

AMS FALL SOUTHEASTERN

OCTOBER 10-11, 2020

CATEGORICAL GROUPS,
THEIR MORPHISMS, AND
HIGHER ALGEBRAIC STRUCTURES

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PLAN

1. General Def's of Categorical Groups & morphisms
2. Crossed Modules & Presentations
3. Postnikov Invariants
4. Classification / Homotopy Category
5. Commutative Structures
6. An Application
7. Categorical Rings

Categorical Groups

Definition of a categorical group \mathcal{C} . Axioms:

- \mathcal{C} is a (small) groupoid (Standing assumption)
- group-like monoidal structure:

1. $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ multiplication

2. $I : * \longrightarrow \mathcal{C}$

3. $(-)^{-1} : \mathcal{C} \longrightarrow \mathcal{C}$

$$\begin{array}{ccc}
 & (\text{Id}, (-)^{-1}) & ((-)^{-1}, \text{Id}) \\
 \mathcal{C} & \longrightarrow \mathcal{C} \times \mathcal{C} & \longleftarrow \mathcal{C} \\
 \downarrow I & \downarrow \sigma & \downarrow \sigma \\
 & * & \\
 \uparrow I & & \uparrow I
 \end{array}$$

$$X \otimes X^{-1} = I = X^{-1} \otimes X$$

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \xrightarrow{(\otimes, \text{Id})} \mathcal{C} \times \mathcal{C}$$

$$\begin{array}{ccc}
 \text{Id}, \otimes & \downarrow & \downarrow \otimes \\
 \mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C}
 \end{array}$$

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$

unit object

inversion

associativity

$$\mathcal{C} \xrightarrow{(I, \text{id})} \mathcal{C} \times \mathcal{C} \xleftarrow{(\text{Id}, I)} \mathcal{C}$$

$$\begin{array}{ccc}
 & \downarrow \sigma & \downarrow \sigma \\
 \text{Id} & & \text{Id} \\
 & \downarrow \otimes & \downarrow \otimes \\
 & \mathcal{C} &
 \end{array}$$

$$I \otimes X = X = X \otimes I$$

STRICT

Categorical Groups

Definition of a categorical group \mathcal{C} . Axioms:

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$$\begin{array}{ccc} & (\text{Id}, (-)^{-1}) & ((-)^{-1}, \text{Id}) \\ \mathcal{C} & \longrightarrow \mathcal{C} \times \mathcal{C} & \longleftarrow \mathcal{C} \end{array}$$

$$\begin{array}{ccc} & \swarrow & \searrow \\ & \mathcal{C} & * \\ & \swarrow & \searrow \\ & \mathcal{C} & \mathcal{C} \end{array}$$

$$X \otimes X^{-1} \cong I \cong X^{-1} \otimes X$$

unit object
inversion
associativity

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \xrightarrow{(\otimes, \text{Id})} \mathcal{C} \times \mathcal{C}$$

$$\mathcal{C} \xrightarrow{(I, \text{id})} \mathcal{C} \times \mathcal{C} \xleftarrow{(\text{Id}, I)} \mathcal{C}$$

$$\begin{array}{ccc} \text{Id}, \otimes & \downarrow & \downarrow \otimes \\ \mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \end{array}$$

$$\begin{array}{ccc} & \swarrow & \searrow \\ & \mathcal{C} & * \\ & \swarrow & \searrow \\ & \mathcal{C} & \mathcal{C} \end{array}$$

