Hermitian-holomorphic (2)-Gerbes and tame symbols

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Abstract

The tame symbol of two invertible holomorphic functions can be obtained by computing their cup product in Deligne cohomology, and it is geometrically interpreted as a holomorphic line bundle with connection. In a similar vein, certain higher tame symbols later considered by Brylinski and McLaughlin are geometrically interpreted as holomorphic gerbes and 2-gerbes with abelian band and a suitable connective structure.

In this paper we observe that the line bundle associated to the tame symbol of two invertible holomorphic functions also carries a fairly canonical hermitian metric, hence it represents a class in a Hermitian holomorphic Deligne cohomology group.

We put forward an alternative definition of hermitian holomorphic structure on a gerbe which is closer to the familiar one for line bundles and does not rely on an explicit "reduction of the structure group." Analogously to the case of holomorphic line bundles, a uniqueness property for the connective structure compatible with the hermitian-holomorphic structure on a gerbe is also proven. Similar results are proved for 2-gerbes as well.

We then show the hermitian structures so defined propagate to a class of higher tame symbols previously considered by Brylinski and McLaughlin, which are thus found to carry corresponding hermitian-holomorphic structures. Therefore we obtain an alternative characterization for certain higher Hermitian holomorphic Deligne cohomology groups.

Contents

1 Introduction

The aim of this work is two-fold. For an analytic manifold X we investigate geometric objects corresponding to the elements of certain low-degree Hermitian-Holomorphic Deligne cohomology groups. These groups, denoted here $H^k_{\mathcal{D}_{h.h.}}(X, l)$, for two integers k and l, were defined in [9] and, in a slightly different fashion, later in [1]. It is already an observation by Deligne (cf. [14]) that $H^2_{\mathcal{D}_{h.h.}}(X, 1) \cong \widehat{\operatorname{Pic} X}$, the group of isomorphism classes of holomorphic line bundles with hermitian fiber metric. Here we define an appropriate notion of hermitian structure on a gerbe (or 2-gerbe) bound by $\mathcal{O}^\times_X$ and show that the corresponding (equivalence) classes are in bijective correspondence with the elements of $H^k_{\mathcal{D}_{h.h.}}(X, 1)$, for $k = 3, 4$.

As a second result and application, we show that the torsors and (2-)gerbes underlying the cup products in ordinary Deligne cohomology studied by Brylinski-McLaughlin [10, 12] can be equipped in a rather natural way with the above mentioned hermitian structures, thus producing classes in the Hermitian-Holomorphic variant. More precisely, we modify the cup product at the level of Deligne complexes to land into a Hermitian-Holomorphic one. This modification is actually quite a natural one from the point of view of Mixed Hodge Structures.

1.1 Background notions

To explain things a little bit more, let X be an analytic manifold and let $A \subseteq \mathbb{R}$ be a subring—typically $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. For any integer j, set $A(j) = (2\pi\sqrt{-1})^j A$ and let $A(j)^\bullet_\mathcal{D}$ be the Deligne complex

$$A(j)_X \hookrightarrow \mathcal{O}_X \to \underline{\Omega}^1_X \to \cdots \to \underline{\Omega}^{j-1}_X .$$

It is well known that (at the level of the derived category) there are maps $A(j)^\bullet_\mathcal{D} \otimes A(k)^\bullet_\mathcal{D} \to A(j+k)^\bullet_\mathcal{D}$ inducing a cup product in cohomology

$$H^p_\mathcal{D}(X, A(j)) \otimes H^q_\mathcal{D}(X, A(k)) \xrightarrow{\cup} H^{p+q}_\mathcal{D}(X, A(j+k)) ,$$

where we have used the notation $H^p_\mathcal{D}(X, A(j)) = \mathbf{H}^p(X, A(j)^\bullet_\mathcal{D})$ for the *Deligne cohomology* groups, and $\mathbf{H}^\bullet(X, -)$ denotes hypercohomology.

The question of obtaining a geometric picture of the cup product in cohomology is a very interesting one. A chief foundational example is the following. For $A = \mathbb{Z}$ the product

$$\mathbb{Z}(1)^\bullet_\mathcal{D} \otimes \mathbb{Z}(1)^\bullet_\mathcal{D} \longrightarrow \mathbb{Z}(2)^\bullet_\mathcal{D} \tag{1.1}$$

corresponds to the morphism

$$\mathcal{O}^\times_X \otimes \mathcal{O}^\times_X \longrightarrow \big(\mathcal{O}^\times_X \xrightarrow{d\log} \underline{\Omega}^1_X\big) \tag{1.2}$$

via the quasi-isomorphisms $\mathbb{Z}(1)^\bullet_\mathcal{D} \xrightarrow{\simeq} \mathcal{O}^\times_X[-1]$ and $\mathbb{Z}(2)^\bullet_\mathcal{D} \xrightarrow{\simeq} \big(\mathcal{O}^\times_X \xrightarrow{d\log} \underline{\Omega}^1_X\big)[-1]$. Deligne gave a geometric construction of (1.2)and the ensuing cup product

$$\mathcal{O}^\times_X(X) \otimes \mathcal{O}^\times_X(X) \xrightarrow{\cup} \mathbf{H}^1\big(X, \mathcal{O}^\times_X \xrightarrow{d\log} \underline{\Omega}^1_X\big)$$

in his work on tame symbols, cf. [13]: If f and g are two invertible functions on X, namely two elements of $\mathcal{O}_X^\times$, their cup product corresponds to a $\mathcal{O}_X^\times$-torsor, denoted $\bigl(f,g\bigr]$, equipped with an analytic connection. Furthermore, if X is a Riemann surface, the complex $\bigl(\mathcal{O}_X^\times \xrightarrow{d\log} \underline{\Omega}_X^1\bigr)$ is quasi-isomorphic to $\mathbb{C}^\times$ and the product is interpreted as the *holonomy* of the connection. For X equal to a punctured disk D_p centered at p, if f and g are holomorphic on D_p, meromorphic at p, the holonomy of $\bigl(f,g\bigr]$ computes the *tame symbol*

$$(f,g)_p = (-)^{v(f)v(g)}\bigl(f^{v(g)}/g^{v(f)}\bigr)(p)\,,$$

where $v(f)$ is the valuation of f at p, cf. [2, 13, 20]. This justifies the use of the name *tame symbol* for $\bigl(f,g\bigr]$.

A particularly pleasant property is that when f and $1-f$ are both invertible a calculation [13] using the classical Euler's dilogarithm Li_2 shows that $\bigl(f,1-f\bigr]$ is isomorphic to the trivial torsor equipped with the trivial connection d, namely the unit element in the group $\mathbf{H}^1\bigl(X,\mathcal{O}_X^\times \xrightarrow{d\log} \underline{\Omega}_X^1\bigr)$. From this one also builds an interpretation of the symbol associated to f and g in terms of Mixed Hodge Structures [13].

In this particular example there appear degree 1 and 2 Deligne cohomology groups: specifically, it is made use of the fact that invertible functions determine elements in the group $H^1_{\mathcal{D}}(X,\mathbb{Z}(1)) \cong \mathcal{O}_X^\times(X)$, and, given f and g, the class of the torsor with connection $\bigl(f,g\bigr]$ is an element of $H^2_{\mathcal{D}}(X,\mathbb{Z}(2)) \cong \mathbf{H}^1\bigl(X,\mathcal{O}_X^\times \xrightarrow{d\log} \underline{\Omega}_X^1\bigr)$. It is therefore natural to investigate the geometric objects corresponding to similar cup products of higher degree. The case of $\bigl(f,L\bigr]$, where f is again an invertible function and L is an $\mathcal{O}_X^\times$-torsor, so it determines a class in $H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) \cong H^1(X,\mathcal{O}_X^\times)$, was already considered in ref. [13], where it is interpreted in terms of a gerbe $\mathcal{G}$ over X.

This idea has been further pursued by Brylinski-McLaughlin, [10, 12]. In their study of degree 4 characteristic classes they considered the symbols $\bigl(f,L\bigr] \in H^3_{\mathcal{D}}(X,\mathbb{Z}(2))$ and, for a pair of $\mathcal{O}_X^\times$-torsors, $\bigl(L,L'\bigr] \in H^4_{\mathcal{D}}(X,\mathbb{Z}(2))$. The corresponding geometric objects are identified with a gerbe (resp. a 2-gerbe) both equipped with the appropriate analog of a connection. Furthermore, the obvious map $\mathbb{Z}(2)^\bullet_{\mathcal{D}} \to \mathbb{Z}(1)^\bullet_{\mathcal{D}}$ induces a corresponding map $H^k_{\mathcal{D}}(X,\mathbb{Z}(2)) \to H^k_{\mathcal{D}}(X,\mathbb{Z}(1))$ which simply forgets the connection. Therefore elements in the groups $H^k_{\mathcal{D}}(X,\mathbb{Z}(1))$, for $k=3,4$ correspond to equivalence classes of (2-)gerbes bound by $\mathcal{O}_X^\times$, cf.[7, 10, 12]. Thus in the end several Deligne cohomology groups have a concrete interpretation in terms of geometric data.

Hermitian-Holomorphic Deligne cohomology, as defined by Brylinski, cf. [9], is an enhanced version of Deligne cohomology. For all positive integers l Brylinski introduces certain complexes $C(l)^\bullet$, and defines the Hermitian-Holomorphic Deligne cohomology groups as the sheaf hypercohomology groups: $H^k_{\mathcal{D}_{h.h.}}(X,l) = \mathbf{H}^k(X,C(l)^\bullet)$. The complex $C(l)^\bullet$ has a map $C(l)^\bullet \to \mathbb{Z}(l)^\bullet_{\mathcal{D}}$, thus there is an obvious map $H^k_{\mathcal{D}_{h.h.}}(X,l) \to H^k_{\mathcal{D}}(X,\mathbb{Z}(l))$ forgetting the extra-structure.

A primary example is provided by Deligne's observation mentioned before, cf. [14], that

$$\widehat{\mathrm{Pic}\,X} \cong \mathbf{H}^2\bigl(X,\mathbb{Z}(1)_X \to \mathcal{O}_X \to \underline{\mathcal{E}}^0_X\bigr)\,, \tag{1.3}$$

where $\widehat{\mathrm{Pic}\,X}$ is the set of isomorphism classes of $\mathcal{O}_X^\times$-torsors with hermitian metric, and $\underline{\mathcal{E}}^0_X$ is the sheaf of smooth real-valued functions on X. The complex in (1.3) is quasi-isomorphic to $C(1)^\bullet$, therefore

$$\widehat{\mathrm{Pic}\,X} \cong H^2_{\mathcal{D}_{h.h.}}(X,1)\,.$$

In fact, both complexes are quasi-isomorphic to the complex $\bigl(\mathcal{O}_X^\times \oplus \underline{\mathbb{T}}_X \to \underline{\mathbb{C}}_X^\times\bigr)[-1]$, [12, 9], which encodes the reduction of the torsor structure from $\mathcal{O}_X^\times$ to $\underline{\mathbb{T}}_X$ afforded by the hermitian metric.

Concerning higher degrees, Brylinski-McLaughlin [12, 11] gave a geometric interpretation for some of the groups $H^k_{\mathcal{D}_{h.h.}}(X,l)$, $k=3,4$ and $l=1,2$ in terms of classes of gerbes and 2-gerbes bound by $\underline{\mathbb{T}}_X$ and equipped with a concept of connection valued in an appropriate Hodge filtration of the de Rham complex of X.

1.2 Statement of the results

In this work we take on the same question of a geometric interpretation for some Hermitian-Holomorphic Deligne cohomology groups from a holomorphic view-point which, we believe, is complementary to that of Brylinski-McLaughlin. We define a hermitian structure on a $\mathcal{O}_X^\times$-gerbe $\mathcal{G}$ as the assignment of a $\underline{\mathcal{E}}^0_{U,+}$-torsor (the "+" denotes positive functions) to any object P of $\mathcal{G}_U$ subject to several conditions spelled out in Definition 5.2.1. We prove that classes of gerbes with hermitian structures in this sense correspond to elements of $H^3_{\mathcal{D}_{h.h.}}(X,1) \cong \mathbf{H}^3\bigl(X,\mathbb{Z}(1)_X \to \mathcal{O}_X \to \underline{\mathcal{E}}^0_X\bigr)$, in complete analogy with (1.3). Moreover we can define a type $(1,0)$-connective structure on $\mathcal{G}$ by requiring that to any object P of $\mathcal{G}_U$ be assigned a $F^1\underline{A}^1_U$-torsor, essentially repeating the steps in ref. [12]. (Here $\underline{A}^\bullet_U$ is the smooth $\mathbb{C}$-valued de Rham complex, and F^1 is the first Hodge filtration.) Then a notion of compatibility between the hermitian structure and the connective one is defined, and in fact we prove there is only one such type $(1,0)$ connective structure compatible with a given hermitian structure, up to equivalence. This result is analogous to the corresponding statement for hermitian holomorphic line bundles, that there is a unique connection — the *canonical or Griffiths connection* — compatible with both structures.

Similar results are available for 2-gerbes: we define a hermitian structure for a $\mathcal{O}^\times_X$-2-gerbe G as the assignment of a $\underline{\mathcal{E}}^0_{U,+}$*-gerbe* for each object P of G_U, subject to several conditions spelled out in Definition 5.5.1. Analogously to the simpler case of gerbes, we have a concept of type $(1,0)$ connectivity compatible with the hermitian structure and a uniqueness result up to equivalence.

A second line of results is more specific to the tame symbols we encountered before. Alongside with the map of complexes

$$\mathbb{Z}(1)^\bullet_{\mathcal{D}} \otimes \mathbb{Z}(1)^\bullet_{\mathcal{D}} \longrightarrow \mathbb{Z}(2)^\bullet_{\mathcal{D}}$$

we define a companion map

$$\mathbb{Z}(1)^\bullet_{\mathcal{D}} \otimes \mathbb{Z}(1)^\bullet_{\mathcal{D}} \longrightarrow 2\pi\sqrt{-1} \otimes C(1)^\bullet \tag{1.4}$$

so that it is possible to obtain a different cup product valued in Hermitian-Holomorphic Deligne cohomology:

$$H^i_{\mathcal{D}}(X,\mathbb{Z}(1)) \otimes H^j_{\mathcal{D}}(X,\mathbb{Z}(1)) \xrightarrow{\cup} 2\pi\sqrt{-1} \otimes H^{i+j}_{\mathcal{D}_{h.h.}}(X,1)\,.$$

An immediate consequence is that for f and g invertible, and L, L' line bundles, the torsor $\big(f,g\big]$ and the gerbe $\big(f,L\big]$ support natural hermitian structures of the type discussed above, in addition to the analytic connection (or connective) ones associated with the cup product in standard Deligne cohomology. The same conclusions are valid for the 2-gerbe $\big(L,L'\big]$. It turns out that supporting both structures is an easy consequence of the commutativity of the following diagram:

$$\begin{tikzcd}
H^i_{\mathcal{D}}(X,\mathbb{Z}(1)) \otimes H^j_{\mathcal{D}}(X,\mathbb{Z}(1)) \arrow[r, "\cup"] \arrow[d, "\cup"'] & 2\pi\sqrt{-1} \otimes H^{i+j}_{\mathcal{D}_{h.h.}}(X,1) \arrow[d, "forget"] \\
H^{i+j}_{\mathcal{D}}(X,\mathbb{Z}(2)) \arrow[r, "forget"] & H^{i+j}_{\mathcal{D}}(X,\mathbb{Z}(1))
\end{tikzcd}$$

Indeed, forgetting either structure, brings us back to the same underlying object.

The map (1.4) has a rather natural definition from the point of view of Mixed Hodge Structures, whose role in the matter was mentioned in relation with the product (1.1), see [13]. Namely, there is a "universal" MHS $\mathscr{M}^{(2)}$ corresponding to an iterated extension of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$ by $\mathbb{Z}(2)$, where in this case $\mathbb{Z}(n)$ denotes a Hodge-Tate structure. To $\mathscr{M}^{(2)}$ we can associate a tensor — the "big period" — $P(\mathscr{M}^{(2)}) \in \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$, cf. [17]. The period is in fact a multiple of the extension class of $\mathscr{M}^{(2)}$, and it belongs to the kernel $\mathscr{I} = \ker\big(m\colon \mathbb{C}\otimes_{\mathbb{Q}}\mathbb{C} \to \mathbb{C}\big)$ of the multiplication map. We find the map (1.4) corresponds to the image of $P(\mathscr{M}^{(2)})$ under the "imaginary part" projection $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{R}(1)$ given by $a \otimes b \mapsto \operatorname{Im}(a)\operatorname{Re}(b)$. On the other hand, the standard one (1.1) involves the projection onto the Kähler differentials $\mathscr{I} \to \mathscr{I}/\mathscr{I}^2$ given by $a \otimes b \mapsto a\,db$.

