mathscr{X}$ and $\mathscr{H}$, respectively,[(2)] S is defined by sending (X, Y) to $\pi_{\mathscr{G}}(s(X)) \otimes_{\mathscr{G}} F(Y)$.

Using the explicit characterization for morphisms $(X, Y) \to (X', Y') \in \mathrm{Mor}(\mathscr{X} \wedge^{\mathscr{C}} \mathscr{G})$ found in [**Bre90**, §6.7], it can be directly verified that S is indeed a functor. The same calculation shows S is compatible with the equivalences in $\mathscr{X} \wedge^{\mathscr{C}} \mathscr{H}$

$$(X, \pi(C) \otimes_{\mathscr{H}} Y) \xrightarrow{\sim} (X \cdot C, Y)$$

resulting from the action of $\mathscr{C}$. Thus S is well defined.

It is also easy to verify diagrams of $(\mathscr{C}, G_0)$-torsors as (6.3.4.1) and (6.3.4.2) (with the additional simplification that most 2-morphisms are trivial since there are no

---

(2) Y is what we would normally write as (Q, t) when emphasizing the fact is is a (H_1, H_0)-torsor, which is not important, at the moment.

non trivial morphisms in G_0) induce corresponding diagrams of $(\mathscr{H},\mathscr{G})$-torsors. For example, from a morphism

$$\begin{tikzcd}
\mathscr{X} \arrow[r] \arrow[dr, bend right, "s"'] & \mathscr{Y} \arrow[d, "t"] \\
& G_0
\end{tikzcd}$$

to which we append $\pi_{\mathscr{G}} \colon G_0 \to \mathscr{G}$, we get

$$\begin{tikzcd}
\mathscr{X} \wedge^{\mathscr{C}} \mathscr{H} \arrow[r] \arrow[dr, bend right, "(\pi_{\mathscr{G}}\circ s)\wedge F"'] & \mathscr{Y} \wedge^{\mathscr{C}} \mathscr{H} \arrow[d, "(\pi_{\mathscr{G}}\circ t)\wedge F"] \\
& \mathscr{G}
\end{tikzcd}$$

The latter is 2-commutative, due to the definition of $\wedge^{\mathscr{C}}$ in the context of stacks.

From the exact sequences of homotopy sheaves above it follows that

$$\pi_0(\textsc{Tors}(\mathscr{H},\mathscr{G})) \simeq \pi_0(\mathfrak{F}),$$

so again following [**Bre94**, Chap. 8] we have that both $\textsc{Tors}(\mathscr{H},\mathscr{G})$ and $\mathfrak{F}$ can be considered as 2-gerbes over the same base $\mathsf{S} \downarrow \pi_0$.

Moreover, we have already observed the automorphism gr-stack (as automorphism gr-stack of the trivial torsor) of $\mathfrak{F}$ is $\mathscr{K}$. It is immediately verified the same is true for $\textsc{Tors}(\mathscr{H},\mathscr{G})$. Hence, as 2-gerbes over π_0, they are "banded" by the same gr-stack and there is a 2-functor $G\colon \mathfrak{F} \to \textsc{Tors}(\mathscr{H},\mathscr{G})$. Hence they are necessarily 2-equivalent over $\mathsf{S} \downarrow \pi_0$. But this implies they are equivalent over S tout-court.

This ends the proof of the proposition and hence of the theorem. □

6.3.12 Remark. — In the course of the proof a longer sequence was obtained, namely

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\mathscr{K}) \arrow[r] & \pi_1(\mathscr{H}) \arrow[r] & \pi_1(\mathscr{G}) \arrow[dll] \\
& \mathscr{K} \arrow[r] & \mathscr{H} \arrow[r, "F"] & \mathscr{G} \arrow[dll, "\iota"] \\
& \mathfrak{F} \arrow[r, "\Delta"] & \textsc{Tors}(\mathscr{H}) \arrow[r, "F_*"] & \textsc{Tors}(\mathscr{G})
\end{tikzcd}$$

Note that the first three (non-zero) terms are (abelian) groups, the next three are 2-groups, whereas the last three are 2-stacks but lack a group structure, weak or otherwise.

This sequence ought to be considered (the geometric version of) the counterpart of the sequence [**Bre90**, (3.9.1)] for 2-groups. A more detailed analysis will appear in [**AN08c**].

Combining with results and remarks from [**Bre92**], one can argue for an extension one step to the right if $\textsc{Tors}(\mathscr{H},\mathscr{H})$, or equivalently, $\mathfrak{F}$, is a *3*-group, that is, if

$F\colon \mathscr{H} \to \mathscr{G}$ is, in the appropriate sense discussed in loc. cit., a crossed module of gr-stacks.

6.3.13 Remark. — Sequence (6.3.10.1) is the exact counterpart of the homotopy fiber sequence of ref. [**Noo05**, Theorem 9.1], where one regards crossed modules as 2-categories with one object, namely considers their suspension. When applied to a gr-stack $\mathscr{H}$, this process gives rise to its the naïve suspension $\mathscr{H}[1]$, namely the fibered bicategory with one object such that the composition of 1-morphisms is given by the group-like structure of $\mathscr{H}$ (cf. [**Bre94**]). Note that in (6.3.10.1) we have the correct geometric suspensions $\mathrm{T{\scriptsize ORS}}(\mathscr{H})$ and $\mathrm{T{\scriptsize ORS}}(\mathscr{G})$ of $\mathscr{H}$ and $\mathscr{G}$, respectively, instead. As mentioned in [**Bre94**], $\mathrm{T{\scriptsize ORS}}(\mathscr{H})$ is obtained by taking the associated 2-stack (in fact 2-gerbe) of the naïve suspension $\mathscr{H}[1]$.

These considerations and Theorem 6.3.10 make it suggestive to consider some portion of the exact sequence (6.3.10.1) as a candidate to play the rôle of an "exact triangle"

$$\mathscr{H} \xrightarrow{F} \mathscr{G} \longrightarrow \mathfrak{F} \longrightarrow \mathscr{H}[1]$$

for the non-abelian derived category. This is only suggestive in that the "cone," that is $\mathfrak{F}$, is a 2-stack—an object related to a complex of length three. Moreover, as we have already observed it is not a group object.

We conclude with the following observation. First, using Proposition 6.3.3, the obvious morphism $\mathrm{T{\scriptsize ORS}}(\mathscr{K}) \longrightarrow \mathrm{T{\scriptsize ORS}}(\mathscr{C})$ determined by $\mathscr{K} \to \mathscr{C}$ factors through a morphism

$$\mathrm{T{\scriptsize ORS}}(\mathscr{K}) \longrightarrow \mathrm{T{\scriptsize ORS}}(\mathscr{C}, G_0).$$

Now, if $F\colon \mathscr{H} \to \mathscr{G}$ is essentially surjective, then the induced map $\pi_0(\mathscr{H}) \to \pi_0(\mathscr{G})$ is an epimorphism. To put it differently, $\pi_0(\mathfrak{F}) = *$, and $\mathfrak{F}$ becomes locally connected, hence a 2-gerbe directly over the site S. Therefore we obtain an equivalence of 2-gerbes

$$\mathrm{T{\scriptsize ORS}}(\mathscr{K}) \xrightarrow{\sim} \mathfrak{F}.$$

In other words, when $\mathscr{H} \to \mathscr{G}$ is essentially surjective—so that taken together with its homotopy kernel it gives rise to a short exact sequence in the sense of section 6.2—the homotopy fiber can be identified with $\mathrm{T{\scriptsize ORS}}(\mathscr{K})$, i.e. the suspension of the kernel.

6.4. Exact sequence in non-abelian cohomology. — We now consider again a short exact sequence

$$\mathscr{K} \xrightarrow{\iota} \mathscr{H} \xrightarrow{p} \mathscr{G}$$

of gr-stacks and show there is a corresponding long exact sequence in non-abelian cohomology. The definition of short-exact was given in section 6.2. Note, the fibration condition was not included, and it is our purpose here to point out how our characterization of weak morphisms allows us to dispense of the fibration condition.

For the definition of non-abelian cohomology, we use the one given in [**Bre90**, §4], supplemented by the explicit cocyclic formulas recalled in sections 3.3.

We will make the assumption (which, thanks to Proposition 5.3.7 is not a restriction) that $\mathscr{K}$, $\mathscr{H}$, and $\mathscr{G}$ are associated to crossed modules $K_\bullet$, $H_\bullet$, and $G_\bullet$, respectively. By implicitly making use of Proposition 5.3.7 we indifferently write $\mathrm{H}^i(G_\bullet)$ or $\mathrm{H}^i(\mathscr{G})$, for $i = -1, 0, 1$.

We will not dwell on the interpretation of $\mathrm{H}^1(\mathscr{G})$ except to note, after loc. cit., that it should be interpreted as $\pi_0(\mathrm{T{\scriptsize ORS}}(\mathscr{G})(*))$ (classes of equivalences of global objects over S). This part will be taken up in detail in the forthcoming [**AN08a**].

As for the other degrees, it follows from the considerations in section 3.3 that $\mathrm{H}^0(\mathscr{G}) = \pi_0(\mathscr{G}(*))$ (isomorphism classes of global objects of $\mathscr{G}$), whereas $\mathrm{H}^{-1}(\mathscr{G}) \simeq \mathrm{H}^0(\pi_1(\mathscr{G}))$ (ordinary abelian sheaf cohomology). The latter identification follows from the definition and [**Jar89**].

6.4.1 Proposition. — *The short exact sequence*

$$\mathscr{K} \xrightarrow{\iota} \mathscr{H} \xrightarrow{p} \mathscr{G}$$

induces an exact sequence in cohomology
(6.4.1.1)

$$0 \longrightarrow \mathrm{H}^{-1}(K_\bullet) \longrightarrow \mathrm{H}^{-1}(H_\bullet) \longrightarrow \mathrm{H}^{-1}(G_\bullet) \longrightarrow \mathrm{H}^0(K_\bullet) \longrightarrow \mathrm{H}^0(H_\bullet) \longrightarrow \mathrm{H}^0(G_\bullet) \longrightarrow \mathrm{H}^1(K_\bullet) \longrightarrow \mathrm{H}^1(H_\bullet) \longrightarrow \mathrm{H}^1(G_\bullet)$$

Proof. — Let $[H_\bullet, E, G_\bullet]$ be a butterfly corresponding to p, e.g. by the fiber product construction. Let

$$H_\bullet \xleftarrow{Q} E_\bullet \xrightarrow{P} G_\bullet$$

be the fraction determined by it, and let $\mathscr{E}$ be the gr-stack associated to $E_\bullet$. Since $\mathscr{H} \to \mathscr{G}$ is essentially surjective, then $\jmath$ is an epimorphism. Moreover, the projection $H_1 \times G_1 \to G_1$ is also evidently so. At the level of simplicial groups we have an epimorphism $\underline{E_\bullet} \to \underline{G_\bullet}$. Moreover, it follows that the morphism of gr-stacks

$$P \colon \mathscr{E} \longrightarrow \mathscr{G}$$

arising from the strict morphism P' is actually not only essentially surjective, but also a fibration.