+ Additional constraints

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$$

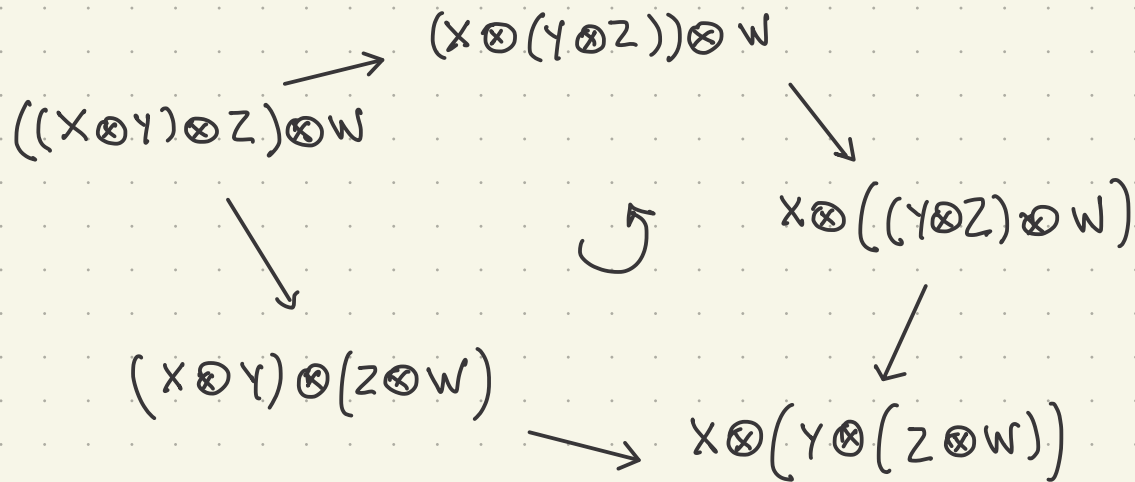
$$I \otimes X \cong X \cong X \otimes I$$

For
Example:

$$\begin{array}{ccc}
 e \times e \times e & \xrightarrow{(\otimes, \text{Id})} & e \times e \\
 \text{Id}, \otimes \downarrow & \alpha \swarrow & \downarrow \otimes \\
 e \times e & \xrightarrow{\otimes} & e \\
 (X \otimes Y) \otimes Z & \cong & X \otimes (Y \otimes Z)
 \end{array}$$

associativity

Famous Mac Lane's Pentagon:



+ Other diagrams for other data

In general, \mathcal{C} will always be assumed to be mon strict

Morphisms

\mathcal{C}, \mathcal{D} categorical groups. A morphism $\mathcal{C} \rightarrow \mathcal{D}$ is a

- monoidal functor $(F, \lambda) :$
1. $F: \mathcal{C} \rightarrow \mathcal{D}$ (underlying functor)
 2. Natural $\lambda_{x,y}: F(x) \otimes F(y) \rightarrow F(x \otimes y)$

Plus:

1. $F(I_{\mathcal{C}}) \xrightarrow{\sim} I_{\mathcal{D}}$
2. $F(x^{-1}) \xrightarrow{\sim} F(x)^{-1}$

Compatibly with the rest of the diagrams.

$$F(x) \otimes F(y) \rightarrow F(x \otimes y)$$

$$\downarrow \qquad \qquad \downarrow$$
$$F(x') \otimes F(y') \rightarrow F(x' \otimes y')$$

whenever $x \rightarrow x', y \rightarrow y'$
in \mathcal{C}

such that, for all
 $x, y, z \in \text{Ob } \mathcal{C}$

$$(F(x) \otimes F(y)) \otimes F(z) \rightarrow F(x \otimes y) \otimes F(z) \rightarrow F((x \otimes y) \otimes z)$$

$$\downarrow \qquad \qquad \qquad \qquad \qquad \qquad \downarrow$$
$$F(x) \otimes (F(y) \otimes F(z)) \rightarrow F(x) \otimes (F(y \otimes z)) \rightarrow F(x \otimes (y \otimes z))$$

Examples

- Let R be a comm. ring with 1. $\text{Pic}(R)^{\mathbb{Z}}$ is the (Picard) groupoid of "graded R -lines"

Objects: (n, L) , $n \in \mathbb{Z}$ and L is an invertible R -module

(Non STRICT)