Another consequence of the previous diagram is that $\big(f,g\big]$, $\big(f,L\big]$, and $\big(L,L'\big]$ come equipped with two connection (or connective) structures. If the unitary connection in a line bundle L is also analytic, then L is flat. In the case of $\big(f,g\big]$ we find there is an obstruction to this type of compatibility. This can be cast in cohomological terms, which allows to extend these considerations to $\mathcal{O}^\times_X$-gerbes and 2-gerbes. We find that the obstruction vanish, so compatibility can always be achieved.

1.3 Outline of the paper

This work is organized as follows. In section 2 we make some preliminaries observations about Deligne complexes and cohomology and collect a few needed facts. We recall the definition of Hermitian-Holomorphic Deligne cohomology and state some of its properties in section 3. Alongside Brylinski's complex $C(l)^\bullet$, we use a complex quasi-isomorphic to it, denoted $D(l)^\bullet_{h.h.}$, which for a line bundle directly encodes the data defining the *canonical connection.*

In section 4 we recall the definition of the tame symbol $\big(f,g\big]$ for two invertible functions and some of its properties. We define the modified product (1.4) and show that through it, the torsor associated to $\big(f,g\big]$ also comes equipped with a hermitian structure. As mentioned before, the product (1.4) and its relation with the standard for Deligne complexes become more clear when analyzed in terms of Hodge Structures. In order to do this, we felt necessary to recall a few elementary facts and calculations concerning Hodge-Tate structures that are certainly well-known to experts. For this reason, and also because this development lies somewhat aside this work's main lines, we present this material in appendix A. This presentation relies in part on the Heisenberg group picture of the Deligne torsor, which we have recalled in section 4.2.

Section 5 is the main part of this work. There we redefine the notion of hermitian structure (modeled after that of connective structure) and prove that equivalence classes of these are classified by the groups $H^k_{\mathcal{D}_{h.h.}}(X,1)$. We then apply this classification to the Hermitian structures and the product (1.4) for the higher versions of the tame symbols considered by Brylinski-McLaughlin.

The interplay between the analytic connection (or connective) structures arising from standard Deligne cohomology and their hermitian counterparts defined here is analyzed in sections 4.4 and 5.7.

Finally, section 6 is devoted to some concluding remarks.

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2 Preliminaries

2.1 Notation and conventions

If z is a complex number, then $\pi_p(z) \overset{\text{def}}{=} \frac{1}{2}(z+(-1)^p\bar{z})$, and similarly for any other complex quantity, e.g. complex valued differential forms. For a subring A of $\mathbb{R}$ and an integer j, $A(j)=(2\pi\sqrt{-1})^j\,A$ is the Tate twist of A. We identify $\mathbb{C}/\mathbb{Z}(j)\cong\mathbb{C}^\times$ via the exponential map $z\mapsto\exp(z/(2\pi\sqrt{-1})^{j-1})$, and $\mathbb{C}/\mathbb{R}(j)\cong\mathbb{R}(j-1)$.

If X is a complex manifold, $\underline{A}^\bullet_X$ and $\underline{\Omega}^\bullet_X$ denote the de Rham complexes of sheaves of smooth $\mathbb{C}$-valued and holomorphic forms, respectively. We denote by $\underline{\mathcal{E}}^\bullet_X$ the de Rham complex of sheaves of real valued differential forms and by $\underline{\mathcal{E}}^\bullet_X(j)$ the twist $\underline{\mathcal{E}}^\bullet_X\otimes_{\mathbb{R}}\mathbb{R}(j)$. We set $\mathcal{O}_X\equiv\underline{\Omega}^0_X$ as usual. When needed, $\underline{A}^{p,q}_X$ will denote the sheaf of smooth (p,q)-forms. We use the standard decomposition $d=\partial+\bar\partial$ according to types. Furthermore, we introduce the differential operator $d^c=\partial-\bar\partial$ (contrary to the convention, we omit the factor $1/(4\pi\sqrt{-1})$). We have $2\partial\bar\partial=d^cd$. The operator d^c is an imaginary one and accordingly we have the rules

$$d\pi_p(\omega)=\pi_p(d\omega)\,,\quad d^c\pi_p(\omega)=\pi_{p+1}(d^c\omega)$$

for any complex form ω.

An open cover of X will be denoted by $\mathfrak{U}_X$. If $\{U_i\}_{i\in I}$ is the corresponding collection of open sets, we write $U_{ij}=U_i\cap U_j$, $U_{ijk}=U_i\cap U_j\cap U_k$, and so on. More generally we can also have $\mathfrak{U}_X=\{U_i\to X\}_{i\in I}$, where the maps are regular coverings in an appropriate category. In this case intersections are replaced by $(n+1)$-fold fibered products $U_{i_0i_1\cdots i_n}=U_{i_0}\times_X\cdots\times_X U_{i_n}$.

If $\underline{F}^\bullet$ is a complex of abelian sheaves on X, its Čech resolution with respect to a covering $\mathfrak{U}_X\to X$ is the double complex

$$\mathsf{C}^{p,q}(\underline{F})\overset{\text{def}}{=}\check{C}^q(\mathfrak{U}_X,\underline{F}^p)\,,$$

where the q-cochains with values in $\underline{F}^p$ are given by $\prod\underline{F}^p(U_{i_0\cdots i_n})$. The Čech coboundary operator is denoted δ. The convention we use is to put the index along the Čech resolution in the *second* place, so if we denote by d the differential in the complex $\underline{F}^\bullet$, the total differential is given by $D=d+(-1)^p\delta$ on the component $\check{C}^q(\mathfrak{U}_X,\underline{F}^p)$ of the total simple complex. Furthermore, recall that the Koszul sign rule causes a sign being picked whenever two degree indices are formally exchanged. For Čech resolutions of complexes of sheaves it leads to the following conventions. If $\underline{G}^\bullet$ is a second complex of sheaves on X, then one defines the cup product

$$\cup:\mathsf{C}^{p,q}(\underline{F})\otimes\mathsf{C}^{r,s}(\underline{G})\longrightarrow\check{C}^{q+s}(\mathfrak{U}_X,\underline{F}^p\otimes\underline{G}^r)\subset\mathsf{C}^{p+r,q+s}(\underline{F}\otimes\underline{G})$$

of two elements $\{f_{i_0,\dots,i_q}\} \in \mathsf{C}^{p,q}(\underline{F})$ and $\{g_{j_0,\dots,j_s}\} \in \mathsf{C}^{r,s}(\underline{G})$ by

$$(-1)^{qr}\, f_{i_0,\dots,i_q} \otimes g_{i_q,i_{q+1},\dots,i_{q+s}}\,.$$

For a given complex of abelian objects, say $\mathsf{C}^\bullet$, the symbol σ^i denotes sharp truncation at the index i: $\sigma^i \mathsf{C}^p = 0$ for $p < i$.

2.2 Deligne cohomology

There are several models for the complexes to use to compute Deligne cohomology [15, 2]. For $A \subset \mathbb{R}$ and an integer j the latter is the hypercohomology:

$$H^\bullet_{\mathcal{D}}(X, A(j)) = \mathbf{H}^\bullet(X, A(j)^\bullet_{\mathcal{D}})\,.$$

Here $A(p)^\bullet_{\mathcal{D}}$ is the Deligne complex

$$A(j)^\bullet_{\mathcal{D}} = A(j)_X \xrightarrow{\imath} \mathcal{O}_X \xrightarrow{d} \underline{\Omega}^1_X \xrightarrow{d} \cdots \xrightarrow{d} \underline{\Omega}^{j-1}_X \tag{2.1}$$

$$\xrightarrow{\simeq} \operatorname{Cone}\big(A(j)_X \oplus F^j \underline{\Omega}^\bullet_X \xrightarrow{\imath - \jmath} \underline{\Omega}^\bullet_X\big)[-1]\,, \tag{2.2}$$

where $F^j \underline{\Omega}^\bullet_X$ in eqn. (2.2) is the Hodge ("stupid") filtration on the de Rham complex. The symbol $\xrightarrow{\simeq}$ denotes a quasi-isomorphism. In view of Beĭlinson formula for the cup product on cones to be recalled below [3], Deligne complexes acquire a family of cup-products (depending on a real parameter α)

$$A(j)^\bullet_{\mathcal{D}} \otimes A(k)^\bullet_{\mathcal{D}} \xrightarrow{\cup_\alpha} A(j+k)^\bullet_{\mathcal{D}}\,.$$

Cup products related to different values of the parameter α are related by homotopy-commutative diagrams, hence they induce a well defined graded commutative cup-product in cohomology

$$H^p_{\mathcal{D}}(X, A(j)) \otimes H^q_{\mathcal{D}}(X, A(k)) \xrightarrow{\cup} H^{p+q}_{\mathcal{D}}(X, A(j+k))\,. \tag{2.3}$$

In order to explicitly compute cup products, the model given by eq. (2.1) leads to simpler formulas (when it can be used). If $f \in A(j)^\bullet_{\mathcal{D}}$ and $g \in A(k)^\bullet_{\mathcal{D}}$, then from ref. [15] we quote:

$$f \cup g = \begin{cases} f\cdot g & \deg f = 0\,, \\ f \wedge dg & \deg f > 0 \text{ and } \deg g = k\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

The following examples are well known and will frequently recur in the following.

Example 2.2.1. For $A = \mathbb{Z}$ it is immediately verified that $\mathbb{Z}(1)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \mathcal{O}^\times_X[-1]$ via the standard exponential sequence, so that $H^k_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^{k-1}(X, \mathcal{O}^\times_X)$. In particular $H^1_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^0(X, \mathcal{O}^\times_X)$, the global invertibles on X, and $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong \operatorname{Pic}(X)$, the Picard group of line bundles over X.

Example 2.2.2. $\mathbb{Z}(2)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \big(\mathcal{O}^\times_X \xrightarrow{d\log} \underline{\Omega}^1_X\big)[-1]$. A fundamental observation by Deligne (see ref. [2]) is that $H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$ is identified with the group of isomorphism classes of holomorphic line bundles with (holomorphic) connection. This is easily understood from a Čech cohomology point of view. Using the cover $\mathfrak{U}_X = \{U_i\}_{i\in I}$, a class in

$$H^2_{\mathcal{D}}(X, \mathbb{Z}(2)) \cong \mathbf{H}^1(X, \mathcal{O}^\times_X \xrightarrow{d\log} \underline{\Omega}^1_X)$$

is represented by a pair (ω_i, g_{ij}) with $\omega_i \in \underline{\Omega}^1_X(U_i)$ and $g_{ij} \in \mathcal{O}^\times_X(U_{ij})$ satisfying the relations

$$\omega_j - \omega_i = d\log g_{ij}\,, \quad g_{ij} g_{jk} = g_{ik}\,.$$

The Čech representative for the actual class in $H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$ is obtained (up to a multiplication by $2\pi\sqrt{-1}$) by extracting local logarithms $\log g_{ij}$, see ref. [15] for full details.

For *real* Deligne cohomology, i.e. when $A = \mathbb{R}$, other models quasi-isomorphic to those in eqs. (2.1) and (2.2) are available. Since the maps

$$\big(\mathbb{R}(j) \to \underline{\Omega}^\bullet_X\big) \xrightarrow{\simeq} \big(\mathbb{R}(j) \to \mathbb{C}\big) \xrightarrow{\simeq} \mathbb{R}(j-1) \xrightarrow{\simeq} \underline{\mathcal{E}}^\bullet_X(j-1)$$

are all quasi-isomorphisms in the derived category, cf. [15], we have

$$\mathbb{R}(j)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \mathrm{Cone}\big(F^j\underline{\Omega}^\bullet_X \to \underline{\mathcal{E}}^\bullet_X(j-1)\big)[-1]\,. \tag{2.5}$$

Moreover, we can use smooth forms thanks to the fact that the inclusion $\underline{\Omega}^\bullet_X \hookrightarrow \underline{A}^\bullet_X$ is a filtered quasi-isomorphism with respect to the filtrations $F^j\underline{\Omega}^\bullet_X \hookrightarrow F^j\underline{A}^\bullet_X$. Here $F^j\underline{A}^\bullet_X$ is the subcomplex of $\underline{A}^\bullet_X$ comprising forms of type (p,q) where p is at least j, so that $F^j\underline{A}^n_X = \oplus_{p\geq j}\underline{A}^{p,n-p}_X$.

Let (ω_1,η_1) be an element of degree n in $\mathbb{R}(j)^\bullet_{\mathcal{D}}$—this means that $\omega_1 \in F^j\underline{\Omega}^n_X$ and $\eta_1 \in \underline{\mathcal{E}}^{n-1}_X(j-1)$—and (ω_2,η_2) any element in $\mathbb{R}(k)^\bullet_{\mathcal{D}}$. A product is given by the formula (cf. ref. [15]):

$$(\omega_1,\eta_1)\,\tilde{\cup}\,(\omega_2,\eta_2) = \big(\omega_1\wedge\omega_2, (-1)^n\,\pi_p\omega_1\wedge\eta_2 + \eta_1\wedge\pi_q\omega_2\big)\,. \tag{2.6}$$

Example 2.2.3. $H^1_{\mathcal{D}}(X,\mathbb{R}(1))$ is the group of real valued functions η on X such that there exists a holomorphic one-form ω such that $\pi_0\omega = d\eta$. In other words, it is the group of those real smooth functions η such that $\partial\eta$ is holomorphic. In particular, if f is holomorphic and invertible, then the class in $H^1_{\mathcal{D}}(X,\mathbb{R}(1))$ determined by f is represented by $(d\log f, \log|f|)$.

2.3 Cones

We recall here a variant of Beĭlinson's formula for the cup product on certain diagrams of complexes. (For full details see refs. [1, 3, 15].)

For $i = 1,2,3$ consider the diagrams of complexes

$$\mathcal{D}_i \overset{\mathrm{def}}{=} X^\bullet_i \xrightarrow{f_i} Z^\bullet_i \xleftarrow{g_i} Y^\bullet_i \tag{2.7}$$

and set

$$C(\mathcal{D}_i) = \mathrm{Cone}(X^\bullet_i \oplus Y^\bullet_i \xrightarrow{f_i - g_i} Z^\bullet_i)[-1]\,,\quad i = 1,2,3\,.$$

Suppose there are product maps $X^\bullet_1 \otimes X^\bullet_2 \xrightarrow{\cup} X^\bullet_3$, and similarly for $Y^\bullet_i$, and $Z^\bullet_i$. We assume the products to be compatible with the f_i, g_i only up to homotopy, namely there exist maps

$$h\colon \big(X_1\otimes X_2\big)^\bullet \longrightarrow Z_3^{\bullet-1}\,,\quad k\colon \big(Y_1\otimes Y_2\big)^\bullet \longrightarrow Z_3^{\bullet-1}$$

such that

$$f_3\circ\cup - \cup\circ(f_1\otimes f_2) = d\,h + h\,d\,,\quad g_3\circ\cup - \cup\circ(g_1\otimes g_2) = d\,k + k\,d\,,$$

with obvious meaning of the symbols. The following lemma establishes a variant of Beĭlinson's product formula [3].

Lemma 2.3.1. *For $(x_i,y_i,z_i) \in X^\bullet_i \oplus Y^\bullet_i \oplus Z_i^{\bullet-1}$, $i = 1,2$, and a real parameter α, the following formula:*

$$\begin{aligned}(x_1,y_1,z_1)\cup_\alpha(x_2,y_2,z_2) = \Big(&x_1\cup x_2, y_1\cup y_2,\\ &(-1)^{\deg(x_1)}\big((1-\alpha)f_1(x_1)+\alpha g_1(y_1)\big)\cup z_2\\ &+ z_1\cup\big(\alpha f_2(x_2)+(1-\alpha)g_2(y_2)\big)\\ &- h(x_1\otimes x_2)+k(y_1\otimes y_2)\Big)\,.\end{aligned} \tag{2.8}$$

defines a family of products

$$C(\mathcal{D}_1)\otimes C(\mathcal{D}_2) \xrightarrow{\cup_\alpha} C(\mathcal{D}_3)\,.$$

These products are homotopic to one another, and graded commutative up to homotopy. The homotopy formula is the same as that found in ref. [3].

Proof. Direct verification. □

If the maps f_i, g_i above are strictly compatible with the products, namely the homotopies h and k are zero, (2.8) reduces to the formulas found in [3, 15]. Homotopy commutativity at the level of complexes ensures the corresponding cohomologies will have genuine graded commutative products.