With the observation that the homotopy kernel of P is still $\mathscr{K}$ (since that of P' is the crossed module $[H_1 \to \ker \jmath]$), we can apply the results of [**Bre90**, §5.1], in particular the sequence (5.1.3). It follows there is a long exact cohomology sequence

$$0 \longrightarrow \mathrm{H}^{-1}(K_\bullet) \longrightarrow \mathrm{H}^{-1}(E_\bullet) \longrightarrow \mathrm{H}^{-1}(G_\bullet) \longrightarrow \mathrm{H}^0(K_\bullet) \longrightarrow \mathrm{H}^0(E_\bullet) \longrightarrow \mathrm{H}^0(G_\bullet) \longrightarrow \mathrm{H}^1(K_\bullet) \longrightarrow \mathrm{H}^1(H_\bullet) \longrightarrow \mathrm{H}^1(G_\bullet)$$

Now, Q' is a quasi-isomorphism, or equivalently the corresponding morphism

$$Q\colon \mathscr{E} \longrightarrow \mathscr{H}$$

is an equivalence. It follows that $\mathrm{H}^i(\mathscr{E}) \simeq \mathrm{H}^i(\mathscr{H})$ (one can use the "intelligent filtration"

$$\pi_1(\mathscr{H})[1] \longrightarrow \mathscr{H} \longrightarrow \pi_0(\mathscr{H})[0],$$

see [**Bre90**, §5.3], for this purpose) which proves the proposition. □

The point of the proposition, or rather of its proof, is that we can dispense of the fibration condition in the definition of exact sequence, since by the very construction of a weak morphism we can always replace the essentially surjective morphism $p\colon \mathscr{H} \to \mathscr{G}$ with a fibration. The butterfly diagram construction of the weak morphism offers a canonical way to accomplish it.

7. Braided and abelian butterflies

In this section we specialize our discussion to butterflies in the abelian category of abelian sheaves over S. We obtain an explicit characterization of the derived category of complexes of length two of abelian sheaves over S, see [**Del73**]. We are going to do so by introducing various customary commutativity conditions (braided, symmetric, Picard) on gr-stacks over S, before devoting ourselves to the fully abelian situation. While these conditions are all well known, our approach, we believe, is new, even in the set-theoretic case.

7.1. Butterflies and braidings. — The first, indeed, weakest, possible commutativity condition that can be imposed on a gr-stack (or gr-category, for that matter) is the simple existence of a *braiding* isomorphism, i.e. a natural isomorphism implementing a formal commutativity condition.

7.1.1. — A braiding takes the form of a collection of functorial isomorphisms

$$s_{X,Y}\colon X \otimes_{\mathscr{G}} Y \xrightarrow{\sim} Y \otimes_{\mathscr{G}} X \tag{7.1.1.1}$$

satisfying the well-known two hexagon diagrams of ref. [**JS93**].

The braiding is called *symmetric* or (symmetric monoidal) if in addition the condition

$$s_{Y,X} \circ s_{X,Y} = \mathrm{Id}_{X \otimes_{\mathscr{G}} Y} \tag{7.1.1.2}$$

is satisfied for all objects X and Y of $\mathscr{G}$. Furthermore, we say the braiding is *Picard,* or that $\mathscr{G}$ is a *Picard stack* (or gr-category in the pointwise case) if in addition to the symmetry condition it satisfies the condition

$$s_{X,X} = \mathrm{Id}_{X \otimes_{\mathscr{G}} X} \tag{7.1.1.3}$$

for all objects X of $\mathscr{G}$. For convenience, we use the terminology "Braided," "Symmetric" and "Picard," where others (notably, Breen, see, e.g. [**Bre94, Bre99**]) use "Braided," "Picard" and "Strictly Picard."

7.1.2. — Let $G_\bullet$ be a crossed module. The previous conditions make sense for the (strict) group law on the groupoid determined by $G_\bullet$, and (7.1.1.1), in particular, gives rise to the braiding map $\{-,-\}\colon G_0 \times G_0 \to G_1$ such that

$$\partial\{x,y\} = y^{-1}x^{-1}yx \tag{7.1.2.1}$$

for all $x, y \in G_0$. (This follows immediately from the request that $s_{x,y}$ be an isomorphism from xy to yx.) As pointed out in [**Bre94**], if all other commutativity conditions are imposed s becomes a full lift of the commutator map from G_0 to G_1.

It is well known that for a group to be abelian is equivalent to the condition that the multiplication map be a group homomorphism. A similar approach can also be adopted in the context of gr-stacks. Indeed it is a relatively simple exercise to show that the existence of a braiding on $\mathscr{G}$ is equivalent to the fact that the tensor operation $\otimes_{\mathscr{G}}\colon \mathscr{G}\times\mathscr{G}\to\mathscr{G}$ is a morphism of gr-stacks, that is, an additive functor. (This uses the fact every object in a gr-category or -gr-stack is regular, see [**Sí75**].)

7.1.3. — If $\otimes_{\mathscr{G}}\colon \mathscr{G}\times\mathscr{G}\to\mathscr{G}$ is a morphism of gr-stacks, by the equivalence in Theorem 4.3.1 there must be a butterfly diagram

$$\begin{tikzcd}
G_1 \times G_1 \arrow[dd, "\partial"'] \arrow[dr, "\alpha"] & & G_1 \arrow[dl, "\beta"'] \arrow[dd, "\partial"] \\
& P \arrow[dl, "\rho"'] \arrow[dr, "\sigma"] & \\
G_0 \times G_0 & & G_0
\end{tikzcd} \tag{7.1.3.1}$$

from $G_\bullet \times G_\bullet$ to $G_\bullet$.

7.1.4. — The butterfly (7.1.3.1) has some interesting additional properties. Let

$$\iota_1, \iota_2\colon \mathscr{G} \longrightarrow \mathscr{G}\times\mathscr{G}$$

be the two injections sending X to $(X, I_{\mathscr{G}})$ (respectively to $(I_{\mathscr{G}}, X)$). In any gr-stack or gr-category the existence of the functorial isomorphisms (3.1.1.1) can be recast as the requirement that the composite of the multiplication law with ι_1 or ι_2 be isomorphic to the identity functor of $\mathscr{G}$ to itself. Translated in the language of butterflies, this means that the butterfly obtained by pre-composing (7.1.3.1) with the (strict) morphisms of crossed modules $\iota_1, \iota_2\colon G_\bullet \to G_\bullet \times G_\bullet$ (defined in the same way as for $\mathscr{G}$) must be isomorphic to the identity morphism. Since pre-composition with a strict morphism means pulling back, we arrive at the conclusion that the extension on the NE-SW diagonal of (7.1.3.1) must split when restricted to either factor in $G_0 \times G_0$, that is, when pulled back by either ι_1 or ι_2.[3] In other words, since the extension

$$G_1 \longrightarrow \iota_i^* P \longrightarrow G_0$$

$i = 1, 2$, splits, there must exist two group homomorphisms

$$s_1, s_2 \colon G_0 \longrightarrow P \tag{7.1.4.1}$$

such that $\rho \circ s_i = \iota_i$, $i = 1, 2$, as maps $G_0 \to G_0 \times G_0$.

Since the composed split butterfly corresponds to the identity morphism, we must have

$$\sigma \circ s_i = \mathrm{id}_{G_0} \tag{7.1.4.2}$$

and

$$s_i(\partial g) = \alpha(\iota_i(g))\beta(g), \quad g \in G_1, \tag{7.1.4.3}$$

for $i = 1, 2$, where we have used the explicit relation between strict morphisms and split butterflies analyzed in sect .4.5.

7.1.5. — The existence of the two homomorphisms s_1 and s_2 implies (7.1.3.1) is a *strong* butterfly in the sense of Definition 4.1.6. Indeed, let

$$\tau \colon G_0 \times G_0 \longrightarrow P$$

be defined by $\tau(x, y) = s_1(x)s_2(y)$, for $x, y \in G_0$. We have

$$\rho \circ \tau(x, y) = \rho(s_1(x)s_2(y)) = (x, 1)(1, y) = (x, y),$$

therefore τ provides a global set-theoretic splitting, as required.

7.1.6. — Analyzing how far τ is from being a group homomorphism, leads to consider the combination

$$s_2(y)^{-1}s_1(x)^{-1}s_2(y)s_1(x). \tag{7.1.6.1}$$

It is immediate that

$$\rho(s_2(y)^{-1}s_1(x)^{-1}s_2(y)s_1(x)) = (1, y^{-1})(x^{-1}, 1)(1, y)(x, 1) = 1$$

so that there exists $c(x, y) \in G_1$ such that

$$\beta(c(x, y)) = s_2(y)^{-1}s_1(x)^{-1}s_2(y)s_1(x). \tag{7.1.6.2}$$

Moreover, by applying σ:

$$\partial c(x, y) = \sigma\beta c(x, y) = \sigma(s_2(y)^{-1}s_1(x)^{-1}s_2(y)s_1(x)) = y^{-1}x^{-1}yx,$$

[3] We commit the abuse of language of still denoting the components of the strict morphism $\iota_i \colon G_\bullet \to G_\bullet \times G_\bullet$ by the same letter

which should be compared with (7.1.2.1). Thus (7.1.6.2) defines a braiding in the standard way. Note that with the previous choices the failure for τ to be a homomorphism is measured as:

$$\tau(x_0x_1, y_0y_1)\beta(c(x_1, y_0)^{y_1}) = \tau(x_0, y_0)\tau(x_1, y_1).$$

7.1.7. — For the interpretation of (7.1.6.2) as a braiding to be complete, two more checks are necessary. First, the two hexagon diagrams of [**JS93**] must be satisfied. It is well known that in the case of a strict 2-group, i.e. crossed module, they reduce to the cocycle conditions

$$\{x, yz\} = \{x, y\}^z\{x, z\}$$

and

$$\{xy, z\} = \{y, z\}\{x, z\}^y$$

for the braiding map. With (7.1.6.2), the above cocycle conditions become an immediate consequence of the fact that s_1 and s_2 are homomorphisms. This simple fact is left as an exercise to the reader.

Second, the functoriality condition for the isomorphisms (7.1.1.1) is expressed in terms of the braiding by two relations

$$\{x, \partial h\} = h^{-1}h^x$$

and

$$\{\partial g, y\} = g^{-y}g,$$

where $x, y \in G_0$ and $g, h \in G_1$. It is immediately verified that both conditions are satisfied by (7.1.6.2) as a consequence of (7.1.4.3).

We can summarize the situation so far in the following proposition.

7.1.8 Proposition. — *Let* $\mathscr{G} \simeq [G_1 \xrightarrow{\partial} G_0]$. *Then the following are equivalent.*

- $\mathscr{G}$ *is braided.*
- $G_\bullet$ *is braided.*
- *There is a butterfly* (7.1.3.1) *equipped with prescribed splittings* (7.1.4.1) *such that* $\rho \circ s_i = \iota_i$ *and both* (7.1.4.2) *and* (7.1.4.3) *hold.*

7.1.9 Remark. — Changing either or both splittings (7.1.4.1) has the effect of replacing the "braiding" (7.1.6.2) with an equivalent one. However, it is more appropriate to consider s_1 and s_2 as part of the structure.

7.2. Symmetric crossed modules and 2-groups. — The exchange of s_1 and s_2 in (7.1.6.2) replaces $c(x,y)$ with $c'(x,y) = c(y,x)^{-1}$. According to [**JS93**], this is still a braiding, and the underlying tensor category is called *symmetric* if it so happens that $c' = c$. We want then to interpret the symmetry and Picard conditions in terms of the characterization provided by Proposition 7.1.8.