Morphisms: $(n, L) \rightarrow (n', L')$ iff $n = n'$, $\alpha: L \cong L'$ (iso. of R -modules)

Monoidal structure:

$$(n, L) \otimes (n', L') = (n+n', L \otimes_R L')$$

CROSSED MODULES

Associated Cat. Group (STRICT)

NOTATION: $[C_1 \rightarrow C_0]^{\sim}$

A crossed Module:

* $C_1 \xrightarrow{\partial} C_0$ (group homomorphism)

* Action: $C_0 \times C_1 \rightarrow C_1$ $(x, a) \mapsto {}^x a$

* Axioms

1. $\partial({}^x a) = x(\partial a)x^{-1}$

2. $\partial a b = a b a^{-1}$

$x \in C_0, a \in C_1$
 $b \in C_1$

- From ∂ form an action groupoid

$$C_1 \times C_0 \begin{matrix} \xrightarrow{s} \\ \xrightarrow{t} \end{matrix} C_0 \quad (a, x) \begin{matrix} \mapsto x \\ \mapsto (\partial a)x \end{matrix}$$

- Monoidal structure (group like)

$$\begin{matrix} (a, x) & (b, y) & (a {}^x b, xy) \\ \downarrow & \downarrow & \downarrow \\ x & \otimes & y \\ & & \downarrow \\ & & xy \end{matrix} \quad \begin{matrix} C_1 \times C_0 \\ C_0 \end{matrix}$$

1. NORMAL SUBGROUPS

$$N \triangleleft G$$

SPLICE

$$\begin{matrix} E & \xrightarrow{\partial} & G \\ \downarrow & & \uparrow \\ & N & \end{matrix}$$

2. CENTRAL EXTENSIONS

$$A \hookrightarrow E \twoheadrightarrow N$$

$$\begin{matrix} \text{St}(R) & \rightarrow & \text{GL}(R) \\ \downarrow & & \uparrow \\ & & E(R) \end{matrix}$$

EVERY CAT. GROUP IS EQUIVALENT TO A STRICT ONE

(COMING FROM A CROSSED MODULE)

(FOLK THEOREM — BAUES, BREEN, ..., NOOHI — E.A., ...)

$$[C_1 \rightarrow C_0]^\sim \xrightarrow{\sim} \mathcal{C}$$

Presentation
of \mathcal{C}

• If \mathcal{C} is strict: $G_\bullet = N\mathcal{C}$, the nerve of \mathcal{C} , is a simplicial group: $\Delta^{op} \rightarrow \text{Grp}$ Functor

Take the Moore Complex $M_\bullet(G_\bullet)$: $M_m(G) = \bigcap_{0 < i \leq m} \ker(d_i)$
 $\partial = d_0$

CROSSED
MODULE
 $C_1 \rightarrow C_0$

• General \mathcal{C} : (Several ways, one will come up later). For now:

1. $\mathcal{C} \mapsto \Omega^{-1}\mathcal{C} = \text{bicategory w/ a single object } *$, $\text{Hom}_{\mathcal{B}}(*, *) = \mathcal{C}$
" \mathcal{B} ← suspension

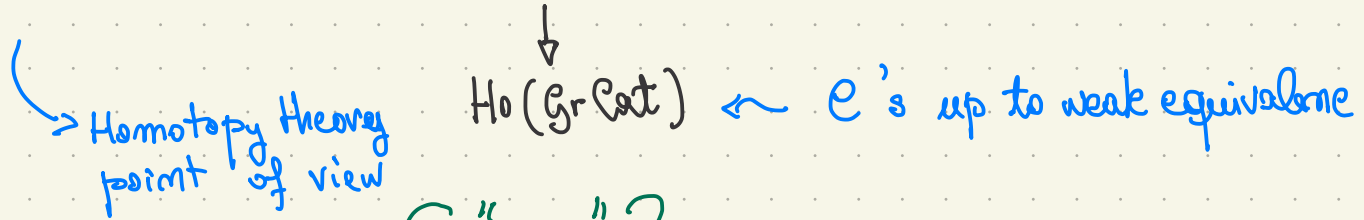
2. $N\mathcal{B}$: geometric nerve (rigidified bicat. nerve) $X := N\mathcal{B} \in \text{sSet}$, Reduced Simplicial Set

3. Loop functor: $G(X) \in \text{sGrp}$

4. Continue as above

Question: What do Categorical Groups form?