3 Hermitian holomorphic Deligne cohomology

3.1 Metrized line bundles

Let X be a complex manifold. Consider a holomorphic line bundle L on X with hermitian fiber metric ρ or, equivalently, an invertible sheaf L equipped with a map $\rho\colon L \to \underline{\mathcal{E}}^0_{X,+}$ to (the sheaf of) positive real smooth functions, see ref. [21] for the relevant formalism. Let $\widehat{\mathrm{Pic}(X)}$ denote the group of isomorphism classes of line bundles with hermitian metric. A basic observation by Deligne (cf. [14]) is that $\widehat{\mathrm{Pic}\,X}$ can be identified with the second hypercohomology group:

$$\mathbf{H}^2\big(X, \mathbb{Z}(1)_X \xrightarrow{\imath} \mathcal{O}_X \xrightarrow{-\pi_0} \underline{\mathcal{E}}^0_X\big)\,. \tag{3.1}$$

This is easy to see in Čech cohomology. Suppose s_i is a trivialization of $L|_{U_i}$, with transition functions $g_{ij} \in \mathcal{O}^\times_X(U_{ij})$ determined by $s_j = s_i g_{ij}$. Let ρ_i be the value of the quadratic form associated to ρ on s_i, namely $\rho_i = \rho(s_i)$. Then we have $\rho_j = \rho_i\, |g_{ij}|^2$. Taking logarithms, we see that

$$\big(2\pi\sqrt{-1}c_{ijk}, \log g_{ij}, \tfrac{1}{2}\log\rho_i\big)\,,$$

where $2\pi\sqrt{-1}c_{ijk} = \log g_{jk} - \log g_{ik} + \log g_{ij} \in \mathbb{Z}(1)$, is a cocycle representing the class of the pair (L,ρ).

3.1.1 Canonical connection

Recall for later use that the *canonical connection,* [19] on a metrized line bundle (L,ρ) is the unique connection compatible with both the holomorphic and hermitian structures. In Čech cohomology with respect to the cover $\mathfrak{U}_X$ as above, the canonical connection on (L,ρ) corresponds to a collection of $(1,0)$ forms $\xi_i \in \underline{A}^{1,0}_X(U_i)$ satisfying the relations

$$\xi_j - \xi_i = d\log g_{ij} \tag{3.2}$$

$$\pi_0(\xi_i) = \tfrac{1}{2} d\log\rho_i\,. \tag{3.3}$$

The latter just means $\xi_i = \partial\log\rho_i$, in more familiar terms. The global 2-form

$$c_1(\rho) = \eta_i \equiv \bar\partial\partial\log\rho_i \tag{3.4}$$

represents the first Chern class of L in $H^2(X,\mathbb{R}(1))$. The class of $c_1(\rho)$ is in fact a pure Hodge class in $H^{1,1}(X)$—the image of the first Chern class of L under the map $H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) \to H^2_{\mathcal{D}}(X,\mathbb{R}(1))$ induced by $\mathbb{Z}(1)\to\mathbb{R}(1)$. It only depends on the class of (L,ρ) in $\widehat{\mathrm{Pic}(X)}$.

3.2 Hermitian holomorphic complexes

In ref. [9] Brylinski introduced the complexes

$$C(l)^\bullet = \mathrm{Cone}\big(\mathbb{Z}(l)_X \oplus (F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l)) \longrightarrow \underline{\mathcal{E}}^\bullet_X(l)\big)[-1]\,. \tag{3.5}$$

Definition 3.2.1. The hypercohomology groups

$$H^p_{\mathcal{D}_{h.h.}}(X,l) \overset{\mathrm{def}}{=} \mathbf{H}^p(X, C(l)) \tag{3.6}$$

are the *Hermitian holomorphic Deligne* cohomology groups.

By the remark after eq. (2.5), the complex

$$\widetilde{\mathbb{R}(l)}^\bullet_{\mathcal{D}} = \mathrm{Cone}\,\big(F^l\underline{A}^\bullet_X \to \underline{\mathcal{E}}^\bullet_X(l-1)\big)[-1]\,.$$

also computes the real Deligne cohomology. Then consider the complex

$$D(l)^\bullet_{h.h.} = \mathrm{Cone}\big(\mathbb{Z}(l)^\bullet_{\mathcal{D}} \oplus (F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l)) \longrightarrow \widetilde{\mathbb{R}(l)}^\bullet_{\mathcal{D}}\big)[-1]\,. \tag{3.7}$$

In ref. [1] we prove

Lemma 3.2.2. *The complexes $C(l)^\bullet$ and $D(l)^\bullet_{h.h.}$ are quasi-isomorphic, hence we also have*

$$H^p_{\mathcal{D}_{h.h.}}(X,l) = \mathbf{H}^p(X, D(l)^\bullet_{h.h.})\,.$$

Remark 3.2.3. The complex $F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l)$ appearing in both (3.5) and (3.6) can be rewritten in terms of the complex $G(l)^\bullet$ of ref. [14]. Set

$$G(l)^\bullet = 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underline{A}^{(l,l)}_X \xrightarrow{\ d\ } \underline{A}^{(l+1,l)}_X \oplus \underline{A}^{(l,l+1)}_X \xrightarrow{\ d\ } \cdots\,.$$

Then we have $F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l) = G(l)^\bullet \cap \underline{\mathcal{E}}^\bullet_X(l)\,.$

For certain ranges of values of the cohomology index the groups $H^p_{\mathcal{D}_{h.h.}}(X,l)$ are fairly ordinary. Indeed we have the following easy

Lemma 3.2.4. *For $p \leq 2l-1$ we have*

$$H^p_{\mathcal{D}_{h.h.}}(X,l) \cong H^{p-1}(X, \mathbb{R}(l)/\mathbb{Z}(l))\,.$$

Proof. Using either $C(l)^\bullet$ or $D(l)^\bullet_{h.h.}$, we see that they are quasi-isomorphic to

$$\mathrm{Cone}\big(F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l) \longrightarrow \mathbb{R}(l)/\mathbb{Z}(l)\big)[-1]\,,$$

which leads to the triangle

$$\mathbb{R}(l)/\mathbb{Z}(l)[-1] \longrightarrow D(l)^\bullet_{h.h.} \longrightarrow F^l\underline{A}^\bullet_X \cap \sigma^{2l}\underline{\mathcal{E}}^\bullet_X(l) \xrightarrow{+1}\,.$$

The statement follows. □

In general these groups are interesting when $p \geq 2l$. The most important example is:

Lemma 3.2.5.

$$\widehat{\mathrm{Pic}(X)} \cong H^2_{\mathcal{D}_{h.h.}}(X,1)\,.$$

Proof. We have quasi-isomorphisms

$$\mathbb{Z}(1)_X \xrightarrow{\ \imath\ } \mathcal{O}_X \xrightarrow{-\pi_0} \underline{\mathcal{E}}^0_X \xrightarrow{\simeq} D(1)^\bullet_{h.h.} \xrightarrow{\simeq} C(1)^\bullet\,.$$

Indeed, note that $D(1)^\bullet_{h.h.}$ can be rewritten as

$$\mathrm{Cone}\big(\mathbb{Z}(1)^\bullet_{\mathcal{D}} \to \widetilde{\mathbb{R}(1)}^\bullet_{\mathcal{D}}/(F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1))\big)[-1]$$

and

$$\widetilde{\mathbb{R}(1)}^\bullet_{\mathcal{D}}/(F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1)) \xrightarrow{\simeq} \mathrm{Cone}\big(F^1\underline{A}^\bullet_X/F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1) \xrightarrow{-\pi_0} \underline{\mathcal{E}}^\bullet_X\big)[-1]\,.$$

By direct verification, the latter complex is quasi-isomorphic to $\underline{\mathcal{E}}^0_X[-1]$. Thus

$$D(1)^\bullet_{h.h.} \xrightarrow{\simeq} \mathrm{Cone}\big(\mathbb{Z}(1)^\bullet_{\mathcal{D}} \to \underline{\mathcal{E}}^0_X[-1]\big)[-1] \xrightarrow{\simeq} \mathbb{Z}(1)_X \to \mathcal{O}_X \to \underline{\mathcal{E}}^0_X\,.$$

□

Since hermitian holomorphic Deligne complexes can be expressed as cones of diagrams of the form (2.7), they admit cup products, and hence there is a cup product for hermitian holomorphic Deligne cohomology [9]:

$$H^p_{\mathcal{D}_{h.h.}}(X,l) \otimes H^q_{\mathcal{D}_{h.h.}}(X,k) \xrightarrow{\ \cup\ } H^{p+q}_{\mathcal{D}_{h.h.}}(X,l+k)\,.$$

3.3 Explicit cocycles

Use of the seemingly more complicated complex (3.7) in place of the one in (3.5) is justified by the fact that the data comprising the canonical connection can be characterized cohomologically, as follows:

Lemma 3.3.1. *Let (L, ρ) be a metrized line bundle on X. Assume (L, ρ) to be trivialized with respect to the open cover $\mathfrak{U}_X$ of X as before. The data:*

$$\xi_i \in \underline{A}^{(1,0)}_X(U_i)\,, \quad \tfrac{1}{2}\log\rho_i \in \underline{\mathcal{E}}^0_X(U_i)\,, \quad \eta_i \in \underline{A}^{(1,1)}_X(U_i)\,,$$
$$2\pi\sqrt{-1}c_{ijk} \in \mathbb{Z}(1)_X(U_{ijk})\,, \quad \log g_{ij} \in \mathcal{O}_X(U_{ij})$$

represent a degree 2 cocycle with values in $\operatorname{Tot}\check{C}^\bullet(\mathfrak{U}_X, D(1)^\bullet_{h.h.})$ *if and only if the relations* (3.2), (3.3), (3.4), *plus those in sect. 3.1, defining the canonical connection are satisfied.*

Proof. One need only unravel the cone defining $D(1)^\bullet_{h.h.}$ as follows:

$$\begin{array}{ccccccc}
\mathbb{Z}(1)_X & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle 0\oplus\pi_0} & & \Big\downarrow & & \\
 & & F^1\underline{A}^1_X \oplus \underline{\mathcal{E}}^0_X & \longrightarrow & F^1\underline{A}^2_X \oplus \underline{\mathcal{E}}^1_X & \longrightarrow & \cdots \\
 & & & & \Big\uparrow{\scriptstyle \jmath\oplus 0} & & \\
 & & & & F^1\underline{A}^2_X \cap \underline{\mathcal{E}}^2_X(1) & \longrightarrow & \cdots
\end{array} \tag{3.8}$$

and carefully chase the diagram. ☐

On the other hand, the hermitian holomorphic Deligne complex in the form (3.5) corresponds to "reducing the structure group" from $\mathbb{C}^\times$ to $\mathbb{T}$. This can be made explicit for $l = 1$ and a line bundle $L \to X$ by choosing sections t_i of the smooth bundle corresponding to L such that $\rho(t_i) = 1$. Clearly the resulting smooth transition functions will be sections of $\underline{\mathbb{T}}_X$ over U_{ij}. See refs. [9] and [12] for more details.

4 Tame symbol and hermitian structure

Let X be a complex analytic manifold and $U \subset X$ open. Let f and g two invertible holomorphic functions on U. The tame symbol [13] $\big(f, g\big]$ associated to f and g is a $\mathcal{O}^\times_X|_U$-torsor equipped with an analytic connection.

4.1 Cup product and Deligne torsor

(See [13, 15].) We consider f and g as elements of $H^1_{\mathcal{D}}(U, \mathbb{Z}(1))$. Then $\big(f, g\big] = f \cup g \in H^2_{\mathcal{D}}(U, \mathbb{Z}(2))$. Consider the cover $\mathfrak{U}_X$ of X so that U is covered by $\{U \cap U_i\}_{i\in I}$ and choose representatives $(2\pi\sqrt{-1}\,m_{ij}, \log_i f)$ and $(2\pi\sqrt{-1}\,n_{ij}, \log_i g)$ for f and g, respectively. Then, using (2.4), the cup product is represented by the cocycle:

$$\Big((2\pi\sqrt{-1})^2 m_{ij}n_{jk}\,,\ -2\pi\sqrt{-1}\,m_{ij}\,\log_j g\,,\ \log_i f\,\frac{dg}{g}\Big)\,. \tag{4.1}$$

Under the quasi-isomorphism with the complex $\big(\mathcal{O}^\times_X \to \underline{\Omega}^1_X\big)$ (which essentially amounts to a division by $2\pi\sqrt{-1}$) the cocycle (4.1) becomes

$$\big(g^{-m_{ij}}\,, -\frac{1}{2\pi\sqrt{-1}}\log_i f\,\frac{dg}{g}\big)\,. \tag{4.2}$$

In ref. [13] the trivializing section on $U \cap U_i$ corresponding to (4.2) is denoted $\{\log_i f, g\}$. Two trivializations over $U \cap U_i$ and $U \cap U_j$ are related by $\{\log_j f, g\} = \{\log_i f, g\}\, g^{-m_{ij}}$. Furthermore, the analytic connection is defined by the rule:

$$\nabla\{\log_i f, g\} = -\{\log_i f, g\} \otimes \frac{1}{2\pi\sqrt{-1}}\log_i f\,\frac{dg}{g}\,. \tag{4.3}$$

A general section s of $\big(f, g\big]$ can be written as $s = h_i\,\{\log_i f, g\}$, for some $h_i \in \mathcal{O}_U(U_i)$, and therefore

$$\nabla s = \{\log_i f, g\} \otimes \big(dh_i - \frac{1}{2\pi\sqrt{-1}}\log_i f\,\frac{dg}{g}\big)\,. \tag{4.4}$$

4.2 Heisenberg group

An equivalent approach to the Deligne symbol is via the complex three-dimensional Heisenberg group, see refs. [5, 20, 22]. Let $H_{\mathbb{C}}$ denote the group of complex unipotent 3×3 lower triangular matrices. Let

$$H_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & & \\ m_1 & 1 & \\ m_2 & n_1 & 1 \end{pmatrix} \Bigm| m_1, n_1 \in \mathbb{Z}(1)\,,\ m_2 \in \mathbb{Z}(2) \right\} \subset H_{\mathbb{C}}\,.$$

The quotient $H_{\mathbb{C}}/H_{\mathbb{Z}}$ is a $\mathbb{C}/\mathbb{Z}(2)$-bundle over $\mathbb{C}/\mathbb{Z}(1)\times\mathbb{C}/\mathbb{Z}(1)$ via the projection map

$$p\colon \begin{bmatrix} 1 & & \\ x & 1 & \\ z & y & 1 \end{bmatrix} \mapsto ([x],[y])\,,$$

where $x, y, z \in \mathbb{C}$, and the brackets denote the appropriate equivalence classes. (The $\mathbb{C}/\mathbb{Z}(2)$-action is by multiplication with a matrix of the form $\left(\begin{smallmatrix} 1 & & \\ 0 & 1 & \\ z & 0 & 1 \end{smallmatrix}\right)$.)

The twisting of $H_{\mathbb{C}}/H_{\mathbb{Z}}$ is analogous to that of the Deligne torsor in sect. 4.1: the right action of $H_{\mathbb{Z}}$ on $H_{\mathbb{C}}$ amounts to:

$$x \mapsto x + m_1\,, \quad y \mapsto y + n_1\,, \quad z \mapsto z + m_1\cdot y + m_2\,. \tag{4.5}$$

Moreover, the complex form

$$\omega = \frac{1}{2\pi\sqrt{-1}}(dz - x\,dy) \tag{4.6}$$

is invariant under the action of $H_{\mathbb{Z}}$ and defines a $\mathbb{C}/\mathbb{Z}(2)$-connection form on the total space $H_{\mathbb{C}}/H_{\mathbb{Z}}$.

The invertible functions f and g on U define a map $(f,g)\colon U \to \mathbb{C}^\times \times \mathbb{C}^\times$. Then the tame symbol $\big(f,g\big]$ is obtained as the pull-back:

$$\big(f,g\big] = (f,g)^*\big(H_{\mathbb{C}}/H_{\mathbb{Z}}\big)\,,$$

and the section $\{\log_i f, g\}$ corresponds to the class of the matrix

$$\begin{pmatrix} 1 & & \\ \log_i f & 1 & \\ 0 & \log_i g & 1 \end{pmatrix}.$$

Furthermore, the pull-back of the connection form ω on $H_{\mathbb{C}}/H_{\mathbb{Z}}$ along the section $\{\log_i f, g\}$ is the same form as the one in (4.1). More generally, a section s as given at the end of sect. 4.1 corresponds to the class of the matrix

$$\begin{pmatrix} 1 & & \\ \log_i f & 1 & \\ h_i & \log_i g & 1 \end{pmatrix},$$

Pulling back (4.6) along the section gives (4.4).

4.3 Hermitian product structure

Consider the "imaginary part" map

$$\begin{aligned} \mathbb{C}\otimes\mathbb{C} &\longrightarrow \mathbb{R}(1) \\ a\otimes b &\longmapsto -\pi_1(a)\,\pi_0(b) \equiv -\sqrt{-1}\,\mathrm{Im}(a)\,\mathrm{Re}(b)\,, \end{aligned} \tag{4.7}$$

Similarly, we have:

$$\mathcal{O}_X\otimes\mathcal{O}_X \longrightarrow \underline{\mathcal{E}}^0_X(1) \quad f\otimes g \longmapsto -\pi_1(f)\,\pi_0(g)\,. \tag{4.8}$$

Definition 4.3.1. Define the map

$$\begin{aligned} \big(\mathbb{Z}(1)_X \to \mathcal{O}_X\big)\otimes\big(\mathbb{Z}(1)_X \to \mathcal{O}_X\big) &\longrightarrow \big(\mathbb{Z}(2)_X \to \mathcal{O}_X \xrightarrow{-\pi_1} \underline{\mathcal{E}}^0_X(1)\big) \\ &\xrightarrow{\simeq} 2\pi\sqrt{-1}\otimes\big(\mathbb{Z}(1)_X \to \mathcal{O}_X \xrightarrow{-\pi_0} \underline{\mathcal{E}}^0_X\big) \end{aligned} \tag{4.9}$$

by using (4.8) in place of the map $\mathcal{O}_X\otimes\mathcal{O}_X \to \underline{\Omega}^1_X$, $f\otimes g\mapsto f dg$, in (2.4).