Let $T\colon \mathscr{G}\times\mathscr{G} \to \mathscr{G}\times\mathscr{G}$ be the swap functor which exchanges the factors: $(X,Y) \mapsto (Y,X)$. The same letter will denote the corresponding map for $G_\bullet$, as well as G_1 and G_0 separately.

Since T can of course be considered as a strict morphism of crossed modules $G_\bullet \times G_\bullet \to G_\bullet \times G_\bullet$, it can be composed with the butterfly (7.1.3.1) to yield a the new one: $[G_\bullet \times G_\bullet, T^*P, G_\bullet]$. Recall that since we are composing with a strict morphism, the resulting object in the center is just a pullback. The butterfly so obtained corresponds to the additive functor $\mathscr{G}\times\mathscr{G} \to \mathscr{G}$ given by the opposite multiplication law: $(X,Y) \mapsto Y \otimes_{\mathscr{G}} X$.

We consider the natural symmetry condition with respect to T, as spelled in the following definition.

7.2.1 Definition. — The butterfly diagram (7.1.3.1) is *symmetric* if it is isomorphic to its own pullback under T. In other words, if there exists a group isomorphism

$$\psi\colon P \longrightarrow T^*P$$

realizing a morphism of butterflies from $G_\bullet \times G_\bullet$ to $G_\bullet$.

It is easy to see that this definition is equivalent to the standard notion of braided symmetric 2-group. Indeed the condition on the butterfly spelled in the previous definition is the translation in terms of butterfly diagrams of the following way to recast the symmetry condition for a braiding. We state it as a Proposition.

7.2.2 Proposition. — *The braiding s on $\mathscr{G}$ is symmetric if and only if it is a morphism of additive functors*

$$s\colon \otimes_{\mathscr{G}} \Longrightarrow \otimes_{\mathscr{G}} \circ T\colon \mathscr{G}\times\mathscr{G} \longrightarrow \mathscr{G}.$$

As a corollary of Proposition 7.2.2 we have:

7.2.3 Proposition. — *The braiding on $\mathscr{G}$ is symmetric if and only if the butterfly* (7.1.3.1) *has the symmetry property of Definition 7.2.1.*

Proof of Proposition 7.2.2. — That the braiding s can be seen as a natural transformation is an obvious fact. The point is of course that it is a natural transformation of *additive* functors. Writing diagram (3.1.4.1) for $\otimes_{\mathscr{G}}$, $\otimes_{\mathscr{G}} \circ T$, and s, and using the two hexagon diagrams shows the equivalence between (7.1.1.2) and the symmetry condition. This is most easy when working directly with a crossed module and the braiding $\{-,-\}$, and it is left as a task to the reader. $\square$

It is instructive to deduce the symmetry of the braiding at the crossed-module level directly from Definition 7.2.1.

7.2.4. — The existence of ψ in Definition 7.2.1 is equivalent to saying that there should be an automorphism $\psi\colon P \to P$ such that

(7.2.4.1)
$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\rho"'] & P \arrow[d, "\rho"] \\ G_0 \times G_0 \arrow[r, "T"'] & G_0 \times G_0 \end{tikzcd}$$

commutes and it is compatible with all the morphisms of (7.1.3.1). In particular, this implies that ψ fixes G_1 inside P.

Using Grothendieck's theory of extensions, the automorphism ψ in the previous diagram can be equivalently understood as a collection of isomorphisms

$$\psi_{x,y}\colon P_{x,y} \longrightarrow P_{y,x}$$

of G_1-bitorsors $E_{x,y}$ above each point $(x,y) \in G_0 \times G_0$, compatible with the multiplication structure of bitorsors

(7.2.4.2)
$$\begin{tikzcd} P_{x_0,y_0} \wedge^{G_1} P_{x_1,y_1} \arrow[r] \arrow[d, "\psi\wedge\psi"'] & P_{x_0x_1,y_0y_1} \arrow[d, "\psi"] \\ P_{y_0,x_0} \wedge^{G_1} P_{y_1,x_1} \arrow[r] & P_{y_0y_1,x_0x_1} \end{tikzcd}$$

The horizontal arrows are the bitorsor contractions corresponding to the multiplication law in P.

7.2.5. — The butterfly T^*P must split when restricted to one of the factors of $G_0 \times G_0$. Thus, there will exist group homomorphisms $\hat{s}_i\colon G_0 \to \iota_i^* T^* P = (T \circ \iota_i)^* P$, $i = 1, 2$. Now, since $T \circ \iota_1 = \iota_2$, we can rather think of $\hat{s}_i$ as homomorphisms

$$\hat{s}_i\colon G_0 \longrightarrow P$$

such that $\rho \circ \hat{s}_1 = \iota_2$ and $\rho \circ \hat{s}_2 = \iota_1$. In other words, we must have

$$\rho \circ \hat{s}_1 = \rho \circ s_2, \qquad \rho \circ \hat{s}_2 = \rho \circ s_1$$

In general, this implies that $\hat{s}_1$ differ from s_2 by multiplication of an element in G_1 provided by an appropriate crossed homomorphism, and similarly for $\hat{s}_2$ and s_1. However, if we consider that s_1 and s_2 are part of the structure, as they implement the given functorial isomorphisms (3.1.1.1), then they must be simply swapped by T, so that we have

(7.2.5.1)
$$\hat{s}_1 = s_2, \qquad \hat{s}_2 = s_1.$$

7.2.6. — If the butterfly is symmetric, so that the automorphism $\psi\colon P \to P$ as in Definition 7.2.1 exists, we must have that the following diagram (which completes (7.2.4.1))

$$
\begin{tikzcd}
P \arrow[rr, "\psi"] \arrow[dd, "\rho"'] & & P \arrow[dd, "\rho"] \\
& G_0 \arrow[ul, bend right, "s_{1,2}"'] \arrow[ur, bend left, "\hat{s}_{1,2}"] \arrow[dl, bend left, "\iota_{1,2}"] \arrow[dr, bend right, "\iota_{2,1}"'] & \\
G_0 \times G_0 \arrow[rr, "T"'] & & G_0 \times G_0
\end{tikzcd}
\tag{7.2.6.1}
$$

commutes. Taking (7.2.5.1) into account, we obtain that ψ must swap s_1 and s_2:

$$\psi \circ s_1 = s_2, \qquad \psi \circ s_2 = s_1. \tag{7.2.6.2}$$

Thus, in view of the remark at the beginning of this section, ψ replaces the braiding $c(x, y)$ with the symmetric one $c(y, x)^{-1}$.

On the other hand, we have observed that compatibility of (7.2.4.1) with the rest of the butterfly means that ψ must fix G_1, so that

$$c(x, y) = \psi(c(x, y)) = c(y, x)^{-1}$$

that is, the braiding is symmetric in the usual sense.

7.3. Picard crossed modules and 2-groups. — Let $\Delta\colon \mathscr{G} \to \mathscr{G} \times \mathscr{G}$ the diagonal functor. It is obvious that Δ is a strict additive functor. Let Δ also denote the corresponding diagonal functor for $G_\bullet$, as well as the degree-wise diagonal homomorphisms for G_0 and G_1.

7.3.1. — We can pull back the butterfly (7.1.3.1) to $G_\bullet$ via Δ. Since Δ is a strict morphism, the composed butterfly corresponds to the additive functor $\mathscr{G} \to \mathscr{G}$, $X \to X \otimes X$. If the butterfly is symmetric as per Definition 7.2.1, we obtain an automorphism

$$\Delta^*\psi\colon \Delta^* P \longrightarrow \Delta^* P. \tag{7.3.1.1}$$

7.3.2 Definition. — A symmetric crossed module is *Picard* if the automorphism in eq. (7.3.1.1) is the identity.

7.3.3. — This is equivalent to the standard notion of "Picard" since, using Proposition 7.2.2 and pre-composing with $\Delta\colon \mathscr{G} \to \mathscr{G} \times \mathscr{G}$, it becomes the statement that $s * \Delta$, as a transformation from $X \to X \otimes X$ to itself, is the identity—which is condition (7.1.1.3).

7.3.4. — The Picard condition simply means that the automorphisms

$$\psi_{x,x}\colon P_{x,x} \longrightarrow P_{x,x},$$

$x \in G_0$, are equal to the identity. Then, using (7.2.4.2) with $(x_0, y_0) = (x, 1)$ and $(x_1, y_1) = (1, x)$ leads to the condition

$$s_1(x)s_2(x) = s_2(x)s_1(x), \tag{7.3.4.1}$$

which implies that $c(x, x) = 1$. It is also easy to see that if we assume condition (7.3.4.1), and use the fact that $P_{x,1}$ and $P_{1,y}$ are *canonically trivial* as G_1-bitorsors, we obtain an automorphism ψ satisfying (7.2.4.1) and $\psi_{x,x} = \mathrm{id}$ for all $x \in G_0$.

To conclude, let us remark that the stronger condition that the homomorphisms s_1 and s_2 have commuting images, that is

$$[s_1(x), s_2(y)] = 1$$

for all $x, y \in G_1$, implies that $\tau\colon G_0 \times G_0 \to P$ is a homomorphism, so that the braided butterfly (7.1.3.1) splits and (7.1.6.2) shows that the braiding is identically equal to 1. As a consequence, we find that both G_1 and G_0 are abelian and the action of G_0 on G_1 is trivial. The crossed module reduces to a length-two complex of abelian sheaves.

From a homological point of view, this reduces to the extension problem

$$0 \longrightarrow A \longrightarrow G_1 \longrightarrow G_1 \longrightarrow B \longrightarrow 0$$

of abelian sheaves.

7.4. Braided butterflies. — Assume $\mathscr{H}$ and $\mathscr{G}$ are braided gr-stacks, and let $H_\bullet$ and $G_\bullet$ be corresponding braided crossed modules (cf. Proposition 7.1.8). The following definition is, *mutatis mutandis,* the same as the one in [**Noo05**, Definition 12.1]:

7.4.1 Definition. — A butterfly $[H_\bullet, E, G_\bullet]$ is *braided* if the following condition is satisfied:

$$\kappa\{\pi(x), \pi(y)\}_H\, \imath\{\jmath(x), \jmath(y)\}_G = y^{-1}x^{-1}yx,$$

for all $x, y \in E$.

If the butterfly comes from a strict morphism, then being braided corresponds to the usual notion of morphism of braided categorical groups, that is:

$$\{f_0(x), f_0(y)\}_G = f_1(\{x, y\}_H)$$

for all $x, y \in H_0$ (see e.g. [**JS93**]).

7.4.2. — One way to understand Definition 7.4.1 is to notice that

$$\{x,y\}_E \overset{\text{def}}{=} \big(\{\pi(x),\pi(y)\}_H, \{\jmath(x),\jmath(y)\}_G\big) \tag{7.4.2.1}$$

defines a braiding on the crossed module $H_1 \times G_1 \to E$ compatible with the strict morphisms to $H_\bullet$ and $G_\bullet$ in the sense explained above. The condition in the definition is just the statement that $\kappa \cdot \imath(\{x,y\}_E) = y^{-1}x^{-1}yx$.

It is easy to verify that the formula (7.4.2.1) for the braiding on $[H_1 \times G_1 \to E]$ is actually dictated by the above requirements.