Proposition 4.3.2. *The product map (4.9) is well defined, namely it is a map of complexes. Furthermore, it is homotopy graded commutative.*

Proof. The fact that (4.9) is a map of complexes is a direct verification. After ref. [15], consider the map

$$h(f\otimes g) = f\,g\,,\quad f,g\in \mathcal{O}_X\,,$$

and zero otherwise. It provides the required homotopy. $\square$

The target complex of the product map in eq. (4.9) is the complex encoding hermitian structures appearing in sect. 3.1. In other words, up to quasi-isomorphism, we have a product:

$$\mathbb{Z}(1)^\bullet_{\mathcal{D}}\otimes\mathbb{Z}(1)^\bullet_{\mathcal{D}}\longrightarrow 2\pi\sqrt{-1}\otimes D(1)^\bullet_{h.h.}\,.$$

Remark 4.3.3. The map (4.8) provides an explicit homotopy map for the homotopy commutative diagram

$$\begin{array}{ccc}\mathbb{Z}(1)^\bullet_{\mathcal{D}}\otimes\mathbb{Z}(1)^\bullet_{\mathcal{D}} & \longrightarrow & \mathbb{Z}(2)^\bullet_{\mathcal{D}}\\ \big\downarrow & & \big\downarrow\\ \mathbb{R}(1)^\bullet_{\mathcal{D}}\otimes\mathbb{R}(1)^\bullet_{\mathcal{D}} & \longrightarrow & \mathbb{R}(2)^\bullet_{\mathcal{D}}\end{array}$$

where the model (2.5) for $\mathbb{R}(k)^\bullet_{\mathcal{D}}$ is used (see [15]).

Now, in view of Prop. 4.3.2, we have a graded commutative product at the level of cohomology groups. In particular, let f,g be two invertible holomorphic functions on $U\subset X$.

Proposition 4.3.4. *The Deligne torsor underlying $\big(f,g\big]$ admits a hermitian fiber metric.*

Proof. View f and g as elements of $H^1_{\mathcal{D}}(U,\mathbb{Z}(1))$. Taking the product according to (4.9) yields an element in

$$H^2_{\mathcal{D}_{h.h.}}(U,1)\cong\widehat{\operatorname{Pic}(U)}$$

that is, a holomorphic line bundle with hermitian fiber metric (up to isomorphism).

Taking the image of the tame symbol $\big(f,g\big]$ under the map $H^\bullet_{\mathcal{D}}(U,\mathbb{Z}(2))\to H^\bullet_{\mathcal{D}}(U,\mathbb{Z}(1))=\operatorname{Pic}(U)$ induced by $\mathbb{Z}(2)^\bullet_{\mathcal{D}}\to\mathbb{Z}(1)^\bullet_{\mathcal{D}}$ forgets the analytic connection and retains just the line bundle. Similarly, the map $H^2_{\mathcal{D}_{h.h.}}(U,1)\to H^\bullet_{\mathcal{D}}(U,\mathbb{Z}(1))=\operatorname{Pic}(U)$ induced by $D(1)^\bullet_{h.h.}\to\mathbb{Z}(1)^\bullet_{\mathcal{D}}$ forgets the hermitian structure. Clearly both map to the same underlying line bundle. $\square$

Using a Čech cover we can represent f and g as in sect. 4.1. Then the cocycle corresponding to their product in $H^2_{\mathcal{D}_{h.h.}}(U,1)$ is:

$$\Big(2\pi\sqrt{-1}\,m_{ij}\,n_{jk}\,,\ -m_{ij}\log_j g\,,-\frac{1}{2\pi\sqrt{-1}}\,\pi_1(\log_i f)\log|g|\Big)\,. \tag{4.10}$$

This allows us to identify the representative of the hermitian metric, or rather its logarithm, as

$$\frac{1}{2}\log\rho_i = -\frac{1}{2\pi\sqrt{-1}}\,\pi_1(\log_i f)\log|g|\;. \tag{4.11}$$

It follows that if s is the local section at the end of sect. 4.1 then

$$\log\rho(s)=\frac{1}{2\pi\sqrt{-1}}\big(\pi_1(h_i)-\pi_1(\log_i f)\,\log|g|\big)\,. \tag{4.12}$$

4.3.1 Remarks on the Heisenberg bundle

The hermitian metric can be constructed from the more global point of view afforded by the use of the Heisenberg group recalled in sect. 4.2. The hermitian metric on the bundle $H_{\mathbb{C}}/H_{\mathbb{Z}}\to\mathbb{C}^\times\times\mathbb{C}^\times$ is given by the map $\rho: H_{\mathbb{C}}/H_{\mathbb{Z}}\to\mathbb{R}_+$ defined by:

$$\rho\colon\begin{bmatrix}1&&\\x&1&\\z&y&1\end{bmatrix}\longmapsto\exp\frac{1}{2\pi\sqrt{-1}}\big(\pi_1(z)-\pi_1(x)\,\pi_0(y)\big) \tag{4.13}$$

Indeed, using the explicit action (4.5), one checks (4.13) is invariant and provides the required quadratic form. In particular, the quantity

$$-\frac{1}{2\pi\sqrt{-1}}\,\pi_1(x)\,\pi_0(y)$$

is immediately shown to behave as the logarithm of the local representative of a hermitian metric. Thus the hermitian holomorphic line bundle represented by the cocycle (4.10) is the pull-back of $(H_{\mathbb{C}}/H_{\mathbb{Z}},\rho)$ via the map $(f,g)\colon U\to\mathbb{C}^\times\times\mathbb{C}^\times$.

4.3.2 Relations with Mixed Hodge Structures

Both structures, namely the standard cup product $\mathbb{Z}(1)^\bullet_{\mathcal{D}}\otimes\mathbb{Z}(1)^\bullet_{\mathcal{D}}\to\mathbb{Z}(2)^\bullet_{\mathcal{D}}$ given by (2.4), and the modified one $\mathbb{Z}(1)^\bullet_{\mathcal{D}}\otimes\mathbb{Z}(1)^\bullet_{\mathcal{D}}\to 2\pi\sqrt{-1}\otimes D(1)^\bullet_{h.h.}$ of Definition 4.3.1, can be obtained by taking projections of a common object in two different ways.

Let s be a local section of the pull-back

$$\big(f,g\big]=(f,g)^*\big(H_{\mathbb{C}}/H_{\mathbb{Z}}\big)$$

as at the end of sect. 4.1. (The local expression in terms of matrices is given at the end of sect. 4.2.) Equivalently, s can be considered as a (local) lift of the map $(f,g):X\to\mathbb{C}^\times\times\mathbb{C}^\times$ to $H_{\mathbb{C}}/H_{\mathbb{Z}}$.

Let $\mathscr{M}_X^{(2)}$ be the resulting variation of Mixed Hodge Structures on X obtained by pulling back the universal MHS $\mathscr{M}^{(2)}$ on $H_{\mathbb{C}}/H_{\mathbb{Z}}$ via s.

Lemma 4.3.5 (See [17]). *The period $P(\mathscr{M}_X^{(2)})\in\mathcal{O}_X\otimes_{\mathbb{Q}}\mathcal{O}_X$ of $\mathscr{M}_X^{(2)}$ is given by:*

$$\begin{aligned}P(\mathscr{M}_X^{(2)})=&\frac{h}{(2\pi\sqrt{-1})^2}\otimes 1-1\otimes\frac{h}{(2\pi\sqrt{-1})^2}\\&+1\otimes\frac{\log f\,\log g}{(2\pi\sqrt{-1})^2}-\frac{\log g}{2\pi\sqrt{-1}}\otimes\frac{\log f}{2\pi\sqrt{-1}}\end{aligned}$$

Proof. The expression is computed in the appendix for the universal case. □

Notice that the period actually belongs to the kernel of the multiplication map $a\otimes b\to ab$.

Let us now use the map $\mathcal{O}_X\otimes_{\mathbb{Q}}\mathcal{O}_X\to\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_X$. Let $\mathscr{I}_X$ be the kernel of the multiplication map (over $\mathbb{C}$). Then $\underline{\Omega}^1_{X/\mathbb{C}}\cong\mathscr{I}_X/\mathscr{I}_X^2$. The calculations for the following proposition are done in the universal case in the appendix.

Proposition 4.3.6. *The expressions* (4.4) *and* (4.12) *respectively correspond to the images of $P(\mathscr{M}_X^{(2)})$ under the projections $\mathscr{I}_X\subset\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_X\to\underline{\Omega}^1_{X/\mathbb{C}}$, sending $a\otimes b-ab\otimes 1$ to $a\,db$, and $\mathscr{I}_X\subset\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_X\to\underline{\mathcal{E}}^0_X$ given by* (4.8).

4.4 Comparisons

In the previous sections we have shown that the Deligne torsor $\big(f,g\big]$ associated to two invertible functions f and g naturally acquires two structures: the analytic connection ∇ described in section 4.1 via the standard cup product in Deligne cohomology, and the hermitian structure described in section 4.3 via the modified cup product (4.9). We wish to briefly compare the two structures.

First, observe that using the canonical connection (cf. section 3.1.1) a pair (L,ρ) can also be thought of as a triple (L,ρ,∇^ρ), where ∇^ρ is the canonical connection determined by ρ. Equivalently, we can just consider the pair (L,∇^ρ). Also, let us stress that the canonical connection is only a *smooth* connection and is in general far from being analytic (or algebraic).

Thus our question can be reformulated as follows: for a given line bundle L equipped with an analytic connection ∇ and a hermitian fiber metric ρ, how do the pairs (L,∇) and (L,∇^ρ) compare?

The answer is the following well-known

Lemma 4.4.1. *Consider both ∇ and ∇^h as* smooth *connections. Then:*

1. *$\nabla-\nabla^h$ determines a global section of $\underline{A}^{1,0}_X$, and*
2. *this global section is zero, that is, $\nabla=\nabla^h$, if and only if L is unitary flat, namely it defines an element of $H^1(X,\mathbb{R}/\mathbb{Z})$.*

Proof. It is a well-known fact that the difference of two connections is a global one-form. Working in a local setting, let $s \in L|_U$ be a local section, and let $\|s\|$ be its length with respect to the metric. Then $\nabla s = \omega \otimes s$, for $\omega \in \underline{\Omega}^1_X(U)$, whereas $\nabla^\rho s = \partial \log\|s\| \otimes s$, and $\partial \log\|s\|$ gives a local $(1,0)$-form representative of ∇^ρ, cf. section 3.1.1. Clearly, the difference $\omega - \partial \log\|s\|$ gives a global section of $\underline{A}^{1,0}_X$.

As for the second point, one would have $\bar\partial\partial \log\|s\| = 0$, but this represents $c_1(L)$, hence the conclusion. $\square$

In the situation when the two connections agree, that is, the connection is simultaneously analytic and it is the canonical connection associated to a hermitian structure, we say they are *compatible.* The line bundle supporting it is necessarily flat.

Interestingly enough, the previous lemma can be recast into entirely cohomological terms. This is advantageous in dealing with the special case $L = \big(f, g\big]$ of special interest to us, as well as to address the very same question in the case of gerbes later on in this paper.

In the previous lemma we have compared ∇ and ∇^ρ by mapping their respective local representatives in $\underline{A}^{1,0}_X$. It will be more convenient to use the sheaf of imaginary 1-forms instead, namely consider $\pi_1 : \underline{\Omega}^1_X \to \underline{\mathcal{E}}^1_X(1)$ and $d : \underline{\mathcal{E}}^0_X(1) \to \underline{\mathcal{E}}^1_X(1)$. Consider the complex

$$\Lambda(2)^\bullet \overset{\text{def}}{=} \big(\mathbb{Z}(2) \xrightarrow{\imath} \mathcal{O}_X \xrightarrow{-\pi_1\circ d} \underline{\mathcal{E}}^1_X(1)\big)\,,$$

and the obvious maps of complexes

$$\alpha : \mathbb{Z}(2)^\bullet_{\mathcal{D}} \longrightarrow \Lambda(2)^\bullet \quad \text{and} \quad \beta : 2\pi\sqrt{-1} \otimes D(1)^\bullet_{h.h.} \longrightarrow \Lambda(2)^\bullet\,.$$

As usual, the cone:

$$\Gamma(2)^\bullet \overset{\text{def}}{=} \operatorname{Cone}\big(\alpha - \beta\big)[-1]\,,$$

characterizes the elements in $\mathbb{Z}(2)^\bullet_{\mathcal{D}}$ and $2\pi\sqrt{-1}\otimes D(1)^\bullet_{h.h.}$ which agree in $\Lambda(2)^\bullet$. A tedious but straightforward direct verification yields:

Lemma 4.4.2. *We have the quasi-isomorphism:*

$$\Gamma(2)^\bullet \xrightarrow{\simeq} \big(\mathbb{Z}(2) \xrightarrow{\imath} \mathcal{O}_X \xrightarrow{(d,-\pi_1)} \underline{\Omega}^1_X \oplus \underline{\mathcal{E}}^0_X(1) \xrightarrow{\pi_1+d} \underline{\mathcal{E}}^1_X(1)\big) \tag{4.14}$$

Dropping the last term in (4.14), we obtain the truncation

$$\tilde\Gamma(2)^\bullet \overset{\text{def}}{=} \big(\mathbb{Z}(2) \xrightarrow{\imath} \mathcal{O}_X \xrightarrow{(d,-\pi_1)} \underline{\Omega}^1_X \oplus \underline{\mathcal{E}}^0_X(1)\big)\,,$$

which clearly characterizes the elements in $\mathbb{Z}(2)^\bullet_{\mathcal{D}}$ and $2\pi\sqrt{-1} \otimes D(1)^\bullet_{h.h.}$ which agree in $2\pi\sqrt{-1} \otimes \mathbb{Z}(1)^\bullet_{\mathcal{D}}$. (In other words, $\tilde\Gamma(2)^\bullet$ can be obtained by replacing $\Lambda(2)^\bullet$ by $\mathbb{Z}(1)^\bullet_{\mathcal{D}}$ in the previous paragraphs.) In particular, let us denote by $\operatorname{Pic}(X, \nabla, h)$ the second hypercohomology group $\mathbf{H}^2(X, \tilde\Gamma(2)^\bullet)$, namely the subgroup of $H^2_{\mathcal{D}}(X, \mathbb{Z}(2)) \times \widehat{\operatorname{Pic}(X)}$ of classes of pairs (L, ∇) and (L, ρ) mapping to the same element of $\operatorname{Pic}(X) \cong H^2_{\mathcal{D}}(X, \mathbb{Z}(1))$. Then lemma 4.4.1 has the following reformulation:

Lemma 4.4.3. *There is an exact sequence:*

$$0 \longrightarrow H^1(X, \mathbb{R}/\mathbb{Z}) \longrightarrow \operatorname{Pic}(X, \nabla, h) \longrightarrow E^1(X)(1)\,, \tag{4.15}$$

where $E^1(X)(1)$ are the global sections of $\underline{\mathcal{E}}^1_X(1)$. Thus compatible connections are necessarily flat.

Proof. The complex $\tilde\Gamma(2)^\bullet$ is a quotient of $\Gamma(2)^\bullet$, namely we have the exact sequence:

$$0 \longrightarrow \underline{\mathcal{E}}^1_X(1)[-3] \longrightarrow \Gamma(2)^\bullet \longrightarrow \tilde\Gamma(2)^\bullet \longrightarrow 0\,,$$

and from the resulting long exact cohomology sequence:

$$0 \to \mathbf{H}^2(X, \Gamma(2)^\bullet) \to \mathbf{H}^2(X, \tilde\Gamma(2)^\bullet) \to E^1(X)(1) \to \cdots$$

It was noted above that $\mathbf{H}^2(X, \tilde\Gamma(2)^\bullet) \cong \operatorname{Pic}(X, \nabla, h)$, whereas for $\Gamma(2)^\bullet$ we have

$$\mathbf{H}^2(X, \Gamma(2)^\bullet) \cong H^1(X, \mathbb{R}(2)/\mathbb{Z}(2))\,.$$

The latter isomorphism follows either from a direct computation, or noticing that $\Gamma(2)^\bullet$ is a quotient of $D(2)^\bullet_{h.h.}$ (see eq. (3.7)) and

$$\mathbf{H}^2(X, \Gamma(2)^\bullet) \cong H^2_{\mathcal{D}_{h.h.}}(X, 2)$$

and then using lemma 3.2.4. $\square$

4.4.1 Comparing $\big(f,g\big]$ and $\big(f,g\big]_{h.h.}$

Suppose now L is the Deligne torsor determined by two invertible functions f and g. Clearly, the symbols $\big(f,g\big]$ and $\big(f,g\big]_{h.h.}$ taken together determine an element of $\mathrm{Pic}(X,\nabla,h)$, since the underlying torsor in $\mathrm{Pic}(X)\cong H^2_{\mathcal{D}}(X,\mathbb{Z}(1))$ is the same. This element can be represented by the cocycle

$$\Big((2\pi\sqrt{-1})^2 m_{ij}n_{jk}\,,\; -2\pi\sqrt{-1}\,m_{ij}\log_j g\,,\; \log_i f\,\frac{dg}{g}\oplus -\pi_1(\log_i f)\log|g|\Big)$$

with values in $\tilde{\Gamma}(2)^\bullet$.

Following Goncharov ([18]) let us define for any two invertibles f and g the 1-form

$$r_2(f,g)\overset{\mathrm{def}}{=}\pi_1(d\log f)\log|g|-\log|f|\,\pi_1(d\log g)\,. \tag{4.16}$$

This is clearly globally defined where f and g are invertible.

We finally obtain the following comparison.

Proposition 4.4.4. *The analytic connection in $\big(f,g\big]$ and the canonical one associated to the hermitian structure in $\big(f,g\big]_{h.h.}$ are compatible if and only if $r_2(f,g)=0$ in $E^1(X)(1)$.*

Proof. Let $\omega_i=\log_i f\;dg/g$ and $\sigma_i=-\pi_1(\log_i f)\log|g|$. The connecting homomorphism from $\tilde{\Gamma}(2)^\bullet$ to $\underline{\mathcal{E}}^1_X(1)$, that is the last map to the right in the sequence (4.15), amounts to computing $\pi_1(\omega_i)+d\sigma_i$. A straightforward calculation yields

$$\pi_1(\omega_i)+d\sigma_i=-r_2(f,g)\,.$$

□

5 Hermitian holomorphic gerbes and 2-gerbes

5.1 Higher tame symbols

Brylinski and McLaughlin considered higher degree versions of the tame symbol construction, [10, 12], namely cup products of higher degree Deligne cohomology classes: $\big(f,L\big]$ for f a holomorphic invertible function and L a holomorphic line bundle, and $\big(L,L'\big]$ for a pair of holomorphic line bundles. The geometric interpretation of the symbols so obtained, also put forward in refs. [10, 12], is that $\big(f,L\big]$ is a gerbe on X with band ($\equiv$ lien) $\mathcal{O}^\times_X$ and a holomorphic connective structure. A similar statement holds for the 2-gerbe $\big(L,L'\big]$.

5.1.1 Cup products

From the point of view of cohomology classes, one computes the relevant cup products. Using (2.4), we find that $\big(f,L\big]\in H^3_{\mathcal{D}}(X,\mathbb{Z}(2))$ is represented by the cocycle

$$\big(g_{jk}^{-m_{ij}}\,,-\frac{1}{2\pi\sqrt{-1}}\log_i f\;d\log g_{ij}\big)\,, \tag{5.1}$$

having made the standard choices for $\log_i f$ and the transition functions g_{ij} of L with respect to the choice of a cover $\mathfrak{U}_X$. Similarly, if g'_{ij} are the transition functions of L', and $2\pi\sqrt{-1}c_{ijk}$ represents $c_1(L)$ with respect to the cover $\mathfrak{U}_X$, then $\big(L,L'\big]\in H^4_{\mathcal{D}}(X,\mathbb{Z}(2))$ is represented by the cocycle

$$\big({g'_{kl}}^{-c_{ijk}}\,,-\frac{1}{2\pi\sqrt{-1}}\log g_{ij}\;d\log g'_{jk}\big)\,. \tag{5.2}$$

5.1.2 Hermitian variant

If we use the product

$$\mathbb{Z}(1)^\bullet_{\mathcal{D}}\otimes\mathbb{Z}(1)^\bullet_{\mathcal{D}}\longrightarrow D(1)^\bullet_{h.h.}$$

introduced in sect. 4.3, for f, L and L' as above we have

$$\begin{aligned} H^1_{\mathcal{D}}(X,\mathbb{Z}(1))\otimes H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) &\longrightarrow H^3_{\mathcal{D}_{h.h.}}(X,1)\\ f\otimes[L] &\longmapsto \big(f,L\big]_{h.h.}\end{aligned}$$

Using the same Čech data as before, the symbol $\big(f,L\big]_{h.h.}$ is represented by the cocycle

$$\big(g_{jk}^{-m_{ij}}\,,-\frac{1}{2\pi\sqrt{-1}}\pi_1(\log_i f)\,\pi_0(\log g_{ij})\big)\,. \tag{5.3}$$

Similarly, with L and L' we have the product

$$\begin{aligned} H^2_{\mathcal{D}}(X,\mathbb{Z}(1))\otimes H^2_{\mathcal{D}}(X,\mathbb{Z}(1)) &\longrightarrow H^4_{\mathcal{D}_{h.h.}}(X,1)\\ [L]\otimes[L'] &\longmapsto \big(L,L'\big]_{h.h.} \end{aligned}$$

and the representing cocycle

$$\big({g'_{kl}}^{-c_{ijk}}\,,-\frac{1}{2\pi\sqrt{-1}}\pi_1(\log g_{ij})\,\pi_0(\log g'_{jk})\big)\,. \tag{5.4}$$

Similarly to the proof of prop. 4.3.4, the maps of complexes $\mathbb{Z}(2)^\bullet_{\mathcal{D}}\to\mathbb{Z}(1)^\bullet_{\mathcal{D}}$ and $D(1)^\bullet_{h.h.}\to\mathbb{Z}(1)^\bullet_{\mathcal{D}}$ induce corresponding maps on the symbols $\big(f,L\big]$ and $\big(f,L\big]_{h.h.}$, moreover their images agree in $H^3_{\mathcal{D}}(X,\mathbb{Z}(1))$. An identical statement holds for $\big(L,L'\big]$ and $\big(L,L'\big]_{h.h.}$.

5.2 Gerbes with Hermitian structure

Let $\mathcal{G}$ be a gerbe on X with band $\mathcal{O}^\times_X$ ([16]). After [7, 8], its class is an element of $H^3_{\mathcal{D}}(X,\mathbb{Z}(1))\cong H^2(X,\mathcal{O}^\times_X)$. Let $\underline{\mathcal{E}}^0_{X,+}$ be the sheaf of real positive smooth functions on X.

Definition 5.2.1. A *hermitian structure* on $\mathcal{G}$ consists of the following data:

1. For each object P in $\mathcal{G}_U$, is assigned a $\underline{\mathcal{E}}^0_{U,+}$-torsor $\underline{\mathrm{herm}}(P)$ (a $\mathbb{R}_+$-principal bundle). The assignment must be compatible with the restriction functors $i^*\colon\mathcal{G}_U\to\mathcal{G}_V$ arising from $i\colon V\hookrightarrow U$ in the cover $\mathfrak{U}_X$ of X.

2. For each morphism $f\colon P\to Q$ in $\mathcal{G}_U$ a corresponding morphism $f_*\colon\underline{\mathrm{herm}}(P)\to\underline{\mathrm{herm}}(Q)$ of $\underline{\mathcal{E}}^0_{U,+}$-torsors.[1] This map must be compatible with compositions of morphisms in $\mathcal{G}_U$ and with the restriction functors.

   For an object P of $\mathcal{G}_U$, an automorphism $\varphi\in\underline{\mathrm{Aut}}(P)$ is identified with a section of $\mathcal{O}^\times_X$ over U. We then require that

$$\begin{aligned} \varphi_*\colon\underline{\mathrm{herm}}(P) &\xrightarrow{\simeq}\underline{\mathrm{herm}}(P)\\ h &\longmapsto h\cdot|\varphi|^2 \end{aligned} \tag{5.5}$$

   where the latter is the $\underline{\mathcal{E}}^0_{U,+}$-action on the torsor $\underline{\mathrm{herm}}(P)$.

Theorem 5.2.2. *Equivalence classes of $\mathcal{O}^\times_X$-gerbes with hermitian structure are classified by the group*

$$\mathbf{H}^3\big(X,\mathbb{Z}(1)_X\to\mathcal{O}_X\to\underline{\mathcal{E}}^0_X\big)\,.$$

Proof. Let $\mathcal{G}$ be an $\mathcal{O}^\times_X$-gerbe on X with hermitian structure as per definition 5.2.1. Choose a full decomposition (see [7]) with objects P_i of $\mathcal{G}_{U_i}$ and isomorphisms $f_{ij}\colon P_j|_{U_{ij}}\to P_i|_{U_{ij}}$ with respect to a cover $\mathfrak{U}_X$ of X. By a standard procedure (see refs.[7, 8]) these data determine a cochain $g_{ijk}\in\underline{\mathrm{Aut}}(P_i)|_{U_{ijk}}\cong\mathcal{O}^\times_X|_{U_{ijk}}$ satisfying the cocycle condition and determining a class in $H^2(X,\mathcal{O}^\times_X)$. Furthermore, choose sections r_i of the torsors $\underline{\mathrm{herm}}(P_i)$ above U_i. From condition 2 in definition 5.2.1 we have that there must exist $\rho_{ij}\in\underline{\mathcal{E}}^0_{X,+}|_{U_{ij}}$ such that:

$$f_{ij\,*}(r_j)=r_i\cdot\rho_{ij}\,. \tag{5.6}$$

On the 3-skeleton of the cover we have that on one hand

$$f_{ij\,*}\circ f_{jk\,*}(r_k)=f_{ij\,*}(r_j)\cdot\rho_{jk}=r_i\cdot\rho_{ij}\,\rho_{jk}\,, \tag{5.7}$$

[1] A $\underline{\mathcal{E}}^0_{U,+}$-torsor will in general be automatically trivializable. However, in this context it is convenient to "forget" the actual trivializing map.

whereas on the other hand, since $f_{ij}\circ f_{jk} = g_{ijk}\circ f_{ik}$, we have

$$(f_{ij}\circ f_{jk})_*(r_k) = g_{ijk\,*}\circ f_{ik\,*}(r_k) = g_{ijk\,*}(r_i\cdot\rho_{ik}) = r_i\cdot\left|g_{ijk}\right|^2\rho_{ik}\,. \tag{5.8}$$

Equating the right hand sides of eqs. (5.7) and (5.8), and extracting the appropriate logarithms, we see we have obtained a Čech cocycle representing a class in

$$\check{\mathbf{H}}^3\big(\mathfrak{U}_X, \mathbb{Z}(1)_X \to \mathcal{O}_X \to \underline{\mathcal{E}}^0_X\big)\,. \tag{5.9}$$

Conversely, let a class in $H^3_{\mathcal{D}_{h.h.}}(X,1)$ be given, and assume we represent it via the choice of $\mathfrak{U}_X$ by a degree 2 Čech cocycle with values in the complex

$$\mathbb{Z}(1)_X \to \mathcal{O}_X \to \underline{\mathcal{E}}^0_X\,,$$

which we write as

$$\big(2\pi\sqrt{-1}c_{ijkl}\,,\ \log g_{ijk}\,,\ \frac{1}{2}\log\rho_{ij}\big)\,.$$

This cocycle determines, via the map $D(1)^\bullet_{h.h.}\to\mathbb{Z}(1)^\bullet_{\mathcal{D}}$, a cocycle $\{g_{ijk}\}\in\check{C}^2(\mathfrak{U}_X,\mathcal{O}^\times_X)$ which can be used, according to refs. [7, 8], to glue the local stacks $\mathsf{Tors}(\mathcal{O}_{U_i})$ into a global $\mathcal{G}$, in fact a gerbe. Given a $\mathcal{O}^\times_{U_i}$-torsor P_i, namely an object of $\mathcal{G}_{U_i}\cong\mathsf{Tors}(\mathcal{O}_{U_i})$, define a hermitian structure by:

$$\underline{\mathrm{herm}}(P_i) = \text{trivial } \underline{\mathcal{E}}^0_{U_i,+} - \text{torsor}$$

Then use ρ_{ij} to glue $\underline{\mathrm{herm}}(P_i)$ and $\underline{\mathrm{herm}}(P_j)$ over U_{ij}, namely *define* an isomorphism via eq. (5.6). Since the isomorphisms $P_k\to P_i$ and $P_k\to P_j\to P_i$ differ by the equivalence determined by g_{ijk}, we see using (5.5) that the condition

$$\rho_{ij}\,\rho_{jk} = \left|g_{ijk}\right|^2\rho_{ik}\,,$$

ensuing from the cocycle condition, ensures the compatibility of this definition over U_{ijk}. □

Corollary 5.2.3. *Using the quasi-isomorphism*

$$D(1)^\bullet_{h.h.}\xrightarrow{\simeq}\big(\mathbb{Z}(1)_X\to\mathcal{O}_X\to\underline{\mathcal{E}}^0_X\big)\,,$$

the class of a gerbe with hermitian structure is in fact in $H^3_{\mathcal{D}_{h.h.}}(X,1)$.

We will see (cf. sect. 5.3) this group also automatically classifies a special type of connective structure on $\mathcal{G}$.

5.3 Hermitian connective structure

The structure defined in sect. 5.2 can be supplemented by a variant of Brylinski's connective structure [8] by taking into account the first Hodge filtration as in ref. [9]. Let $\mathcal{G}$ be an $\mathcal{O}^\times_X$ gerbe over X.

Definition 5.3.1. A type $(1,0)$ *connective structure* on $\mathcal{G}$ is the assignment to each object P of $\mathcal{G}_U$ of a $F^1\underline{A}^1_U$-torsor $\underline{\mathrm{Co}}(P)$ compatible with restriction functors and morphisms of objects. In particular, for $\varphi\in\underline{\mathrm{Aut}}(P)$, we require that

$$\begin{aligned}\varphi_*\colon \underline{\mathrm{Co}}(P) &\xrightarrow{\simeq}\underline{\mathrm{Co}}(P)\\ \nabla &\longmapsto \nabla + d\log\varphi\end{aligned} \tag{5.10}$$

where ∇ is a section of $\underline{\mathrm{Co}}(P)$ over U.[2]

Definition 5.3.2. Let $\mathcal{G}$ be equipped with a hermitian structure. A type $(1,0)$ connective structure on $\mathcal{G}$ is *compatible* with the hermitian structure if for each object P of $\mathcal{G}$ there is an isomorphism of torsors

$$\begin{aligned}\underline{\mathrm{herm}}(P) &\longrightarrow \underline{\mathrm{Co}}(P)\\ r &\longmapsto \nabla_r\end{aligned}$$

such that for a positive function ρ on U

$$r\cdot\rho\longmapsto\nabla_r+\partial\log\rho\,.$$

(In other words, $\nabla_{r\cdot\rho}=\nabla_r+\partial\log\rho$.)

[2] Note that $d\log\varphi$ is holomorphic, hence of type $(1,0)$.

Connective structures of type $(1,0)$ are classified as follows.

Theorem 5.3.3. *Let again $D(1)^\bullet_{h.h.}$ be the complex given by (3.7) for $l=1$. Equivalence classes of connective structures on a $\mathcal{O}_X^\times$-gerbe $\mathcal{G}$ compatible with a given hermitian structure are classified by the group*

$$\mathbf{H}^3\big(X, D(1)^\bullet_{h.h.}\big)\,.$$

We have the following analog of the existence and uniqueness of the canonical connection on an invertible sheaf.

Corollary 5.3.4. *A connective structure compatible with a hermitian structure on a gerbe $\mathcal{G}$ is uniquely determined up to equivalence.*

Proof. It is an immediate consequence of the fact that the groups in Theorems 5.2.2 and 5.3.3, being computed from quasi-isomorphic complexes, are actually the same (and equal to $H^3_{\mathcal{D}_{h.h.}}(X,1)$.) ☐

Remark 5.3.5. The group $\mathbf{H}^3\big(X, D(1)^\bullet_{h.h.}\big) \cong H^3_{\mathcal{D}_{h.h.}}(X,1)$ is *not* equal to Brylinski's

$$\mathbf{H}^3\big(X, \mathbb{Z}(1) \to \underline{\mathcal{E}}^0_X(1) \to \underline{\mathcal{E}}^1_X(1)\big)\,,$$

cf. ref. [9, Proposition 6.9 (1)]. (In fact there is an epimorphism $C(1)^\bullet \to \big(\mathbb{Z}(1) \to \underline{\mathcal{E}}^0_X(1) \to \underline{\mathcal{E}}^1_X(1)\big)$ with non-trivial kernel.) It follows that the notion of "hermitian gerbes with hermitian connective structure" in loc. cit. is not identical to our notion of $\mathcal{O}_X^\times$-gerbe with hermitian structure and compatible type $(1,0)$ connective structure.