7.4.3. — The weak morphism counterpart of a braided butterfly is that of a weak morphism (i.e. additive functor between associated stacks) that is compatible with the braidings—or braided, for short. So, $F\colon \mathscr{H} \to \mathscr{G}$ is braided if all objects X, Y of $\mathscr{H}$ we have

$$\begin{tikzcd}
F(X) \otimes_{\mathscr{G}} F(Y) \arrow[r, "\lambda_{X,Y}"] \arrow[d, "s_{F(X),F(Y)}"'] & F(X \otimes_{\mathscr{H}} Y) \arrow[d, "F(s_{X,Y})"] \\
F(Y) \otimes_{\mathscr{G}} F(X) \arrow[r, "\lambda_{Y,X}"'] & F(Y \otimes_{\mathscr{H}} X)
\end{tikzcd}$$

7.4.4 Proposition. — *The butterfly $[H_\bullet, E, G_\bullet]$ is braided, that is, it satisfies the condition of Definition 7.4.1 if and only if the corresponding weak morphism is a morphism of braided gr-stacks.*

Proof. — Let

$$H_\bullet \xleftarrow{Q'} E_\bullet \xrightarrow{P'} G_\bullet$$

be the factorization of the butterfly in terms of strict morphisms (with Q' a quasi-isomorphism). Let

$$\mathscr{H} \xleftarrow{Q} \mathscr{E} \xrightarrow{P} \mathscr{G} \tag{7.4.4.1}$$

be the corresponding additive morphisms of gr-stacks, with F factorized as $P \circ Q^{-1}$.

If the butterfly is braided, then previous considerations show that P' and Q' are strictly braided morphisms and then P and Q are braided morphisms of gr-stacks. Thus F is braided.

Conversely, let us assume F is braided. First, in the decomposition (7.4.4.1), $\mathscr{E}$ is braided and Q is also braided. This actually follows from the diagram:

$$\begin{tikzcd}
\mathscr{H} \times \mathscr{H} \arrow[d, "\otimes"'] & \mathscr{E} \times \mathscr{E} \arrow[l, "Q\times Q"'] \arrow[d, "\otimes"] \\
\mathscr{H} & \mathscr{E} \arrow[l, "Q"]
\end{tikzcd}$$

(Note that the diagram will only be 2-commutative.) Choosing a quasi-inverse Q^* for Q we obtain that $\otimes_{\mathscr{E}}$ is an additive functor, or equivalently that $\mathscr{E}$ is braided. That Q itself is braided follows from the next lemma, whose proof is left to the reader.

7.4.5 Lemma. — *Let C and D be braided gr-categories. An additive functor*

$$(F, \lambda) \colon \mathsf{D} \to \mathsf{C}$$

is braided if and only if the following diagram

$$\begin{tikzcd} \mathsf{D} \times \mathsf{D} \arrow[r, "F \times F"] \arrow[d, "\otimes"'] & \mathsf{C} \times \mathsf{C} \arrow[d, "\otimes"] \arrow[dl, phantom, "\lambda"] \\ \mathsf{D} \arrow[r, "F"'] & \mathsf{C} \end{tikzcd}$$

(which defines (F, λ) to be an additive functor) is in fact a morphism of additive functors.

Using the factorization (7.4.4.1) again, we conclude that $P \simeq F \circ Q$ is braided. It follows from Proposition 7.1.8 that $H_\bullet$, $G_\bullet$, and $E_\bullet$ are braided.

Now, both P (resp. Q) arise from *strict* morphisms P' (resp. Q'), and we want to conclude that P' and Q' themselves are braided. Observe that P' gives rise to a morphism of (necessarily braided) groupoids $\mathsf{E} \to \mathsf{G}$ and moreover one has the (2-commutative) diagram of prestacks

$$\begin{tikzcd} \mathsf{E} \arrow[r, "P'"] \arrow[d, "a"'] & \mathsf{G} \arrow[d, "a"] \\ \mathscr{E} \arrow[r, "P"'] & \mathscr{G} \end{tikzcd}$$

where the vertical arrows are the "associated stack" functors. Since these are equivalences, a moment's thought will convince the reader that P braided implies that so is P'. The situation with Q, Q' is analogous. Thus P' and Q' are braided and strict, so the butterfly is braided, as wanted. □

The results of [**Noo05**, §12] are still valid in the present context: thus braided butterflies compose according to the rules of section 4.1; braided crossed modules over S form a bicategory $\underline{\mathsf{BrXMod}}$, and there is a forgetful functor $\underline{\mathsf{BrXMod}} \to \underline{\mathsf{XMod}}$.

We note a particular case of braided butterfly. Assume that $G_\bullet$ is fully abelian in the sense described in the last couple of paragraphs in section 7.2. Then the condition on the braiding reduces to:

$$\kappa\{\pi(x), \pi(y)\}_H = y^{-1}x^{-1}yx.$$

It is also easy to see that the NE-SW diagonal of the butterfly is a *central* extension of H_0 by G_1. If the butterfly is an equivalence, so that the other diagonal is also an extension, it is easy to verify that the braiding $\{-,-\}_H$ is Picard, as expected. This remark will be of some relevance in the discussion of butterflies in abelian categories. The following statements give a slightly more general take on the same theme. The proof is very easy, using Definition 7.4.1, and we will leave it out.

7.4.6 Lemma. — *Let $[H_\bullet, E, G_\bullet]$ be a braided butterfly.*

1. *If the corresponding weak morphism $\mathscr{H} \to \mathscr{G}$ is essentially surjective, or equivalently if $\jmath\colon E \to G_0$ is a sheaf epimorphism, then $H_\bullet$ symmetric (resp. Picard) implies $G_\bullet$ symmetric (resp. Picard).*
2. *If $\mathscr{H} \to \mathscr{G}$ has trivial homotopy kernel, or equivalently if $\kappa\colon H_1 \to E$ is injective, then $G_\bullet$ symmetric (resp. Picard) implies $H_\bullet$ symmetric (resp. Picard).*

Combining the two statements above into one, we obtain that if $[H_\bullet, E, G_\bullet]$ is a braided *reversible* butterfly, then obviously $H_\bullet$ is symmetric (resp. Picard) if and only if $G_\bullet$ is symmetric (resp. Picard).

8. Butterflies and abelian categories

In this section we specialize our discussion to the case of abelian crossed modules, that is, complexes of length two in an abelian category. This topic has been treated in some generality in ref. [**Noo05**, §12.3]. We want to expand on this topic in the case of $\mathsf{Ch}(\mathsf{S})$, the abelian category of complexes of abelian sheaves over S.

8.1. Crossed modules in an abelian category. — Let A be an abelian category.

8.1.1. — Crossed modules in A are simply (cohomological) complexes of length two of objects of A without additional requirements. Typically, a complex will be denoted as $A^\bullet\colon A^{-1} \to A^0$, restoring the upper-index convention—and following ref. [**Del73**].

8.1.2. — Complexes of length 2 in A, supported in degrees $[-1, 0]$ form an abelian sub-category $\mathsf{Ch}^{[-1,0]}(\mathsf{A})$ of the abelian category $\mathsf{Ch}(\mathsf{A})$ of complexes of objects of A. If A equal the category of abelian sheaves over S we write directly $\mathsf{Ch}(\mathsf{S})$ and $\mathsf{Ch}^{[-1,0]}(\mathsf{S})$.

It is immediate that the notions of (strict) morphism and 2-morphism explained in sect. 3.2 simply reduce to the standard ones of morphism of complexes and (chain) homotopy between such morphisms, respectively.

8.1.3. — Since we can consider a complex $A^\bullet$ as a crossed module with trivial braiding, it follows from our previous analyses that a "braided" butterfly from $B^\bullet$ to $A^\bullet$ is a diagram

$$
\begin{tikzcd}
B^{-1} \arrow[dd, "d"'] \arrow[dr, "\kappa"] & & A^{-1} \arrow[dl, "\imath"'] \arrow[dd, "d"] \\
& E \arrow[dl, "\pi"'] \arrow[dr, "\jmath"] & \\
B^0 & & A^0
\end{tikzcd}
\tag{8.1.3.1}
$$

of abelian sheaves such that the NE-SW diagonal is an extension. The difference with Definition 4.1.3 is that all compatibility requirements for the various actions are dropped.

As such, this definition makes sense in any abelian category A, as noted in ref. [**Noo05**].

8.1.4 Definition. — Let A be an abelian category. A butterfly in A is a diagram of the form (8.1.3.1) of objects of A such that the NE-SW diagonal is short exact, and the NW-SE diagonal is a complex.

Comparing with section 5.4, we see that for length-two complexes the butterfly diagram (8.1.3.1) provides a canonical choice for the complex $E^\bullet$ quasi-isomorphic to $B^\bullet$.

8.2. 2-categories of Picard and abelian objects. — From now on, we set A equal to the category of abelian sheaves over S.

We have already mentioned the category $\mathsf{Ch}^{[-1,0]}(\mathsf{S})$ of length-two complexes. Morphisms are simply morphisms of complexes in the usual sense.

$\mathsf{Ch}^{[-1,0]}(\mathsf{S})$ neglects the homotopies. When they are included, we actually obtain a 2-category, to be denoted $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})_{\mathrm{str}}$. Objects and 1-morphisms are the same as $\mathsf{Ch}^{[-1,0]}(\mathsf{S})$, and 2-morphisms are chain homotopies between strict morphisms. Thus $\mathsf{Ch}^{[-1,0]}(\mathsf{S})$ is tautologically the 1-category obtained from the 2-category $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})_{\mathrm{str}}$ by simply forgetting the 2-morphisms.

The bicategory $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})$ has still the same objects, plus abelian butterflies as 1-morphisms, and morphisms of (abelian) butterflies as 2-morphisms. (This is modeled on the definitions of section 5.1.) Let us denote by $\mathsf{Hom}(B^\bullet, A^\bullet)$ the morphism groupoid from $B^\bullet$ to $A^\bullet$ in $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})$. In a similar way, let us denote by $\mathsf{Hom}(B^\bullet, A^\bullet)_{\mathrm{str}}$ the morphism groupoid of the 2-category $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})_{\mathrm{str}}$.

We mention that the groupoids $\mathsf{Hom}(B^\bullet, A^\bullet)$ acquire an extra structure: they are symmetric gr-categories, since, thanks to the abelianness of everything involved, butterflies such as (8.1.3.1) can be added, there is an inverse, and an identity (the zero butterfly corresponding to the identity morphism). The formulas are identical to the ones worked out in [**Noo05**] and will not be repeated here: there is no change in passing from the set-theoretic context to that of sheaves over the site S.

$\underline{\mathsf{Pic}}(\mathsf{S})$ will denote the 2-category of Picard stacks over S. This is a sub-2-category of $\underline{\mathsf{Gr\text{-}Stacks}}(\mathsf{S})$.

The gr-stack associated to a Picard crossed module is evidently a Picard stack. In particular so is the stack associated to a complex $A^\bullet\colon A^{-1}\to A^0$. (Considering $\mathrm{T{\scriptsize ORS}}(A^{-1},A^0)$, for instance, immediately gives the Picard structure.)