Proof of Theorem 5.3.3. Choose a cover $\mathfrak{U}_X$ as usual and let (P_i, f_{ij}, r_i) be a decomposition of $\mathcal{G}$ and its hermitian structure as in the proof of Theorem 5.2.2.

If $\mathcal{G}$ has a compatible type $(1,0)$ connective structure, we have a map $\underline{\mathrm{herm}}(G_{U_i}) \ni r_i \mapsto \nabla_i \in \underline{\mathrm{herm}}(G_{U_i})$. For every isomorphism f_{ij} the compatibility condition from Definition 5.3.2 determines a form

$$\xi_{ij} = \partial \log \rho_{ij} \in F^1\underline{A}^1_X(U_{ij})$$

satisfying the condition

$$\xi_{jk} - \xi_{ik} + \xi_{ij} = d\log g_{ijk}\,. \tag{5.11}$$

The imaginary 2-form $\eta_{ij} \overset{\text{def}}{=} \bar\partial \xi_{ij} = \bar\partial\partial \log \rho_{ij}$ then is a cocycle with values in $F^1\underline{A}^2_X \cap \underline{\mathcal{E}}^2_X(1)$.

Altogether, g_{ijk}, $\frac{1}{2}\log\rho_{ij}$, ξ_{ij} and η_{ij} determine a cocycle of total degree 3 in the Čech resolution $\check{C}^\bullet(\mathfrak{U}_X, D(1)^\bullet_{h.h.})$.

Conversely, given a degree 3 cocycle with values in $D(1)^\bullet_{h.h.}$, a gerbe $\mathcal{G}$ with hermitian structure can be obtained by gluing trivial $\mathcal{O}^\times_{U_i}$-torsors and $\underline{\mathcal{E}}^0_{U_i,+}$ torsors as in Theorem 5.2.2. Furthermore, define a map by assigning the trivial $F^1\underline{A}^1_{U_i}$-torsor to the trivial $\underline{\mathcal{E}}^0_{U_i,+}$-torsor by

$$r \longmapsto \nabla_r \equiv \partial \log r\,.$$

Clearly, this defines a type $(1,0)$ connective structure compatible with the hermitian structure on $\mathcal{G}$. ☐

Remark 5.3.6. Note the proof of Theorem 5.3.3 that $d\eta_{ij} = 0$, hence we obtain a class

$$[\eta_{ij}] \in \mathbf{H}^3\big(X, F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1)\big)$$

which can be associated to $\mathcal{G}$ via the obvious map

$$D(1)^\bullet_{h.h.} \longrightarrow F^1\underline{A}^\bullet_X \cap \sigma^2\underline{\mathcal{E}}^\bullet_X(1)\,.$$

This class plays the same role for $\mathcal{G}$ as the (global) imaginary form $c_1(\rho) = \bar\partial\partial\log\rho_i$ for a metrized line bundle (L,ρ).

Remark 5.3.7 *(*Hermitian curving*).* An equivalent degree 3 cocycle can be obtained by introducing the cochain $K_i \in \underline{A}^{1,1}_X \cap \underline{\mathcal{E}}^2_X(1)(U_i)$ of imaginary 2-forms such that

$$\bar\partial\partial \log \rho_{ij} = K_j - K_i\,,$$

and the imaginary 3-form $\Omega_i \equiv \Omega|_{U_i}$ such that

$$dK_i = \Omega|_{U_i}\,,$$

where $\Omega \in F^1A^3(X) \cap E^3(X)(1)$ (global sections). We can regard K_i as the hermitian *curving* and Ω as the hermitian 3-*curvature,* respectively, of the type $(1,0)$ hermitian connection.

5.4 The symbol $\big(f,L\big]_{h.h.}$

Given an invertible function f and a line bundle L we have seen there is a product $\big(f,L\big]_{h.h.}\in H^3_{\mathcal{D}_{h.h.}}(X,1)$. We briefly give a geometric construction of the corresponding hermitian-holomorphic gerbe.

We need to recall from [12] the construction of the gerbe $\mathcal{C}$ underlying $\big(f,L\big]$. $\mathcal{C}$ is the stackification of the following pre-stack $\mathcal{C}^0$. For $U\hookrightarrow X$ objects of the category $\mathcal{C}^0_U$ are non vanishing sections of $L|_U$. If $s\in L|_U$, and non vanishing, it is denoted $\big(f,s\big]$ as an object of $\mathcal{C}^0_U$. Given another non vanishing section s' of L over U, there is $g\in\mathcal{O}^\times_U$ such that $s'=sg$. Morphisms from $\big(f,s'\big]$ to $\big(f,s\big]$ are given by sections of the Deligne torsor $\big(f,g\big]$ over U. For a third non vanishing section s'', with $s''=s'g'=sgg'$, composition of morphisms in the category $\mathcal{C}^0_U$ corresponds to the K-theoretic property of the Deligne torsor:

$$\big(f,gg'\big]\cong\big(f,g\big]\otimes\big(f,g'\big]\,.$$

Given a trivialization of L by a collection $\{s_i\}$ relative to a cover $\mathfrak{U}_X=\{U_i\}_{i\in I}$, with transition functions $g_{ij}\in\mathcal{O}^\times_X(U_{ij})$, the objects $\big(f,s_i\big]$ and the morphisms

$$\phi_{ij}=\{\log_i f,g_{ij}\}\colon\big(f,s_j\big]\to\big(f,s_i\big]$$

provide a decomposition of $\mathcal{C}$ in the sense of [7]. It follows that the automorphisms

$$h_{ijk}=\phi_{ij}\otimes\phi_{jk}\otimes\phi_{ik}^{-1}=g_{jk}^{-m_{ij}}\in\underline{\mathrm{Aut}}(\big(f,s_i\big]|_{U_{ijk}})\cong\mathcal{O}^\times_X(U_{ijk})\tag{5.12}$$

represent the cohomology class of $\mathcal{C}$ in $H^3_{\mathcal{D}}(X,\mathbb{Z}(1))\cong H^2(X,\mathcal{O}^\times_X)$.

Now define a *hermitian structure* on $\mathcal{C}$ as follows. To an object $\big(f,s\big]$ of $\mathcal{C}_U$ we assign

$$\big(f,s\big]\rightsquigarrow\underline{\mathrm{herm}}(\big(f,s\big])=\text{trivial }\underline{\mathcal{E}}^0_{U,+}\text{-torsor.}\tag{5.13}$$

Then, given a morphism $\big(f,g\big]\ni\phi\colon\big(f,s'\big]\to\big(f,s\big]$ in $\mathcal{C}_U$, with $s'=sg$ as above, we use the hermitian structure on the Deligne torsor underlying $\big(f,g\big]$ defined in sect. 4.3, Proposition 4.3.4. Namely

$$\begin{aligned}\phi_*\colon\underline{\mathrm{herm}}(\big(f,s'\big])&\longrightarrow\underline{\mathrm{herm}}(\big(f,s\big])\\ h&\longmapsto h\cdot\|\phi\|^2\end{aligned}\tag{5.14}$$

where h is a local section of $\underline{\mathrm{herm}}(\big(f,s'\big])$, to be identified with one of $\underline{\mathcal{E}}^0_{U,+}$ and $\|\phi\|$ is the length of the non-vanishing section ϕ. We have the following analog of Proposition 4.3.4:

Proposition 5.4.1. *The class of the gerbe $\mathcal{C}$ underlying the symbol $\big(f,L\big]$ with hermitian structure defined by eqs.* (5.13) *and* (5.14) *is given by the product $\big(f,L\big]_{h.h.}$ in the group $\mathbf{H}^3\big(X,\mathbb{Z}(1)_X\to\mathcal{O}_X\to\underline{\mathcal{E}}^0_X\big)\cong H^3_{\mathcal{D}_{h.h.}}(X,1)$.*

Proof. We need to find the class of the $\mathcal{C}$ as in the proof of Thm. 5.2.2 and show it coincides with $\big(f,L\big]_{h.h.}$ as computed in eq. (5.3). To this end, let us use the decomposition of $\mathcal{C}$ given by the objects $\big(f,s_i\big]$ and morphisms $\phi_{ij}=\{\log_i f,g_{ij}\}\colon\big(f,s_j\big]\to\big(f,s_i\big]$ for non vanishing sections $s_i\in L|_{U_i}$, as before. The class of $\mathcal{C}$ (without extra structures) is represented by the cochain $g_{jk}^{-m_{ij}}$ already appearing in eq. (5.12).

Furthermore, in the hermitian Deligne torsor $\big(f,g_{ij}\big]$ over U_{ij} the logarithm of the length of the section $\phi_{ij}=\{\log_i f,g_{ij}\}$ is given by

$$\sigma_{ij}\equiv\frac{1}{2}\log\|\phi_{ij}\|^2\equiv\frac{1}{2}\log\rho_{ij}=-\frac{1}{2\pi\sqrt{-1}}\,\pi_1(\log_i f)\log|g_{ij}|\ ,$$

cf. eq. (4.11). Thus we have found the total cocycle representing $\big(f,L\big]_{h.h.}$ as in eq. (5.3). Indeed, by computing the Čech coboundary we find

$$\sigma_{ij}-\sigma_{ik}+\sigma_{jk}=-m_{ij}\log|g_{jk}|\ ,$$

as desired. $\square$

5.5 Hermitian 2-Gerbes

Let us briefly extend the considerations outlined in the previous sections to 2-gerbes over X bound by $\mathcal{O}_X^\times$. (An extended exposition of the local geometry of 2-gerbes is to be found in ref. [7]. See also [10] for the abelian case.)

Recall that a 2-gerbe G over X bound by a sheaf of *abelian groups* $\underline{H}$ is a fibered 2-category over X which satisfies the 2-descent condition for objects, and such that for any two objects P and Q in the fiber 2-category G_U over $U \subset X$ the fibered category $\underline{\mathrm{Hom}}(P,Q)$ is a stack. If fact, this fibered category turns out to be an $\underline{H}$-gerbe equivalent to the neutral one $\mathsf{Tors}(\underline{H})$. The properties of interest to us are the following: G is *locally non-empty,* namely there is a cover $\mathfrak{U}_X$ of X such that for $U \subset X$ in the cover, the object set of G_U is non-empty; G is *locally connected,* namely any two objects can be connected by a weakly invertible 1-arrow (that is, invertible up to a 2-arrow); any two 1-arrows can be (locally) joined by a 2-arrow; finally, for every 1-arrow its automorphism group is isomorphic in a specified way to $\underline{H}$.

Once the appropriate notion of isomorphism for 2-gerbes is introduced, isomorphism classes of 2-gerbes bound by $\underline{H}$ are classified by the sheaf cohomology group $H^3(X,\underline{H})$, see, e.g. refs. [7, 10].

In what follows, we shall set $\underline{H} = \mathcal{O}_X^\times$. Hence we can rephrase the previous statement by saying that isomorphism classes of 2-gerbes bound by $\mathcal{O}_X^\times$ are classified by the group

$$H^3(X,\mathcal{O}_X^\times) \cong H^4_{\mathcal{D}}(X,\mathbb{Z}(1))\,.$$

We shall need the local calculation leading to the classification, so we recall it here. Given a 2-gerbe G, let us choose a decomposition by selecting a cover $\mathfrak{U}_X$ of X and a collection of objects P_i in G_{U_i}. There is a 1-arrow

$$f_{ij}\colon P_j \to P_i$$

between their restrictions to $\mathsf{G}_{U_{ij}}$. Furthermore, from the axioms there is a 2-arrow

$$\alpha_{ijk}\colon f_{ij}\circ f_{jk} \Longrightarrow f_{ik}\,.$$

Further restricting over a 4-fold intersection U_{ijkl}, we have two 1-arrows $f_{ij}\circ f_{jk}\circ f_{kl}\colon P_l \to P_i$ and $f_{il}\colon P_l \to P_i$ and between them *two* 2-arrows, namely $\alpha_{ijl}\circ(\mathrm{Id}_{f_{ij}} * \alpha_{jkl})$ and $\alpha_{ikl}\circ(\alpha_{ijk} * \mathrm{Id}_{f_{kl}})$. Since 2-arrows are strictly invertible, it follows again from the axioms that there exists a section h_{ijkl} of $\mathcal{O}_X^\times$ over U_{ijkl} such that

$$\alpha_{ijl}\circ(\mathrm{Id}_{f_{ij}} * \alpha_{jkl}) = h_{ijkl}\circ\alpha_{ikl}\circ(\alpha_{ijk} * \mathrm{Id}_{f_{kl}})\,. \tag{5.15}$$

This section is a 3-cocycle and the assignment $\mathsf{G} \mapsto [h]$ gives the classification isomorphism.

In analogy with what was previously done for gerbes, we are going to define a notion of hermitian structure and of type $(1,0)$ *connectivity* for 2-gerbes on X bound by $\mathcal{O}_X^\times$. Brylinski and McLaughlin defined a *concept of connectivity* on a 2-gerbe G over X to be the datum of a compatible class of connective structures on the gerbes $\underline{\mathrm{Hom}}_U(P,Q)$ for two objects P, Q in the fiber G_U. It is possible to introduce several variants of this notion, as done in refs. [10, 12]. Thus a type $(1,0)$ connectivity will just be the requirement that these connective structures take their values in $F^1\underline{A}^1_X -$ torsors.

Let us model the concept of hermitian structure on a 2-gerbe after the one for gerbes given above in definition 5.2.1.

Definition 5.5.1. A *hermitian structure* on a $\mathcal{O}_X^\times$-2-gerbe G over X consists of the following data.

1. To each object P in the fiber 2-category G_U over $U \subset X$ we assign a $\underline{\mathcal{E}}^0_{U,+}$-gerbe $\underline{\mathrm{herm}}(P)$ over U. (As before, $\underline{\mathcal{E}}^0_{U,+}$ is the sheaf of real positive functions on U.)

2. This assignment must be compatible with the inverse image 2-functors $i^*\colon \mathsf{G}_U \to \mathsf{G}_V$, natural transformations $\varphi_{i,j}\colon j^*i^* \Rightarrow (ij)^*$ and modifications $\alpha_{i,j,k}\colon \varphi_{ij,k}\circ(h^* * \varphi_{i,j}) \Rrightarrow \varphi_{i,jk}\circ(\varphi_{j,k} * i^*)$ arising from the inclusions $i\colon V \hookrightarrow U$, $j\colon W \hookrightarrow V$, and $k\colon Z \hookrightarrow W$, in the cover $\mathfrak{U}_X$.

3. For each 1-arrow $f\colon P \to Q$ in G_U a corresponding equivalence $f_*\colon \underline{\mathrm{herm}}(P) \to \underline{\mathrm{herm}}(Q)$ of $\underline{\mathcal{E}}^0_{U,+}$-gerbes. For each 2-arrow $\alpha\colon f \Rightarrow f'$ a corresponding natural transformation $\alpha_*\colon f_* \Rightarrow f'_*$ between equivalences. We ask that this correspondence be compatible with compositions of 1- and 2-arrows. Namely, for 1-arrows $f,f'\colon P \to Q$ and $g,g'\colon Q \to R$ and for 2-arrows $\alpha\colon f \Rightarrow f'$ and $\beta\colon g \Rightarrow g'$ in G_U, which we

compose as $\beta * \alpha\colon g\circ f \Rightarrow g'\circ f'$, we find a diagram of natural transformations

$$
\begin{tikzcd}
g_* \circ f_* \arrow[r, Rightarrow, "\varepsilon(f,g)"] \arrow[d, Rightarrow, "\beta_* * \alpha_*"'] & (g\circ f)_* \arrow[d, Rightarrow, "(\beta*\alpha)_*"] \\
g'_* \circ f'_* \arrow[r, Rightarrow, "\varepsilon(f',g')"'] & (g'\circ f')_*
\end{tikzcd}
\tag{5.16}
$$

of equivalences between the $\underline{\mathcal{E}}^0_{U,+}$-gerbes $\underline{\mathrm{herm}}(P)$ and $\underline{\mathrm{herm}}(R)$ on $U\subset X$.

4. From the axioms, the group of automorphisms of a 1-arrow $f\colon P\to Q$ in G_U is identified with $\mathcal{O}_U^\times$. It follows that such an automorphism α (that is, a 2-arrow from f to itself) can be identified with a section $a\in\mathcal{O}_U^\times$. We then require that the induced natural isomorphism

$$\alpha_*\colon f_* \Longrightarrow f_*\,,\quad \text{where } f_*\colon \underline{\mathrm{herm}}(P)\longrightarrow \underline{\mathrm{herm}}(Q)$$

be identified with a section of $\underline{\mathcal{E}}^0_{U,+}$ via the map

$$a \longmapsto |a|^2 \tag{5.17}$$

and an appropriate labeling of $\underline{\mathrm{herm}}(P)$ and $\underline{\mathrm{herm}}(Q)$ by objects r and s, respectively. In more detail, given an arrow $f_*(r)\to s$ in $\underline{\mathrm{herm}}(Q)$, the action of α via α_* will amount to an automorphism of s. We require that it be $|a|^2$.