8.2.1. — Given two complexes $A^\bullet$ and $B^\bullet$ we define, in analogy to what was done in section 4.1, the groupoid $\mathsf{WM}(B^\bullet,A^\bullet)$ of weak morphisms from $B^\bullet$ to $A^\bullet$, as

$$\mathsf{WM}(B^\bullet,A^\bullet)\stackrel{\text{def}}{=}\mathsf{Hom}_{\underline{\mathsf{Pic}}(\mathsf{S})}(B^{\bullet\sim},A^{\bullet\sim}),$$

that is, as the groupoid of additive functors of Picard stacks from $[B^{-1}\to B^0]^\sim\to[A^{-1}\to A^0]^\sim$.

Thus, there is a natural homomorphism:

$$\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})\longrightarrow\underline{\mathsf{Pic}}(\mathsf{S}).\tag{8.2.1.1}$$

This is just the composition of the natural inclusion of $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})\to\underline{\mathsf{XMod}}(\mathsf{S})$ with $\underline{\mathsf{XMod}}(\mathsf{S})\to\underline{\mathsf{Gr\text{-}Stacks}}(\mathsf{S})$, factoring through $\underline{\mathsf{Pic}}(\mathsf{S})$.

Finally, we can specialize the construction of the bicategory $\underline{\mathsf{XMod}}(\mathsf{S})$ in section 5.1 to Picard crossed modules in the sense of section 7.2. We obtain in this way a bicategory denoted $\underline{\mathsf{PicXMod}}(\mathsf{S})$ whose objects are Picard crossed modules, and morphism groupoids from $H^\bullet$ to $G^\bullet$, denoted $\mathsf{PicB}(H^\bullet,G^\bullet)$, are Picard butterflies and their morphisms.

8.2.2. — We can very well give the same definition of weak morphism of Picard crossed modules, by setting

$$\mathsf{WM}(H^\bullet,G^\bullet)\stackrel{\text{def}}{=}\mathsf{Hom}_{\underline{\mathsf{Pic}}(\mathsf{S})}(H^{\bullet\sim},G^{\bullet\sim}),$$

where $H^\bullet$ and $G^\bullet$ are two objects of $\underline{\mathsf{PicXMod}}(\mathsf{S})$. There also is an obvious homomorphism

$$\underline{\mathsf{PicXMod}}(\mathsf{S})\longrightarrow\underline{\mathsf{Pic}}(\mathsf{S}).\tag{8.2.2.1}$$

8.3. Deligne's results in "La formule de dualité globale". — Theorem 4.3.1 remains true in the present context, in the following alternative form:

***8.3.1 Theorem* (Theorem 4.3.1 for Picard stacks)**. — *For two objects $B^\bullet$, $A^\bullet$ of $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})$ there is an equivalence of groupoids:*

$$\mathsf{WM}(B^\bullet,A^\bullet)\xrightarrow{\sim}\mathsf{Hom}(B^\bullet,A^\bullet).$$

In particular one direction—from weak morphisms to butterflies—corresponds to [**Del73**, Lemme 1.4.13 (II)]. The proof is obtained by translating the proof of Theorem 4.3.1 to abelian butterflies by simply assuming everything is abelian and neglecting the actions. The reader will be able to check that it reduces to Deligne's proof (in one direction). In the other direction, our result provides a direct converse to Deligne's 1.4.13 (II), different from loc. cit., Corollaire 1.4.17.

Proposition 5.3.7 also remains true, upon replacing "gr-stack" with "Picard stack." Namely, we have:

8.3.2 Proposition **(Proposition 5.3.7 for Picard stacks)**
Let $\mathscr{A}$ be a gr-stack. Then there exists a complex $A^\bullet\colon A^{-1} \to A^0$ such that $\mathscr{A}$ is equivalent to the Picard stack $[A^{-1} \to A^0]^\sim$.

This is actually Lemme 1.4.13 (I) of loc. cit. Again, the proof can be obtained from the proof of Proposition 5.3.7 by: (1) replacing the sheafification of the free group on a sheaf of sets with the free abelian group thereof; (2) replacing the coherence argument from [**Lap83**] as used above with the one from [**SR72**] found in [**Del73**]. In this way the proof becomes essentially the same as the one found in Deligne's work.

8.4. 2-category of Picard crossed modules. — A slightly different point of view on the relationship between Proposition 5.3.7 and [**Del73**, Lemme 1.4.13 (I)] is as follows. Combining the former with Proposition 7.1.8 and the considerations on Picard butterflies in sect. 7.2, we can state an alternative version of Proposition 8.3.2:

8.4.1 Proposition **(Proposition 5.3.7 for Picard stacks—2$^{\text{nd}}$ version)**
Let $\mathscr{A}$ be a Picard stack. Then there exists a Picard crossed module $[G^{-1} \to G^0]$ such that $\mathscr{A}$ is equivalent to the Picard stack $[G^{-1} \to G^0]^\sim$.

The crossed module obtained in this way is not necessarily an object of $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})$. However, it is *equivalent* to one. More precisely, the combination of Propositions 8.3.2 and 8.4.1 guarantees that given a Picard crossed module $[G^{-1} \to G^0]$ one can find a complex $A^{-1} \to A^0$ of abelian sheaves such that there is an equivalence

$$[G^{-1} \to G^0]^\sim \xrightarrow{\sim} [A^{-1} \to A^0]^\sim$$

and that moreover this equivalence can be realized by a reversible butterfly, say $[G^\bullet, P, A^\bullet]$.

The foregoing discussion has the following consequence, which can be considered as an alternative statement for Theorem 8.3.1:

8.4.2 Corollary. — *For two objects $H^\bullet$, $G^\bullet$ of $\underline{\mathsf{PicXMod}}(\mathsf{S})$ there is an equivalence of groupoids:*

$$\mathsf{WM}(H^\bullet, G^\bullet) \xrightarrow{\sim} \mathsf{PicB}(H^\bullet, G^\bullet).$$

Putting all together we immediately obtain the following:

8.4.3 Proposition. — *There are biequivalences*

$$\underline{\mathsf{Pic}}(\mathsf{S}) \simeq \underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S}) \simeq \underline{\mathsf{PicXMod}}(\mathsf{S})$$

induced by the homomorphisms (8.2.1.1)*,* (8.2.2.1)*.*

Analogously to ref. [**Del73**, Proposition 1.4.15], it results from the above that there is an equivalence of categories between any of $\underline{\mathsf{Pic}}(\mathsf{S})^\flat$, $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})^\flat$, or $\underline{\mathsf{PicXMod}}(\mathsf{S})^\flat$, and $\mathsf{D}^{[-1,0]}(\mathsf{S})$, the sub-category of the derived category of $\mathsf{Ch}(\mathsf{S})$ consisting of complexes $K^\bullet$ with $\mathrm{H}^i(K^\bullet) \neq 0$ only for $i = -1, 0$.

8.5. Stacky analogs. — Many of the constructions of the previous sections can be sheafified over S.

For two objects $A^\bullet$ and $B^\bullet$ of $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})$, the sheaf-theoretic counterpart of the groupoid $\mathsf{Hom}(B^\bullet, A^\bullet)$, denoted $\mathscr{H}om(B^\bullet, A^\bullet)$, is obtained in the same way as its non-abelian counterpart $\mathscr{B}(-,-)$ in section 4.6, namely by assigning to every object U of S the groupoid $\mathsf{Hom}(B^\bullet|_U, A^\bullet|_U)$.

The same proof as the one for Proposition 4.6.1 yields

8.5.1 Proposition. — *$\mathscr{H}om(B^\bullet, A^\bullet)$ is a stack over S.*

Note that by virtue of the remarks on the additivity of the abelian butterflies, it follows that $\mathscr{H}om(B^\bullet, A^\bullet)$ is a symmetric gr-stack.

There is an obvious fibered analog $\mathfrak{Ch}^{[-1,0]}(\mathsf{S})$ of $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})$ defined exactly as $\mathfrak{XMod}(\mathsf{S})$ in section 5.3. Recall that for each object U of S its fiber over U is $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S}/U)$. Again, this is a fibration in bicategories and, by the previous proposition, $\mathfrak{Ch}^{[-1,0]}(\mathsf{S})$ is a pre-bistack. Analogously to Theorem 5.3.6, we have

8.5.2 Theorem. — *$\mathfrak{Ch}^{[-1,0]}(\mathsf{S})$ is a bistack.*

Sketch of the proof. — We need only show that the 2-descent condition for objects holds. To do this, we can rely upon Theorem 5.3.6 to conclude that given a 2-descent data relative to $\mathfrak{Ch}^{[-1,0]}$ are effective in $\mathfrak{XMod}(\mathsf{S})$.

Thus, let $\{Y_\bullet, A^\bullet, E, \alpha\}$ be a 2-descent datum relative to a hypercover $Y_\bullet \to U$ as in 5.3.3, but with all data in $\mathfrak{Ch}^{[-1,0]}(\mathsf{S})$. It glues to an object $G^\bullet$ of $\mathfrak{XMod}(\mathsf{S})_U = \underline{\mathsf{XMod}}(\mathsf{S}/U)$.

From 5.3.3, there is a reversible butterfly $[A^\bullet, F, G^\bullet|_{Y_0}]$ which implies, using Lemmas 7.4.5 and 7.4.6, that $G^\bullet$ can be made Picard—and F automatically braided.

When we pull back to Y_1, using the descent condition, we potentially obtain *two* different Picard structures on $G^\bullet|_{Y_1}$. One results from $[d_0^*A^\bullet, d_0^*F, G^\bullet|_{Y_1}]$, the other from the composition of $[d_0^*A^\bullet, E, d_1^*A^\bullet]$ with $[d_1^*A^\bullet, d_1^*F, G^\bullet|_{Y_1}]$. It is easy to verify they are in fact the same. More generally:

8.5.3 Claim. — *Let $H^\bullet$ be braided by $\{-,-\}_H$, and $G^\bullet$ have* two *braided structures $\{-,-\}_G$ and $\{-,-\}'_G$. Let E (resp. E') sit in a braided butterfly from $(H^\bullet, \{-,-\}_H)$ to $(G^\bullet, \{-,-\}_G)$ (resp. $(G^\bullet, \{-,-\}_G)$).*

Assume $E \xrightarrow{\sim} E'$. If E (hence E') corresponds to an essentially surjective butterfly, then

$$\{-,-\}_G = \{-,-\}'_G.$$

It follows from the claim that the Picard structure on $G^\bullet$ descends to U. Thus appealing to 8.4.3, we can conclude that $G^\bullet$ is equivalent to a complex $B^\bullet\colon B^{-1}\to B^0$, i.e. an object of $\mathfrak{Ch}^{[-1,0]}(\mathsf{S})$ over U. □

There are obvious fibered analogs of what we have just discussed for Picard crossed modules and Picard stacks, yielding the bistack $\mathfrak{PicXMod}(\mathsf{S})$ and the 2-stack $\mathfrak{Pic}(\mathsf{S})$, respectively, whose fibers over U are $\underline{\mathsf{PicXMod}}(\mathsf{S}/U)$ and $\underline{\mathsf{Pic}}(\mathsf{S}/U)$.

The homomorphisms (8.2.1.1), (8.2.2.1), appropriately localized, provide morphisms of bistacks

$$\mathfrak{Ch}^{[-1,0]}(\mathsf{S})\longrightarrow\mathfrak{Pic}(\mathsf{S}),\qquad \mathfrak{PicXMod}(\mathsf{S})\longrightarrow\mathfrak{Pic}(\mathsf{S}).$$

and finally the stack analog of Proposition 8.4.3 is

8.5.4 Proposition. — *There are biequivalences*

$$\mathfrak{Pic}(\mathsf{S})\simeq\mathfrak{Ch}^{[-1,0]}(\mathsf{S})\simeq\mathfrak{PicXMod}(\mathsf{S}).$$

We conclude with the following analogs of [**Noo05**, Proposition 12.3]. For two abelian groups A,B over S define

$$\mathsf{Ext}(B,A)$$

to be the groupoid whose objects are extensions of B by A and whose morphisms are isomorphisms of extensions.

Let $A^\bullet\colon A^{-1}\to A^0$ and $B^\bullet\colon B^{-1}\to B^0$ be two objects of $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})$. There is an obvious forgetful morphism of groupoids

$$\mathsf{Hom}(B^\bullet,A^\bullet)\longrightarrow\mathsf{Ext}(B^0,A^{-1})$$

which sends a butterfly to its NE-SW diagonal.

8.5.5 Proposition. — *The above map is a fibration. Its homotopy kernel is equivalent to* $\mathsf{Hom}(B^\bullet,A^\bullet)_{\mathrm{str}}$ *and the sequence*

$$\mathsf{Hom}(B^\bullet,A^\bullet)_{\mathrm{str}}\longrightarrow\mathsf{Hom}(B^\bullet,A^\bullet)\longrightarrow\mathsf{Ext}(B^0,A^{-1})$$

is a short-exact sequence (in the sense of section 6.2) of symmetric gr-categories.

Proof. — For the additive structures, we refer to loc. cit. The rest is trivial, except perhaps the fibration condition. If one has an isomorphism $E\to E'$ fitting in an isomorphism of extensions of B^0 by A^{-1}, and there is a butterfly whose NE-SW diagonal is the extension $A^{-1}\to E'\to B^0$, the construction of a butterfly with center E is just a calculation, and it is left to the reader. □

Note that an extension $0\to A\to E\to B\to 0$ can be thought of as a butterfly from $[0\to B]$ to $[A\to 0]$, that is, as a weak morphism $B[0]\to\mathrm{T{\scriptsize ORS}}(A)$. It follows that there is a symmetric gr-stack

$$\mathscr{E}xt(B,A)$$

whose fibers over $U\in\operatorname{Ob}\mathsf{S}$ are $\mathsf{Ext}(B|_U,A|_U)$.

Recall the *locally split butterflies* of Definition 4.7.1. With the obvious changes in notations, let $\mathscr{H}om(B^\bullet, A^\bullet)^\sim_{\mathrm{str}}$ denote the stack of locally split butteflies from $B^\bullet$ to $A^\bullet$. We obtain the following analog of Proposition 8.5.5:

8.5.6 Proposition. — *Let $A^\bullet$ and $B^\bullet$ be two (global) objects of $\underline{\mathsf{Ch}}^{[-1,0]}(\mathsf{S})$. There is a short exact sequence of symmetric gr-stacks*

$$\mathscr{H}om(B^\bullet, A^\bullet)^\sim_{\mathrm{str}} \longrightarrow \mathscr{H}om(B^\bullet, A^\bullet) \longrightarrow \mathscr{E}xt(B^0, A^{-1})$$

The forgetful map from butterflies to extensions is a fibration.

8.6. Ringed sites and locally split butterflies. — Let us suppose $\mathscr{O}_{\mathsf{S}}$ is a sheaf of commutative unital rings over S, so that the pair $(\mathsf{S}, \mathscr{O}_{\mathsf{S}})$ becomes a ringed site. It is known the category $\mathsf{Mod}(\mathscr{O}_{\mathsf{S}})$ of $\mathscr{O}_{\mathsf{S}}$-modules is abelian.

8.6.1. — The constructions of butterflies, morphism of butterflies, etc., in general those of section 8.2 make sense in the category of $\mathscr{O}_{\mathsf{S}}$-modules. A crossed module in $\mathsf{Mod}(\mathscr{O}_{\mathsf{S}})$ will simply be a length-two complex $M^\bullet\colon M^{-1} \to M^0$, where M^{-1} and M^0 are $\mathscr{O}_{\mathsf{S}}$-modules. A butterfly from $N^\bullet$ to $M^\bullet$ is a diagram

$$\begin{tikzcd}
N^{-1} \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & M^{-1} \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& P \arrow[dl, "\pi"'] \arrow[dr, "\jmath"] & \\
N^0 & & M^0
\end{tikzcd}$$

of $\mathscr{O}_{\mathsf{S}}$-modules such that the NE-SW diagonal is an extension in $\mathsf{Mod}(\mathscr{O}_{\mathsf{S}})$, whereas the NW-SE one is a complex. A split butterfly is one where the extension is split in $\mathsf{Mod}(\mathscr{O}_{\mathsf{S}})$. A *locally split* one is a butterfly for which the splitting occur after restricting to a (generalized) cover.

Thus everything is as for the category of abelian sheaves on S, plus the additional requirement of compatibility with the $\mathscr{O}_{\mathsf{S}}$-linear action. According to this new setting, for two complexes $N^\bullet$ and $M^\bullet$ let us denote by $\mathsf{Hom}_{\mathscr{O}_{\mathsf{S}}}(N^\bullet, M^\bullet)$ the groupoid of butterflies, by $\mathsf{Hom}_{\mathscr{O}_{\mathsf{S}}}(N^\bullet, M^\bullet)_{\mathrm{str}}$ that of butterflies corresponding to strict morphisms. Their stack counterparts will be $\mathscr{H}om_{\mathscr{O}_{\mathsf{S}}}(N^\bullet, M^\bullet)$ and $\mathscr{H}om_{\mathscr{O}_{\mathsf{S}}}(N^\bullet, M^\bullet)^\sim_{\mathrm{str}}$, the latter denoting the stack of *locally split* butterflies. We then have $\underline{\mathsf{Ch}}^{[-1,0]}(\mathscr{O}_{\mathsf{S}})$, its strict counterpart $\underline{\mathsf{Ch}}^{[-1,0]}(\mathscr{O}_{\mathsf{S}})_{\mathrm{str}}$, and the stack analog $\mathfrak{Ch}^{[-1,0]}(\mathscr{O}_{\mathsf{S}})$.

8.6.2. — $M \in \operatorname{Ob}\mathsf{Mod}(\mathscr{O}_{\mathsf{S}})$ is *locally free* of rank m if there is a generalized cover $Y \to *$ such that $M|_Y \xrightarrow{\sim} \big(\mathscr{O}_{\mathsf{S}}|_Y\big)^m$. Let us say that $M^\bullet$ is locally free if both M^{-1} and M^0 are.

The following is easy:

8.6.3 Proposition. — *Let $N^\bullet, M^\bullet$ be two locally free complexes. Then every butterfly $[N^\bullet, P, M^\bullet]$ is a locally split butterfly of locally free objects.*

Proof. — In the sequence $0 \to M^{-1} \to P \to N^0 \to 0$ N^0 locally free implies that the sequence is locally split. Then M^{-1} locally free implies that so is P. $\square$

8.6.4 Corollary. — *If $N^\bullet$ and $M^\bullet$ are locally free, then $\mathscr{E}xt_{\mathcal{O}_{\mathsf{S}}}(N^0, M^{-1})$ is equivalent to a point.*

Proof. — Proposition 8.6.3 gives $\mathscr{H}om_{\mathcal{O}_{\mathsf{S}}}(N^\bullet, M^\bullet) \simeq \mathscr{H}om_{\mathcal{O}_{\mathsf{S}}}(N^\bullet, M^\bullet)^{\sim}_{\mathrm{str}}$, then use Proposition 8.5.6. $\square$

Restricting our attention to the locally free objects of $\mathfrak{Ch}^{[-1,0]}(\mathcal{O}_{\mathsf{S}})$, we immediately obtain that they comprise a full fibered sub-bicategory of $\mathfrak{Ch}^{[-1,0]}(\mathcal{O}_{\mathsf{S}})$, which we denote by $\mathfrak{Ch}^{[-1,0]}_{\mathrm{loc.fr.}}(\mathcal{O}_{\mathsf{S}})$. The inclusion

$$\mathfrak{Ch}^{[-1,0]}_{\mathrm{loc.fr.}}(\mathcal{O}_{\mathsf{S}}) \hookrightarrow \mathfrak{Ch}^{[-1,0]}(\mathcal{O}_{\mathsf{S}})$$

is a map of pre-bistacks.

A
The 2-stack of gr-stacks

In the main text, in Theorem 5.3.4, it is claimed that the fibered 2-category GR-STACKS(S) is a 2-stack over S.

Here we provide a sketch of a direct, brute-force approach, proof of this fact. The main reason for providing one at all is that although this fact should be well know to experts, the authors could not find an adequate reference, let alone a proof, to it in the literature.

A.1. Preliminaries. — We begin with writing down a few necessary diagrams, translating some of the definitions recalled in sect. 3.1. Given two gr-stacks $\mathscr{C}$ and $\mathscr{D}$, an additive functor $(F, \lambda)\colon \mathscr{C} \to \mathscr{D}$ corresponds to the diagram

$$\begin{tikzcd}
\mathscr{C} \times \mathscr{C} \arrow[r, "F \times F"] \arrow[d, "\otimes_{\mathscr{C}}"'] \arrow[dr, phantom, "\not\Downarrow\lambda"] & \mathscr{D} \times \mathscr{D} \arrow[d, "\otimes_{\mathscr{D}}"] \\
\mathscr{C} \arrow[r, "F"'] & \mathscr{D}
\end{tikzcd}$$

The compatibility of (F, λ) with the associator morphisms, expressed in diagram (3.1.3.1), can be written as:

$$\begin{aligned}
(\lambda * (\mathrm{Id}, \otimes_{\mathscr{C}})) \circ (\otimes_{\mathscr{D}} * (\mathrm{Id}, \lambda)) \circ (a_{\mathscr{D}} * (F \times F \times F)) \\
= (F * a_{\mathscr{C}}) \circ (\lambda * (\otimes_{\mathscr{C}}, \mathrm{Id})) \circ (\otimes_{\mathscr{D}} * (\lambda, \mathrm{Id}))
\end{aligned} \tag{A.1.1}$$

This expresses the commutativity of the cube one can construct from the diagram above and the ones resulting from the associativity morphisms for $\mathscr{C}$ and $\mathscr{D}$.

A morphism of additive functors $\theta\colon (F,\lambda) \Rightarrow (G,\mu)$ translates into the equality of the following two diagrams:

$$\begin{tikzcd} \mathscr{C}\times\mathscr{C} \arrow[r, "F\times F"] \arrow[d, "\otimes_{\mathscr{C}}"'] \arrow[dr, phantom, "\swarrow\lambda"] & \mathscr{D}\times\mathscr{D} \arrow[d, "\otimes_{\mathscr{D}}"] \\ \mathscr{C} \arrow[r, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, bend right=20, "\Downarrow\theta"] & \mathscr{D} \end{tikzcd}
\qquad
\begin{tikzcd} \mathscr{C}\times\mathscr{C} \arrow[r, bend left, "F\times F"] \arrow[r, "G\times G"'] \arrow[r, phantom, bend left=20, "\Downarrow\theta\times\theta"] \arrow[d, "\otimes_{\mathscr{C}}"] \arrow[dr, phantom, "\swarrow\mu"] & \mathscr{D}\times\mathscr{D} \arrow[d, "\otimes_{\mathscr{D}}"] \\ \mathscr{C} \arrow[r, "G"'] & \mathscr{D} \end{tikzcd}$$

or, in other words:

$$(\theta * \otimes_{\mathscr{C}}) \circ \lambda = \mu \circ (\otimes_{\mathscr{D}} * (\theta \times \theta)), \tag{A.1.2}$$

where $*$ denotes horizontal composition (pasting) of 2-arrows.