Remark 5.5.2. The abstract nonsense of definition 5.5.1 could have more succinctly characterized by saying that the correspondence $\underline{\mathrm{herm}}(\cdot)$ realizes a Cartesian 2-functor between G and the 2-gerbe $\mathsf{Gerbes}(\underline{\mathcal{E}}^0_{X,+})$ on X, shifting to the reader the burden of unraveling the diagrams.

We have the following analog of theorem 5.2.2:

Theorem 5.5.3. *Isomorphism classes of $\mathcal{O}_X^\times$-2-gerbes with hermitian structure in the sense of definition 5.5.1 are classified by the group*

$$\mathbf{H}^4\big(X,\mathbb{Z}(1)_X\to\mathcal{O}_X\to\underline{\mathcal{E}}^0_X\big)\cong H^4_{\mathcal{D}_{h.h.}}(X,1)\,.$$

Proof. Let G be a $\mathcal{O}_X^\times$-2-gerbe on X with hermitian structure as per definition 5.5.1. Forgetting the hermitian structure, G will determine a class in the group $H^4_{\mathcal{D}}(X,\mathbb{Z}(1))\cong H^3(X,\mathcal{O}_X^\times)$, and we have briefly recalled before — cf. eq. (5.15) — how to obtain a 3-cocycle representing the class of G.

To obtain the rest of the cocycle with values in the complex $\mathbb{Z}(1)_X\to\mathcal{O}_X\to\underline{\mathcal{E}}^0_X$ let us make the same choice for a decomposition of G with respect to the cover $\mathfrak{U}_X$: a collection of objects P_i in G_{U_i}, 1-arrows $f_{ij}\colon P_j\to P_i$ between their restrictions and 2-arrows $\alpha_{ijk}\colon f_{ij}\circ f_{jk}\Rightarrow f_{ik}$.

We shall also need a decomposition of the $\underline{\mathcal{E}}^0_{U_i,+}$-gerbes $\underline{\mathrm{herm}}(P_i)$: to this end let us choose objects r_i over U_i and arrows $\xi_{ij}\colon (f_{ij})_*(r_j)\to r_i$ between their restriction to U_{ij}.

Let us consider a triple of objects P_i,P_j,P_k over U_{ijk}. (we are implicitly restricting to the fiber 2-category $\mathsf{G}_{U_{ijk}}$.) We obtain the following diagram in $\underline{\mathrm{herm}}(P_i)|_{U_{ijk}}$:

$$
\begin{tikzcd}
(f_{ij})_*(f_{jk})_*(r_k) \arrow[r, "(f_{ij})_*(\xi_{jk})"] \arrow[d] & (f_{ij})_*(r_j) \arrow[d, "\xi_{ij}"] \\
(f_{ik})_*(r_k) \arrow[r, "\xi_{ik}"'] & r_i \arrow[loop right, "\rho_{ijk}"]
\end{tikzcd}
\tag{5.18}
$$

The left vertical arrow in (5.18) results from the composition of two-arrows

$$(f_{ij})_*\circ(f_{jk})_* \xRightarrow{\varepsilon_{ijk}} (f_{ij}\circ f_{jk})_* \xRightarrow{(\alpha_{ijk})_*} (f_{ik})_*$$

resulting from diagram (5.16) in definition 5.5.1. At the level of objects in the gerbe $\underline{\mathrm{herm}}(P_i)$ diagram (5.16) is of course not commutative, so we obtain a section $\rho_{ijk}\in\underline{\mathrm{Aut}}(r_i)$, which we can identify with a section of the sheaf $\underline{\mathcal{E}}^0_{U,+}$ over U_{ijk}.

Now consider a four-fold intersection U_{ijkl}: we have a cube determined by the objects $r_i, \dots, r_l$ whose faces are built from copies of (5.18). Since this cube brings in the relation (5.15), using the mapping of the $\mathcal{O}_X^\times$ action spelled out in the last point in definition 5.5.1, we get the relation

$$\rho_{jkl}\,\rho_{ikl}^{-1}\,\rho_{ijl}\,\rho_{ijk}^{-1} = |h_{ijkl}|^2 \tag{5.19}$$

which, after taking the appropriate logarithms, defines a Čech cocycle representing a class in

$$\check{\mathbf{H}}^4\big(\mathfrak{U}_X, \mathbb{Z}(1)_X \to \mathcal{O}_X \to \underline{\mathcal{E}}^0_X\big)\,.$$

Details (and diagram chasing) are straightforward and left to the reader.

Conversely, let us be given a class in

$$\mathbf{H}^4\big(X, \mathbb{Z}(1)_X \to \mathcal{O}_X \to \underline{\mathcal{E}}^0_X\big) \cong \mathbf{H}^3\big(X, \mathcal{O}_X^\times \xrightarrow{|\cdot|} \underline{\mathcal{E}}^0_{X,+}\big)\,,$$

and let us assume it is represented by the (multiplicative) Čech cocycle $\big(h_{ijkl}, \rho_{ijk}\big)$. Let just explain the construction of a corresponding 2-gerbe with hermitian structure (up to equivalence). Again, details will be left to the reader.

We first apply the map

$$\big(\mathbb{Z}(1)_X \to \mathcal{O}_X \to \underline{\mathcal{E}}^0_X\big) \longrightarrow \big(\mathbb{Z}(1)_X \to \mathcal{O}_X\big)$$

to the representative Čech cocycle to reconstruct a $\mathcal{O}_X^\times$-2-gerbe G according to refs. [7, 10, 12]. Recall that this is accomplished by gluing the local stacks $\mathsf{Gerbes}(\mathcal{O}_{U_i}^\times)$ using h_{ijkl}. Secondly, we define a hermitian structure as follows. Assign to any object P_i over U_i of the so-determined 2-gerbe G the trivial $\underline{\mathcal{E}}^0_{U_i,+}$-gerbe $\underline{\mathrm{herm}}(P_i) = \mathsf{Tors}(\underline{\mathcal{E}}^0_{U_i,+})$. For a triple of such on U_{ijk} we use $\rho_{ijk} \in \underline{\mathcal{E}}^0_{U_i,+}|_{U_{ijk}}$ as an automorphism of an object r_i in $\underline{\mathrm{herm}}(P_i)$.

Checking that this structure satisfies the properties in definition 5.5.1 and it defines a 2-gerbe with hermitian structure whose class is the one we started with is modeled after the pattern of refs. [7] and [8] and it will be left to the reader. $\square$

As mentioned before, a connectivity on a $\mathcal{O}_X^\times$-2-gerbe is in practice the assignment of compatible connective structures on the local gerbes of morphisms. We have the following definition (see also [9, sect. 7], for the first part):

Definition 5.5.4. Let G be a $\mathcal{O}_X^\times$-2-gerbe on X.

1. A type $(1,0)$ *concept of connectivity* on G is the assignment of a $F^1\underline{A}^1_U$-gerbe $\underline{\mathrm{Co}}(P)$ to each object P in G_U. This assignment will have to satisfy properties analogous to those of definition 5.5.1. Of course, in the last condition, the map (5.17) will have to be replaced by $a \mapsto d\log a$.

2. A type $(1,0)$ concept of connectivity is *compatible* with a hermitian structure if for each object P of G_U there is an equivalence of gerbes
$$\underline{\mathrm{herm}}(P) \longrightarrow \underline{\mathrm{Co}}(P)$$
satisfying the obvious compatibility conditions with the operations of G_U and the restrictions.

The proof of the following theorem can be patterned after an appropriate generalization of the proof of Theorem 5.3.3, so we shall omit it.

Theorem 5.5.5. *Let G be a $\mathcal{O}_X^\times$-2-gerbe with hermitian structure and let $D(1)^\bullet_{h.h.}$ be the complex given by* (3.7) *for $l = 1$. Equivalence classes of type* $(1,0)$ *connectivities on G compatible with the given hermitian structure are classified by the group*

$$\mathbf{H}^4\big(X, D(1)^\bullet_{h.h.}\big)\,.$$

Furthermore, the equivalence class is unique.

5.6 The symbol $\big(L, L'\big]_{h.h.}$

We have seen that given two line bundles L and L' over X their cup product $\big(L, L'\big]_{h.h.}$ defines a class in $H^4_{\mathcal{D}_{h.h.}}(X, 1)$. According to Theorem 5.5.3 it corresponds to an equivalence class of 2-gerbes with hermitian structure. Using the obvious maps of complexes $D(1)^\bullet_{h.h.} \to \mathbb{Z}(1)^\bullet_{\mathcal{D}}$ and $\mathbb{Z}(2)^\bullet_{\mathcal{D}} \to \mathbb{Z}(1)^\bullet_{\mathcal{D}}$, the geometric 2-gerbe G that underlies $\big(L, L'\big]_{h.h.}$ is the same one as for the standard symbol $\big(L, L'\big]$ constructed by Brylinski and McLaughlin.

Recall (see ref. [12] for more details) that objects of G underlying $\big(L, L'\big]$ over $U \subset X$ are the non-vanishing sections s of $L|_U$, denoted $\big(s, L\big]$. Given another non vanishing section $s' \in L|_U$ we have $s' = sg$ for an invertible function g over U. Then the category of morphisms from $\big(s', L\big]$ to $\big(s, L\big]$ is the *gerbe* $\big(g, L\big]$ defined in section 5.4. For a third non vanishing section s'' of L over U, with $s'' = s'\,g'$, the morphism composition functor is given by the equivalence

$$\big(g, L'\big] \otimes \big(g', L\big] \longrightarrow \big(gg', L\big]$$

where on the left hand side we have the contracted product of two (abelian) gerbes. To be precise, it turns out that G is an appropriate "2-stackification" of the 2-pre-stack defined here.

A calculation in ref. [12] shows that with respect to the trivializations $\{g_{ij}\}$ and $\{g'_{ij}\}$ of L and L', respectively, the class of G is represented by the cocycle ${g'_{kl}}^{-c_{ijk}} \in \mathcal{O}^\times_X(U_{ijkl})$, where the cocycle c_{ijk} represents $c_1(L)$.

We can define a hermitian structure on G as follows. To an object $\big(s, L'\big]$ of G_U we assign

$$\big(s, L'\big] \rightsquigarrow \underline{\mathrm{herm}}(\big(s, L'\big]) = \text{trivial } \underline{\mathcal{E}}^0_{U,+}\text{-gerbe.} \tag{5.20}$$

Furthermore, as remarked above we have $\underline{\mathrm{Hom}}_U\big(\big(s', L'\big], \big(s, L'\big]\big) \cong \big(g, L'\big]$. Thus we set

$$\underline{\mathrm{Hom}}_U\big(\underline{\mathrm{herm}}(\big(s', L'\big]), \underline{\mathrm{herm}}(\big(s, L'\big])\big) = \big(g, L'\big]_{h.h.}\,, \tag{5.21}$$

where on the right hand side we use the hermitian structure on the gerbe $\big(g, L'\big]$ as defined in section 5.4. On the left hand side of (5.21) we have the equivalences of the two $\underline{\mathcal{E}}^0_{U,+}$-gerbes.

The proof of the following proposition is a straightforward generalization of the one for proposition 5.4.1.

Proposition 5.6.1. *The class of the $\mathcal{O}^\times_X$-2-gerbe G underlying the symbol $\big(L, L'\big]$ with hermitian structure defined by eqs.* (5.20) *and* (5.21) *is given by the product $\big(L, L'\big]_{h.h.}$ in the group $\mathbf{H}^4\big(X, \mathbb{Z}(1)_X \to \mathcal{O}_X \to \underline{\mathcal{E}}^0_X\big) \cong H^4_{\mathcal{D}_{h.h.}}(X, 1)$.*

5.7 Comparisons and relations with other definitions

Recall from refs. [10, 12], that analytic connective structures on gerbes with band $\mathcal{O}^\times_X$ are classified by the group $H^3_{\mathcal{D}}(X, \mathbb{Z}(2))$. Similarly, for 2-gerbes with the same band, the relevant group is $H^4_{\mathcal{D}}(X, \mathbb{Z}(2))$. In the previous sections we have introduced hermitian structures1 and type-$(1,0)$ connective structures on gerbes and 2-gerbes with band $\mathcal{O}^\times_X$. We define the concept of compatibility analogously to the case of line bundles in sect. 4.4 as follows.

Let $\mathcal{G}$ be a $\mathcal{O}^\times_X$-gerbe on X. Let $\underline{\mathrm{Co}}(\cdot)^{an}$ be a (holomorphic) connective structure on $\mathcal{G}$ in the sense of refs. [10, 12], and let $\underline{\mathrm{Co}}(\cdot)^h$ be a connective structure *on the same gerbe* in the sense of sect. 5.3.

The relevant group classifying $\mathcal{G}$ equipped with both types of connections is therefore $\mathbf{H}^3(X, \tilde{\Gamma}(2)^\bullet)$, where the complex $\tilde{\Gamma}(2)^\bullet$ has been introduced in sect. 4.4.

Definition 5.7.1. We say that $\underline{\mathrm{Co}}(\cdot)^{an}$ and $\underline{\mathrm{Co}}(\cdot)^h$ are *compatible* if for any object P of $\mathcal{G}_U$, $U \subset X$, there is an isomorphism of torsors $\underline{\mathrm{Co}}(P)^{an} \cong \underline{\mathrm{Co}}(P)^h$ (after lambda-extension of $\underline{\mathrm{Co}}(P)^{an}$ from $\underline{\Omega}^1_U$ to $\underline{A}^{1,0}_U$.)

Similarly, if G is a $\mathcal{O}^\times_X$-2-gerbe on X, carrying both types of connective structures, its class is an element of the group $\mathbf{H}^4(X, \tilde{\Gamma}(2)^\bullet)$. We can also repeat the above definition, taking care that now for any object of G over $U \subset X$, $\underline{\mathrm{Co}}(P)^{an} \cong \underline{\mathrm{Co}}(P)^h$ must be an equivalence of gerbes.

The next lemma immediately follows from the definitions.

Lemma 5.7.2. *Let $\Gamma(2)^\bullet$ be the complex defined in sect. 4.4.*

1. *Classes of $\mathcal{O}^\times_X$-gerbes with compatible connective structures in the sense of definition 5.7.1 are classified by the elements of the group $\mathbf{H}^3(X, \Gamma(2)^\bullet)$.*
2. *Similarly, classes of $\mathcal{O}^\times_X$-2-gerbes with compatible connective structures are classified by $\mathbf{H}^4(X, \Gamma(2)^\bullet)$.*

5.7.1 Compatibility and flatness conditions

While these definitions seem to follow the pattern of line bundles analyzed in sect. 4.4, there in an important difference, namely gerbes (or 2-gerbes) satisfying the compatibility condition of definition 5.7.1 are *not necessarily flat!* Moreover, in the present framework the compatibility condition is less special than it was seen in the case of line bundles. This can be seen by way of the following cohomological argument.

The complex $\Gamma(2)^\bullet$ introduced in sect. 4.4 is easily seen to be a quotient of the complex $D(2)^\bullet_{h.h.}$:

$$D(2)^\bullet_{h.h.} \longrightarrow \Gamma(2)^\bullet \longrightarrow 0\,.$$

The kernel is complicated, but up to quasi-isomorphism, it can be reduced (by direct computation) to the one-element complex $\underline{\mathcal{E}}^2_X(1) \cap \underline{A}^{1,1}_X[-4]$ so that we have the triangle:

$$\underline{\mathcal{E}}^2_X(1) \cap \underline{A}^{1,1}_X[-4] \longrightarrow D(2)^\bullet_{h.h.} \longrightarrow \Gamma(2)^\bullet \xrightarrow{+1}$$

Focusing our attention to degree 3 and 4, we get the sequence:

$$0 \to H^2(X, \mathbb{R}(2)/\mathbb{Z}(2)) \to \mathbf{H}^3(X, \Gamma(2)^\bullet) \to E^2(X)(1) \cap A^{1,1}(X) \to H^4_{\mathcal{D}_{h.h.}}(X, 2) \to \mathbf{H}^4(X, \Gamma(2)^\bullet) \to 0\,,$$

where we have used lemma 3.2.4. Moreover, the exact sequence from the proof of lemma 4.4.3 relating $\tilde{\Gamma}(2)^\bullet$ to $\Gamma(2)^\bullet$ yields the following completion of (4.15):

$$0 \to H^1(X, \mathbb{R}(2)/\mathbb{Z}(2)) \to \operatorname{Pic}(X, \nabla, h) \to E^1(X)(1) \to \mathbf{H}^3(X, \Gamma(2)^\bullet) \to \mathbf{H}^3(X, \tilde{\Gamma}(2)^\bullet) \to 0$$

and

$$\mathbf{H}^4(X, \Gamma(2)^\bullet) \xrightarrow{\cong} \mathbf{H}^4(X, \tilde{\Gamma}(2)^\bullet)\,,$$

where we have used that $\underline{\mathcal{E}}^1_X(1)$ is soft. In summary we have:

Proposition 5.7.3.