Let us say that a fibered 2-category $\mathfrak{F}$ over S is *separated* if the 2-morphisms glue, in other words if for any two objects X, Y of $\mathfrak{F}$, the fibered category $\mathscr{H}om_{\mathfrak{F}}(X,Y)$ is a prestack. We will say that $\mathfrak{F}$ is a *2-prestack* if $\mathscr{H}om_{\mathfrak{F}}(X,Y)$ is in fact a stack, that is if 1-morphisms also glue.

A.2. The proof. —

A.2.1 Claim. — GR-STACKS(S) *is separated.*

Proof. — Suppose we are given two gr-stacks $\mathscr{C}$ and $\mathscr{D}$, plus two additive functors $F, G\colon \mathscr{C} \to \mathscr{D}$. Let $U_\bullet \to *$ be a hypercover, and let

$$\theta\colon F|_{U_0} \Longrightarrow G|_{U_0}\colon \mathscr{C}|_{U_0} \longrightarrow \mathscr{D}|_{U_0}$$

be a morphism of additive functors such that

$$d_0^*\theta = d_1^*\theta$$

over U_1. Since STACKS(S) is a 2-stack, there exists a 2-morphism of stacks $\tilde{\theta}$ such that

$$\tilde{\theta}|_{U_0} = \theta,$$

and, by construction, $\tilde{\theta}$ satisfies (A.1.2) on U_0. Since this is an identity between 2-morphisms, it follows that (A.1.2) is then satisfied globally. This proves the claim. $\square$

A.2.2 Claim. — GR-STACKS(S) *is a 2-prestack.*

Proof. — Let $U_\bullet$ be as in the proof of the previous claim.

This time, let us suppose we are given an appropriate descent datum for 1-morphisms, that is, suppose we are given 1-morphisms

$$(F,\lambda)\colon \mathscr{C}|_{U_0} \longrightarrow \mathscr{D}|_{U_0}.$$

over U_0, 2-morphisms

$$\theta\colon d_0^*(F,\lambda) \Longrightarrow d_1^*(F,\lambda)$$

over U_1, such that the relation

$$d_1^*\theta = d_2^*\theta \circ d_0^*\theta$$

holds over U_2.

This implies that in $\textsc{Stacks}(\mathsf{S})$ there exist a 1-morphism $\tilde{F}\colon \mathscr{C} \to \mathscr{D}$ and a 2-morphism $\tau\colon F \Rightarrow \tilde{F}|_{U_0}$ such that

(A.2.1)
$$\begin{tikzcd} d_0^*F \arrow[rr, Rightarrow, "\theta"] \arrow[dr, Rightarrow, "d_0^*\tau"'] & & d_1^*F \arrow[dl, Rightarrow, "d_1^*\tau"] \\ & \tilde{F}|_{U_1} & \end{tikzcd}$$

is a commuting diagram of natural transformations over U_1.

We can use these data to enforce the additivity condition on to $\tilde{F}$. Namely, let us *define* the natural isomorphism

$$\tilde{\lambda}\colon\ \otimes_{\mathscr{D}} \circ(\tilde{F}\times\tilde{F}) \Longrightarrow \tilde{F}\circ\otimes_{\mathscr{C}}$$

over U_0 by:

$$\tilde{\lambda}\circ(\otimes_{\mathscr{D}} * (\tau\times\tau)) = (\tau * \otimes_{\mathscr{C}})\circ\lambda.$$

(Note that the compatibility with the associators, namely (3.1.3.1), or (A.1.1)), follows automatically.) An elementary manipulation using (A.2.1) then shows that $d_0^*\tilde{\lambda} = d_1^*\tilde{\lambda}$ and therefore, by the preceding claim, there exists a global $\tilde{\tilde{\lambda}}$ satisfying

$$\tilde{\tilde{\lambda}}|_{U_0} = \tilde{\lambda}.$$

Again, the compatibility with the associators will follow. Thus, the morphism $\tilde{F}$ is actually additive: the pair $(\tilde{F},\tilde{\tilde{\lambda}})$ satisfies the required properties, which proves the claim. □

Having so far shown that $\textsc{Gr-Stacks}(\mathsf{S})$ is a 2-prestack, the hardest step is to show that $\textsc{Gr-Stacks}(\mathsf{S})$ actually is a 2-stack, as there are many more diagrams to check.

The problem is posed by giving ourselves an appropriate 2-descent datum of gr-stacks. This means that if $U_\bullet$ is the previously introduced hypercover, then we are given the following data:

1. A gr-stack $\mathscr{C}$ over U_0.
2. An additive functor $(F,\lambda)\colon d_0^*\mathscr{C} \to d_1^*\mathscr{C}$ over U_1.

3. A morphism of additive functors
$$\theta\colon d_2^*(F,\lambda)\circ d_0^*(F,\lambda)\Longrightarrow d_1^*(F,\lambda)$$
over U_2.
4. A coherence diagram for θ over U_3, in other words, the diagram formed by the faces of the tetrahedron must commute:

$$\begin{tikzcd}
\mathscr{C}_3 \arrow[rr, "F_{03}"] \arrow[dd, "F_{23}"'] \arrow[ddrr, "F_{13}"] & & \mathscr{C}_0 \\
& & \\
\mathscr{C}_2 \arrow[rr, "F_{12}"'] \arrow[uurr, "F_{02}"] & & \mathscr{C}_1 \arrow[uu, "F_{01}"']
\end{tikzcd}
\qquad \theta_{023}\circ(\theta_{012}*F_{23})=\theta_{013}\circ(F_{01}*\theta_{123})$$

We have used a "missing index" convention: $\mathscr{C}_0=(d_1d_2d_3)^*\mathscr{C}$, $F_{01}=(d_2d_3)^*F$, $\theta_{012}=d_3^*\theta$, and so on.

Again, since $\text{Stacks}(\mathsf{S})$ is a 2-stack, there exists a *stack* $\tilde{\mathscr{C}}$ over S with an equivalence of stacks
$$G\colon \mathscr{C}\xrightarrow{\sim}\tilde{\mathscr{C}}|_{U_0}$$
and a 2-morphism of stacks over U_1:
$$\begin{tikzcd}
\mathscr{C}_1 \arrow[r, "F"] \arrow[dr, bend right, "G_1"'] & \mathscr{C}_0 \arrow[d, "G_0"] \arrow[dl, phantom, "\swarrow\mu"] \\
& \tilde{\mathscr{C}}|_{U_1}
\end{tikzcd}$$
such that the system is coherent over U_2:

(A.2.2)
$$\begin{tikzcd}
\mathscr{C}_2 \arrow[rr, "F_{02}"] \arrow[dr, "F_{12}"] \arrow[ddr, bend right, "G_2"'] & & \mathscr{C}_0 \arrow[ddl, bend left, "G_0"] \\
& \mathscr{C}_1 \arrow[ur, "F_{01}"] \arrow[d, "G_1"] & \\
& \tilde{\mathscr{C}}|_{U_2} &
\end{tikzcd}
\qquad \mu_{12}\circ(\mu_{01}*F_{12})=\mu_{02}\circ(G_0*\theta)$$

We want to show these data can be used to induce a gr-structure on $\tilde{\mathscr{C}}$. This means we have to induce multiplication, inverse, and unit object functors:

$$\otimes_{\tilde{\mathscr{C}}} \colon \tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \longrightarrow \tilde{\mathscr{C}}, \quad * \colon \tilde{\mathscr{C}}^{\mathrm{op}} \longrightarrow \tilde{\mathscr{C}}, \quad \mathbf{1} \longrightarrow \tilde{\mathscr{C}}$$

satisfying the usual diagrams, such as:
(A.2.3)

$$\begin{tikzcd}
\tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \arrow[r, "\otimes_{\tilde{\mathscr{C}}} \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \otimes_{\tilde{\mathscr{C}}}"'] & \tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \arrow[d, "\otimes_{\tilde{\mathscr{C}}}"] \arrow[dl, phantom, "\Swarrow a"] \\
\tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \arrow[r, "\otimes_{\tilde{\mathscr{C}}}"'] & \tilde{\mathscr{C}}
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathbf{1} \times \tilde{\mathscr{C}} \arrow[r] \arrow[dr, bend right] \arrow[dr, phantom, "\Swarrow l"] & \tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \arrow[d, "\otimes_{\tilde{\mathscr{C}}}"] \\
& \tilde{\mathscr{C}}
\end{tikzcd}, \qquad
\begin{tikzcd}
\tilde{\mathscr{C}} \times \mathbf{1} \arrow[r] \arrow[dr, bend right] \arrow[dr, phantom, "\Swarrow r"] & \tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \arrow[d, "\otimes_{\tilde{\mathscr{C}}}"] \\
& \tilde{\mathscr{C}}
\end{tikzcd}$$

and so on. This is done locally (i.e. on U_0) using the gr-structure of $\mathscr{C}$. For example, consider the multiplicative structure. We can induce one via the diagram:

$$\begin{tikzcd}
\mathscr{C} \times \mathscr{C} \arrow[r, "G \times G"] \arrow[d, "\otimes_{\mathscr{C}}"'] \arrow[dr, phantom, "\Swarrow \nu"] & \tilde{\mathscr{C}}|_{U_0} \times \tilde{\mathscr{C}}|_{U_0} \arrow[d, "\tilde{\otimes}"] \\
\mathscr{C} \arrow[r, "G"'] & \tilde{\mathscr{C}}|_{U_0}
\end{tikzcd}$$

where ν is determined by the choice of a quasi-inverse of G. The multiplication morphism $\tilde{\otimes}$ defined by this diagram lives over U_0, and we must show that the gluing data for $\tilde{\mathscr{C}}$ in $\textsc{Stacks}(\mathsf{S})$ yield appropriate gluing data for $\tilde{\otimes}$ along $U_\bullet$, so that it will glue to a global monoidal functor $\otimes_{\tilde{\mathscr{C}}}$.

The two possible pull-backs of the morphism $\tilde{\otimes}$ to U_1 are related by a 2-morphism $\varepsilon \colon \tilde{\otimes}_1 \Rightarrow \tilde{\otimes}_0$ which is defined by the following diagram:

(A.2.4)
$$\begin{tikzcd}
\mathscr{C}_1 \times \mathscr{C}_1 \arrow[rr, "F \times F"] \arrow[dr, "G_1 \times G_1"'] \arrow[ddd, "\otimes_1"'] & \arrow[d, phantom, "\Swarrow \mu \times \mu"] & \mathscr{C}_0 \times \mathscr{C}_0 \arrow[dl, "G_0 \times G_0"] \arrow[ddd, "\otimes_0"] \\
& \tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \arrow[d, bend right, "\tilde{\otimes}_1"'] \arrow[d, bend left, "\tilde{\otimes}_0"] \arrow[d, phantom, "\overset{\varepsilon}{\Longrightarrow}"] & \\
\arrow[r, phantom, "\Swarrow \nu_1"] & \tilde{\mathscr{C}} & \arrow[l, phantom, "\Searrow \nu_0"] \\
\mathscr{C}_1 \arrow[rr, "F"'] \arrow[ur, bend left, "G_1"] \arrow[r, phantom, "\Uparrow \mu"] & & \mathscr{C}_0 \arrow[ul, bend right, "G_0"'] \arrow[l, phantom, "\Swarrow \lambda"]
\end{tikzcd}$$

(In the previous diagram we are still using the "missing index" convention introduced above.) We need the 2-arrow ε to be coherent on U_2, namely that, after pulling everything back to U_2, it satisfies

(A.2.5)
$$d_1^* \varepsilon = d_2^* \varepsilon \circ d_0^* \varepsilon.$$

To see how this may be true, we should consider the pasting diagram resulting from the three possible pullbacks of (A.2.4) to U_2.

(A.2.6)

$$
\begin{tikzcd}
\mathscr{C}_1\times\mathscr{C}_1 \arrow[rrrrrrr, "F_{01}\times F_{01}"] \arrow[drr, "G_1\times G_1"] \arrow[dddd, "\otimes_1"'] & & & & & & & \mathscr{C}_0\times\mathscr{C}_0 \arrow[dll, "G_0\times G_0"'] \arrow[dddd, "\otimes_0"] \\
& & \tilde{\mathscr{C}}\times\tilde{\mathscr{C}} \arrow[rrr, equal] \arrow[dr, equal] \arrow[dddd, "\tilde{\otimes}_1"] & & & \tilde{\mathscr{C}}\times\tilde{\mathscr{C}} \arrow[dll, equal] \arrow[dddd, "\tilde{\otimes}_0"] & & \\
& & & \tilde{\mathscr{C}}\times\tilde{\mathscr{C}} \arrow[dddd, "\tilde{\otimes}_2"] & & & & \\
& & \arrow[rrr, phantom, "\varepsilon_{01}"] & & \mathscr{C}_2\times\mathscr{C}_2 \arrow[uuullll, "F_{12}\times F_{12}"] \arrow[uuurrr, "F_{02}\times F_{02}"'] \arrow[ul, "G_2\times G_2"'] \arrow[dddd, "\otimes_2"] & & & \\
\mathscr{C}_1 \arrow[rrrrrrr, "F_{01}"] \arrow[drr, "G_1"] & & \arrow[r, phantom, "\varepsilon_{12}"] & \arrow[rr, phantom, "\varepsilon_{02}"] & & & & \mathscr{C}_0 \arrow[dll, "G_0"'] \\
& & \tilde{\mathscr{C}} \arrow[rrr, equal] \arrow[dr, equal] & & & \tilde{\mathscr{C}} \arrow[dll, equal] & & \\
& & & \tilde{\mathscr{C}} & & & & \\
& & & & \mathscr{C}_2 \arrow[uuullll, "F_{12}"] \arrow[uuurrr, "F_{02}"'] \arrow[ul, "G_2"'] & & &
\end{tikzcd}
$$

For added clarity, we have not explicitly written the 2-arrows, except for the ε_{ij}, $ij = 01, 02, 12$. In the diagram, the bottom pyramid is the diagram in eq. (A.2.2), whereas the the top pyramid is simply the product of same with itself. The three lateral prisms are the three pull-backs of (A.2.4). All these blocks 2-commute, in the sense that their faces form a system of commutative 2-arrows. Therefore, all the faces of these parts in the diagram in (A.2.6) commute, forcing the central triangular prism to commute as well.

Since relation (A.2.5) is satisfied, we can invoke Claim A.2.2 to conclude that there exists a global functor

$$\otimes_{\tilde{\mathscr{C}}} \colon \tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \longrightarrow \tilde{\mathscr{C}}$$

equipped with a 2-arrow

$$\rho \colon \tilde{\otimes} \Longrightarrow \otimes_{\tilde{\mathscr{C}}}|_{U_0}$$

such that the relation

$$\rho_0 \circ \varepsilon = \rho_1$$

is satisfied over U_1 (again, $\rho_0 = d_1^* \rho$ and $\rho_1 = d_0^* \rho$). As a result, there is a morphism of additive functors over U_0:

$$\begin{tikzcd}
\mathscr{C} \times \mathscr{C} \arrow[r, "G\times G"] \arrow[d, "\otimes_{\mathscr{C}}"'] \arrow[dr, phantom, "\Swarrow_{\tilde{\nu}}"] & \tilde{\mathscr{C}}|_{U_0} \times \tilde{\mathscr{C}}|_{U_0} \arrow[d, "\otimes_{\tilde{\mathscr{C}}}|_{U_0}"] \\
\mathscr{C} \arrow[r, "G"'] & \tilde{\mathscr{C}}|_{U_0}
\end{tikzcd}$$

where $\tilde{\nu} = \nu \circ (\rho^{-1} * (G \times G))$, such that the following diagram holds

(A.2.7)
$$\begin{tikzcd}
\mathscr{C}_1 \times \mathscr{C}_1 \arrow[rr, "F\times F"] \arrow[dr, "G_1\times G_1"'] \arrow[ddd, "\otimes_1"'] & \arrow[d, phantom, "\Swarrow \mu\times\mu"] & \mathscr{C}_0 \times \mathscr{C}_0 \arrow[dl, "G_0\times G_0"] \arrow[ddd, "\otimes_0"] \\
& \tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \arrow[d, "\otimes_{\tilde{\mathscr{C}}}"] \arrow[dl, phantom, "\Swarrow \nu_1"] \arrow[dr, phantom, "\Searrow \nu_0"] & \\
& \tilde{\mathscr{C}} & \\
\mathscr{C}_1 \arrow[rr, "F"'] \arrow[ur, bend left, "G_1"] & \arrow[u, phantom, "\Uparrow \mu"] & \mathscr{C}_0 \arrow[ul, bend right, "G_0"'] \arrow[l, phantom, "\Swarrow \lambda"]
\end{tikzcd}$$

in place of (A.2.4), over U_1.

For $\otimes_{\tilde{\mathscr{C}}}$ to be a true monoidal functor, there must be an associator 2-morphism $\tilde{a}$, as in the first diagram in (A.2.3), satisfying the appropriate coherence condition—the pentagon identity. The cube

$$
\begin{tikzcd}
\mathscr{C}\times\mathscr{C}\times\mathscr{C} \arrow[rr, "{(\otimes,\mathrm{Id})}"] \arrow[dd, "{(\mathrm{Id},\otimes)}"'] \arrow[dr] & & \mathscr{C}\times\mathscr{C} \arrow[dd, "\otimes"'] \arrow[dr] \arrow[d, phantom, "\Uparrow{(\tilde\nu,\mathrm{Id}_G)}"] & \\
& \tilde{\mathscr{C}}\times\tilde{\mathscr{C}}\times\tilde{\mathscr{C}} \arrow[rr, "{(\otimes_{\tilde{\mathscr{C}}},\mathrm{Id})}"] \arrow[dd, "{(\mathrm{Id},\otimes_{\tilde{\mathscr{C}}})}"] \arrow[r, phantom, "\Swarrow a"] & & \tilde{\mathscr{C}}\times\tilde{\mathscr{C}} \arrow[dd, "\otimes_{\tilde{\mathscr{C}}}"] \arrow[dl, phantom, "\Swarrow\tilde\nu"] \\
\mathscr{C}\times\mathscr{C} \arrow[rr, "\otimes"] \arrow[dr] \arrow[dr, phantom, "\Swarrow{(\mathrm{Id}_G,\tilde\nu)}"] & & \mathscr{C} \arrow[dr] \arrow[d, phantom, "\tilde\nu\Uparrow \quad \Swarrow\tilde a"] & \\
& \tilde{\mathscr{C}}\times\tilde{\mathscr{C}} \arrow[rr, "\otimes_{\tilde{\mathscr{C}}}"'] & & \tilde{\mathscr{C}}
\end{tikzcd}
$$

will define the 2-arrow $\tilde{a}$ over U_0. Its two pullbacks to U_1, together with the appropriate number of copies (four) of (A.2.7), and the cube resulting from the fact that (F,λ) is an additive functor (hence compatible with the associators), give rise to a 2-commutative hypercube from which it follows that $d_0^*\tilde{a} = d_1^*\tilde{a}$. Applying Claim A.2.1 implies that there exists a globally well-defined 2-arrow

$$a^{\tilde{\mathscr{C}}}\colon\ \otimes_{\tilde{\mathscr{C}}} \circ(\otimes_{\tilde{\mathscr{C}}},\mathrm{Id}) \Longrightarrow \otimes_{\tilde{\mathscr{C}}} \circ (\mathrm{Id},\otimes_{\tilde{\mathscr{C}}})$$

such that $a^{\tilde{\mathscr{C}}}|_{U_0} = \tilde{a}$.

As for the pentagon identity, it holds locally, that is, on U_0, since it does for a. More precisely, from the cube expressing the pentagon identity for a, and the five copies of the above cube, we can form another hypercubic diagram. Of the two remaining cubes of this tesseract, one corresponds to the operation $(X,Y,Z,W) \to (X\otimes Y)\otimes(Z\otimes W)$, whereas the latter corresponds to the pentagon identity for $\tilde{a}$. Since all the first seven cubes are 2-commutative, so must be the case for the latter. It follows that $a^{\tilde{\mathscr{C}}}$ satisfies the pentagon identity.

This proves that $\tilde{\mathscr{C}}$ has a monoidal functor. The rest of the functors comprising the gr-stack structure are treated in an analogous (and equally lengthy!) way. The same is true for the coherence conditions that should be satisfied by the multiplication functor, the inverse, and the identity. After ref. [**Lap83**], this amounts to the commutativity of

$$
\begin{tikzcd}
(X\otimes X^*)\otimes X \arrow[rr, "a"] \arrow[d, "e\otimes X"'] & & X\otimes(X^*\otimes X) \arrow[d, "X\otimes\eta"] \\
I\otimes X \arrow[r, "l_X"'] & X & X\otimes I \arrow[l, "r_X"]
\end{tikzcd}
$$

written in an object-wise fashion. This corresponds to the 2-commutativity of

$$
\begin{tikzcd}
\tilde{\mathscr{C}}\times\tilde{\mathscr{C}}\times\tilde{\mathscr{C}} \arrow[rr, "(\otimes,\mathrm{Id})"] \arrow[dd, "(\mathrm{Id},\otimes)"'] \arrow[dr, phantom, "\Downarrow\eta"'] & \arrow[d, phantom, "\Downarrow e"] & \tilde{\mathscr{C}}\times\tilde{\mathscr{C}} \arrow[dd, "\otimes"] \arrow[dl, phantom, "\Downarrow l"] \\
& \tilde{\mathscr{C}} \arrow[ul, "(\mathrm{Id},*,\mathrm{Id})"'] \arrow[ur, "(I,\mathrm{Id})"] \arrow[dl, "(\mathrm{Id},I)"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, "\Rightarrow r"] & \\
\tilde{\mathscr{C}}\times\tilde{\mathscr{C}} \arrow[rr, "\otimes"'] & \arrow[r, phantom, "\Leftarrow a"] & \tilde{\mathscr{C}}
\end{tikzcd}
$$

These observations complete the proof. ☐

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