1. *The class of a $\mathcal{O}^\times_X$-gerbe supporting both types of connective structures can be lifted to a class of compatible connective structures on a (possibly equivalent) gerbe.*
2. *A $\mathcal{O}^\times_X$-gerbe with compatible connective structures is flat if the (trivial)* $(1,1)$*-curving is zero (cf. sect. 5.3, remarks 5.3.6 and 5.3.7.)*
3. *A $\mathcal{O}^\times_X$-2-gerbe supporting both types of connective structures is equivalent to a* 2*-gerbe with compatible connective structures. Its class can be lifted to* $H^4_{\mathcal{D}_{h.h.}}(X, 2)$

5.7.2 Comparing $\big(f, L\big]$ and $\big(L, L'\big]$ with their hermitian variants

The higher symbols $\big(f, L\big]$ and $\big(f, L\big]_{h.h.}$ have the same underlying gerbe, and similarly $\big(L, L'\big]$ and $\big(L, L'\big]_{h.h.}$ determine the same 2-gerbe. Let us denote them, respectively, by $\big\{f, L\big\}$ and $\big\{L, L'\big\}$. By construction, they determine classes in $\mathbf{H}^3(X, \tilde{\Gamma}(2)^\bullet)$ and $\mathbf{H}^4(X, \tilde{\Gamma}(2)^\bullet)$, respectively. The proposition specializes to this case as follows:

Corollary 5.7.4. *The connective structures $\underline{\mathrm{Co}}(\cdot)^{an}$ and $\underline{\mathrm{Co}}(\cdot)^h$ on $\big\{f, L\big\}$ are compatible (up to $\underline{\mathcal{E}}^1_U$-torsor automorphism).*

The analytic and hermitian connective structures on the 2*-gerbe* $\big\{L, L'\big\}$ *are compatible.*

Proof. The statement follows at once from the calculations preceding the proposition. □

Remark 5.7.5. As an alternative proof of the corollary, note that a calculation analogous to that of the proof of proposition 4.4.4 from the cocycle representations (5.1) and (5.3), yields the 1-cocycle $r_2(f, g_{ij})$ with values in $\underline{\mathcal{E}}^1_X(1)$, where g_{ij} are the transition functions of L. This cocycle represents the zero class (softness of $\underline{\mathcal{E}}^1_X(1)$), therefore $r_2(f, g_{ij}) = \eta_j - \eta_i$, and this choice is determined up to a global section of $\underline{\mathcal{E}}^1_X(1)$.

Similarly, in the case of $\big\{L, L'\big\}$ we get the 2-cocycle $r_2(g_{ij}, g'_{jk})$ which again represents the zero class.

6 Concluding remarks

In this paper we have put forward a definition for the concept of hermitian structure, and associated compatible connective structure for gerbes and 2-gerbes with band $\mathcal{O}_X^\times$. We have presented classification results in terms of low degree hermitian holomorphic Deligne cohomology groups. Notable examples are provided by higher versions of the classical notion of tame symbol associated to two invertible functions. Indeed, our second main result that there exists a modified version of the cup product in low degree Deligne cohomology taking values in the first hermitian holomorphic Deligne complex, naturally provides the symbols $\big(f, L\big]$ and $\big(L, L'\big]$ with hermitian structures according to our definition.

Two questions naturally arise. Since $\big(f, L\big]$ and $\big(L, L'\big]$ also carry an analytic connective structure, we may ask to what degree the latter and the hermitian one are compatible. Remark 5.3.5 prompts a second obvious question regarding the relation between our classification theorems 5.3.3 and 5.5.5 and others', notably Brylinski's ([9, Proposition 6.9 (1)]).

We have analyzed the compatibility in cohomological terms, first for line bundles (in the sense of $\mathcal{O}_X^\times$-torsors) and then for gerbes and 2-gerbes with band-$\mathcal{O}_X^\times$, with somewhat surprising results. Whereas the compatibility may be regarded as exceptional for a line bundle—and it implies its flatness—it is not so for gerbes (or 2-gerbes). Thus flatness is not a necessary condition. In the specific case of the tame symbols and their generalizations, we have found that while the compatibility of $\big(f, g\big]$ and $\big(f, g\big]_{h.h.}$ (that is, their respective connections) may in general be obstructed, $\big(f, L\big]$ and $\big(f, L\big]_{h.h.}$ can always be made compatible, and $\big(L, L'\big]$ and $\big(L, L'\big]_{h.h.}$ are automatically so.

As for the relation with other notions of "hermitian gerbe" with "hermitian connective structure" (or 2-gerbe) there appear to be subtle differences in the definitions which we can trace to what aspect of line bundles with connection we decide to generalize. Our approach has been to copy the concept of *metrized analytic (or algebraic) line bundle* familiar from Arakelov geometry (cf. ref. [21]). On the other hand, one could describe a metrized $\mathcal{O}_X^\times$-line bundle by means of the $\mathbb{T}$-reduction of its associated smooth line bundle plus a unitary connection. Whereas these two approaches are equivalent in the case of line bundles, they seem to diverge as soon as we move on to gerbes. (And possibly matters worsen in the case of 2-gerbes.) This may also serve to explain the lack of uniqueness found by Hitchin's student D. Chatterjee in his thesis. Although that school's approach to gerbes lacks the categorical input (in fact for them a gerbe is just the "torsor cocycle" in the sense of [7]) the definition of hermitian gerbe is along Brylinski's lines.

Another difference is the following. Our cohomological characterization via the group $H^k_{\mathcal{D}_{h.h.}}(X, 1) \cong \mathbf{H}^k(X, D(1)^\bullet_{h.h.})$, $k = 3, 4$, involves forms of degree two, which points to a natural notion of curving naturally associated with the structures we have defined (cf. remarks 5.3.6 and 5.3.7). This is obviously absent in the truncated group in remark 5.3.5. The cohomological analysis of sect. 5.7, where the group $H^4_{\mathcal{D}_{h.h.}}(X, 2)$ appears, suggests that curvings can be a very nuanced structure, however dealing with them in detail falls outside the scope of the present work.

We hope to further elucidate matters in the future in another publication.

A Remarks on Hodge-Tate structures

The relation between the "imaginary part" map made in sect. 4.3 together with the product $\mathbb{Z}(1)^\bullet_{\mathcal{D}} \otimes \mathbb{Z}(1)^\bullet_{\mathcal{D}} \to 2\pi\sqrt{-1} \otimes D(1)^\bullet_{h.h.}$, and the cup product $\mathbb{Z}(1)^\bullet_{\mathcal{D}} \otimes \mathbb{Z}(1)^\bullet_{\mathcal{D}} \to \mathbb{Z}(2)^\bullet_{\mathcal{D}}$ giving rise to the tame symbol becomes more transparent from the point of view of Hodge-Tate structures.

A.1 A Mixed Hodge Structure

Let us briefly recall the following well known MHS on $\mathbb{C}^3$, see [13, 4]. Consider, as before,

$$\text{(A.1)} \qquad M^{(2)} = \begin{pmatrix} 1 & & \\ x & 1 & \\ z & y & 1 \end{pmatrix}$$

with complex entries x, y, z. Consider also its canonical version

$$\text{(A.2)} \qquad A^{(2)} = \begin{pmatrix} 1 & & \\ x & 2\pi\sqrt{-1} & \\ z & 2\pi\sqrt{-1}\, y & (2\pi\sqrt{-1})^2 \end{pmatrix}.$$

The MHS $\mathscr{M}_2$ corresponding to $M^{(2)}$, or more precisely $A^{(2)}$, comprises the following data. The integer lattice is the $\mathbb{Z}$ span of the columns of $A^{(2)}$, and similarly for $\mathbb{Q}$ and $\mathbb{R}$. Let v_0, v_1, v_2 denote the columns of $A^{(2)}$ starting from the left. The weight spaces are $W_{-2k}\mathscr{M}^{(2)} = \operatorname{span}\langle v_k, \dots, v_2\rangle$ (over the appropriate ring), and the Hodge filtration is given by $F^{-k}\mathscr{M}^{(2)}(\mathbb{C}) = \mathbb{C}\langle e_0, \dots, e_k\rangle$, where the e_i's are the standard basis vectors in $\mathbb{C}^2$. The graded quotients $\operatorname{Gr}^W_{-2k}\mathscr{M}^{(2)}$ are the Tate structures $\mathbb{Z}(0)$, $\mathbb{Z}(1)$, and $\mathbb{Z}(2)$. A change of the generators v_i preserving the structure clearly amounts to a change of $A^{(2)}$ by right multiplication by a lower unipotent matrix over $\mathbb{Z}$ (or $\mathbb{Q}$ or $\mathbb{R}$). This is the same as changing $M^{(2)}$ by a matrix in $H_{\mathbb{Z}}$ (or the appropriate ring thereof) as in sect. 4.2. [3]

The real structure underlying $\mathscr{M}^{(2)}$ is linked to the hermitian structure on the bundle $H_{\mathbb{C}}/H_{\mathbb{Z}}$ as presented in sect. 4.3.1. In [4] the image of $A^{(2)}$ in $\mathrm{GL}_2(\mathbb{C})/\mathrm{GL}_2(\mathbb{R})$ is obtained by computing the matrix

$$B \overset{\text{def}}{=} A\bar{A}^{-1} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} ,$$

(we have dropped the superscript (2) for ease of notation). The logarithm is:

$$\frac{1}{2}\log B = \begin{pmatrix} 1 & & \\ \pi_0(x) & 1 & \\ \pi_1(z) - \pi_1(x)\pi_0(y) & \pi_0(y) & 1 \end{pmatrix} .$$

We immediately recognize the expression of the hermitian form as given in sect. 4.3.1.

A.2 The big period

In ref. [17] Goncharov defines a tensor

$$P(\mathscr{M}) \in \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$$

associated to a MHS (technically, a framed one) $\mathscr{M}$. For the MHS defined by the period matrix (A.1) it is computed as follows. Let f_0, f_1, f_2 be the dual basis to v_0, v_1, v_2. Then, according to ref. [17],

$$P(\mathscr{M}^{(2)}) = \sum_k \langle f_2, M^{(2)} v_k\rangle \otimes_{\mathbb{Q}} \langle f_k, {M^{(2)}}^{-1} v_0\rangle .$$

Performing the calculation we find:

$$\begin{aligned} P(\mathscr{M}^{(2)}) = {} & \frac{z}{(2\pi\sqrt{-1})^2} \otimes 1 - 1 \otimes \frac{z}{(2\pi\sqrt{-1})^2} \\ & + 1 \otimes \frac{xy}{(2\pi\sqrt{-1})^2} - \frac{y}{2\pi\sqrt{-1}} \otimes \frac{x}{2\pi\sqrt{-1}} \end{aligned} \tag{A.3}$$

Clearly, $P(\mathscr{M}^{(2)})$ is invariant under the action (4.5) (over $\mathbb{Q}$). Moreover, $P(\mathscr{M}^{(2)})$ belongs to the kernel $\mathscr{I}$ of the multiplication map $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C}$. As a consequence, we have:

Proposition A.2.1. *The "connection form" (4.6) and the (logarithm of the) hermitian fiber metric on the Heisenberg bundle correspond to the images of $P(\mathscr{M}^{(2)})$ under the two projections*

$$\mathscr{I} \longrightarrow \mathscr{I}/\mathscr{I}^2 = \Omega^1_{\mathbb{C}/\mathbb{Q}}$$

and

$$\mathscr{I} \subset \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathbb{R}(1) ,$$

respectively.

Proof. The images under the two projections are, respectively, equal to

$$-d\Big(\frac{z}{(2\pi\sqrt{-1})^2}\Big) + \frac{x}{2\pi\sqrt{-1}}\, d\Big(\frac{y}{2\pi\sqrt{-1}}\Big)$$

and

$$\frac{1}{(2\pi\sqrt{-1})^2}\big(\pi_1(z) - \pi_1(x)\,\pi_0(y)\big) .$$

□

[3]These data correspond to the case $N = 2$ of a MHS on $\mathbb{C}^N$ defined for any integer N, cf. [4]

A.3 The extension class

The big period can be obtained as a symmetrization of an extension class of MHS. Indeed, the weight -2 subspace $W_{-2}\mathscr{M}^{(2)} \cong \mathscr{M}^{(1)} \otimes 2\pi\sqrt{-1} \equiv \mathscr{M}^{(1)}(1)$ is itself a MHS (twisted by $2\pi\sqrt{-1}$) defined by

$$A^{(1)} = \begin{pmatrix} 1 & \\ y & 2\pi\sqrt{-1} \end{pmatrix} . \tag{A.4}$$

(The data are as for $\mathscr{M}^{(2)}$, replacing 2 by 1.) We thus have an extension of MHS:

$$0 \longrightarrow \mathscr{M}^{(1)}(1) \longrightarrow \mathscr{M}^{(2)} \longrightarrow \mathbb{Z}(0) \longrightarrow 0 . \tag{A.5}$$

Following the procedure explained in ref. [6], it is seen that the class of the extension (A.5) belongs to

$$\mathscr{M}^{(1)}_{\mathbb{C}}(1)/\mathscr{M}^{(1)}_{\mathbb{Q}}(1) ,$$

and it is given by the vector

$$e = -\frac{x}{2\pi\sqrt{-1}}\, v_1 - \frac{z - xy}{(2\pi\sqrt{-1})^2}\, v_2 \tag{A.6}$$

taken modulo $\mathscr{M}^{(1)}_{\mathbb{Q}}$. This computation can be refined by noticing ([6]) that $\mathscr{M}^{(1)}$ is itself an extension,

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathscr{M}^{(1)} \longrightarrow \mathbb{Z}(0) \longrightarrow 0$$

mapping (over $\mathbb{Q}$) to the "universal extension" $\mathscr{H}^{(1)}$:

$$0 \longrightarrow \mathbb{Q}(1) \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^\times \otimes \mathbb{Q} \longrightarrow 0 \tag{A.7}$$

obtained by tensoring the standard exponential sequence by $\mathbb{Q}$. Over the complex numbers, we have

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathbb{C}^\times \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}/\mathbb{Q}(1) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow 0 ,$$

Here we have $\mathscr{H}^{(1)}_{\mathbb{Q}} = \mathbb{C}$ and $\mathscr{H}^{(1)}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$. According to the same principle the class of the extension (A.7) lives in

$$\mathscr{H}^{(1)}_{\mathbb{C}}/\mathscr{H}^{(1)}_{\mathbb{Q}} \cong \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}/\mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{C}^\times . \tag{A.8}$$

The image of (A.6) in $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$ is given by

$$\tilde{e} = -y \otimes x - 2\pi\sqrt{-1} \otimes \frac{z - xy}{2\pi\sqrt{-1}} . \tag{A.9}$$

Taking (A.9) modulo $\mathscr{H}^{(1)}_{\mathbb{Q}} \cong \mathbb{C}$ we finally have

$$(\mathrm{Id} \otimes \exp)(\tilde{e}) = y \otimes e^{-x} + 2\pi\sqrt{-1} \otimes e^{-(z-xy)/2\pi\sqrt{-1}} . \tag{A.10}$$

This is the (image of) the class of the extension (A.5) as computed in ref. [6]. It is easily seen that the element (A.10) is invariant under the transformations (4.5).

Lemma A.3.1. *There is a unique well defined lift of the class* (A.10) *to* $F^0\mathscr{H}^{(1)}_{\mathbb{C}} = \ker(m\colon \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C})$. *This can be obtained by adding to* (A.9) *a (necessarily unique, see ref. [6]) element from* $\mathscr{H}^{(1)}_{\mathbb{Q}} \cong \mathbb{C}$ *to* (A.9). *The lift is*

$$2\pi\sqrt{-1} \otimes 2\pi\sqrt{-1} \cdot P(\mathscr{M}^{(2)}) .$$

Proof. We can identify $\mathscr{H}^{(1)}_{\mathbb{Q}} \cong \mathbb{C}$ inside $\mathscr{H}^{(1)}_{\mathbb{C}}$ via $a \mapsto a \otimes 2\pi\sqrt{-1}$. Thus add any such element to $\tilde{e}$ and consider the image under the multiplication map:

$$m(\tilde{e} + a \otimes 2\pi\sqrt{-1}) = -z + 2\pi\sqrt{-1}a .$$

It is equal to zero iff $a = z/2\pi\sqrt{-1}$, hence

$$\begin{aligned} \tilde{\tilde{e}} &= \tilde{e} + \frac{z}{2\pi\sqrt{-1}} \otimes 2\pi\sqrt{-1} \\ &= -y \otimes x + 2\pi\sqrt{-1} \otimes \frac{xy}{2\pi\sqrt{-1}} + \frac{z}{2\pi\sqrt{-1}} \otimes 2\pi\sqrt{-1} - 2\pi\sqrt{-1} \otimes \frac{z}{2\pi\sqrt{-1}} \end{aligned}$$

is the required element. $\square$

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