- Understanding $\text{GrCat} \subset \text{Cat}$ (as a 2-Category)



Dévissage
(Algebra)

"Exact" ?

$$\pi_1(e)[1] \rightarrow \mathcal{C} \xrightarrow{\omega} \pi_0(e)$$

$[A \rightarrow 1]^\sim$

$$B = \pi_0(e) = \text{ob } \mathcal{C} / \sim \text{ iso}$$

$$X \sim Y \text{ iff } X \xrightarrow{\cong} Y$$

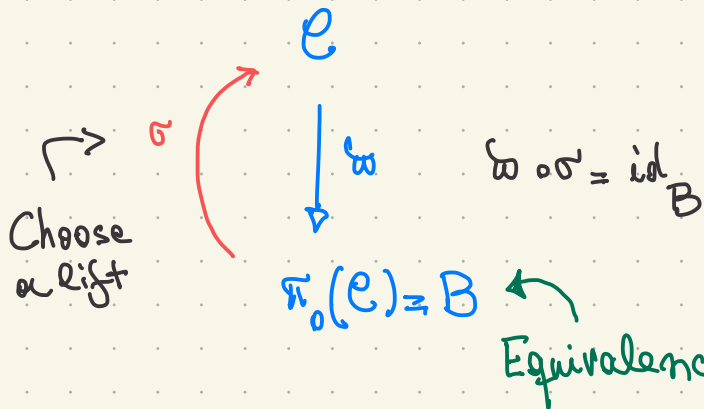
$$A = \pi_1(e) = \text{Aut}_{\mathcal{C}}(I) \text{ (Abelian)}$$

with a presentation

$$0 \rightarrow \overset{\text{ker } \partial}{A} \rightarrow \overset{\partial}{C_1} \rightarrow \overset{\text{coker } \partial}{C_0} \rightarrow B \rightarrow 1$$

Exact on the nose

- Classification of these sequences \leadsto POSTNIKOV INVARIANT $k \in H^3(B, A)$
(HOÀNG XUÂN SINH - 1975)



$$\sigma(x) \otimes \sigma(y) \longrightarrow \sigma(xy)$$

$$(\sigma(x) \sigma(y)) \sigma(z) \longrightarrow \dots \longrightarrow \sigma(xyz)$$

associator $\downarrow \alpha$

$$\sigma(x) (\sigma(y) \sigma(z)) \longrightarrow \dots \longrightarrow \sigma(xyz)$$

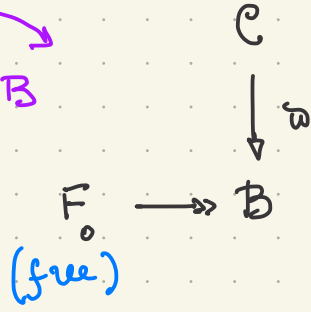
$\downarrow k$

Using the Postnikov Invariant to find a presentation of \mathcal{C}

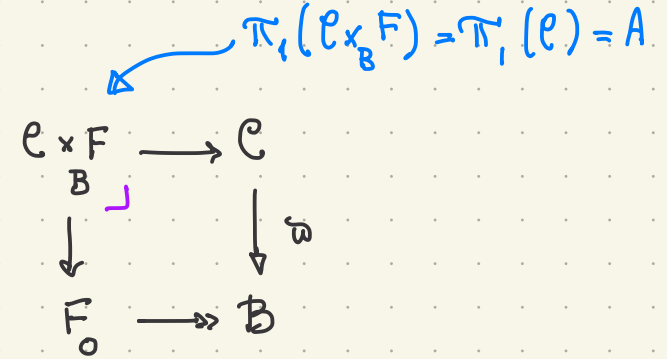
Start:



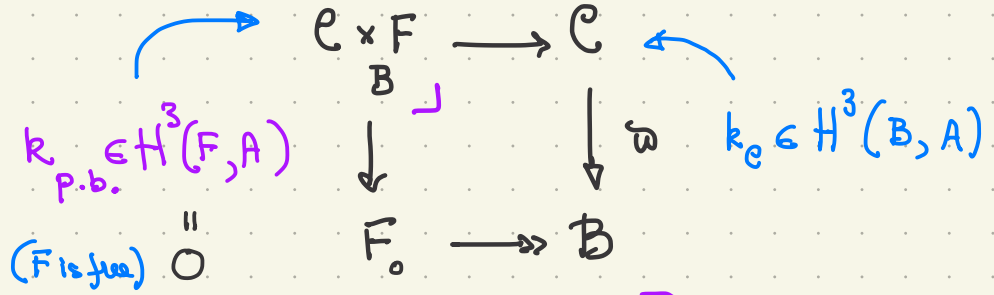
resolve \mathcal{B}



pull back

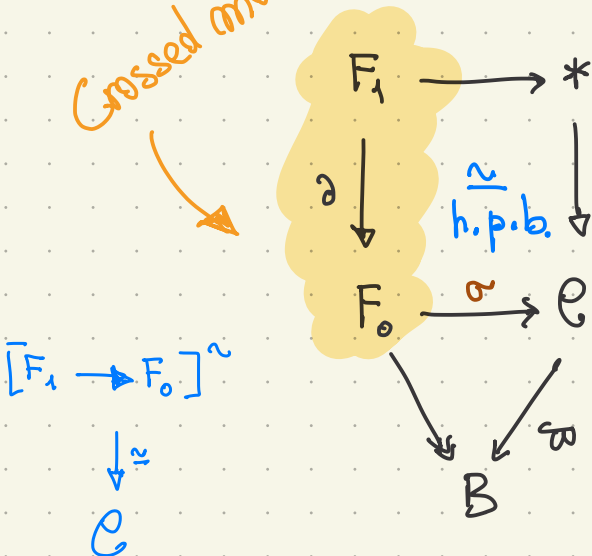


Postnikov inv't's



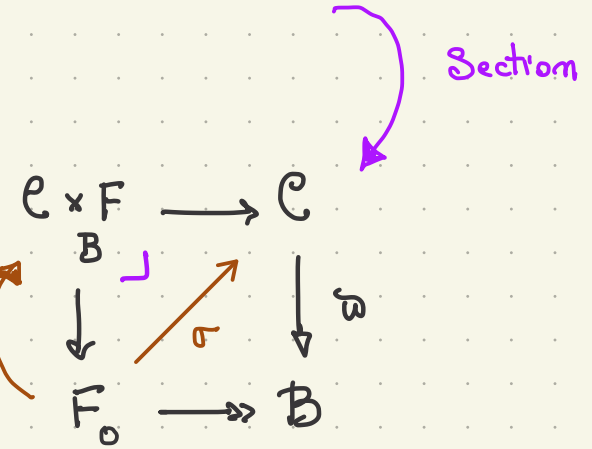
exact up to homotopy

Crossed module



pullback * along σ

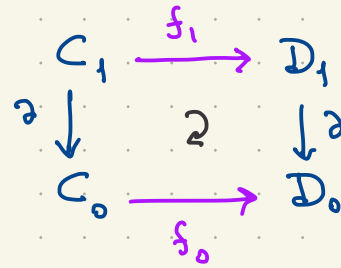
Trace Homoidal functor



Question: What do Categorical Groups form? II

• How to describe morphisms $F: \mathcal{C} \rightarrow \mathcal{D}$ (say, in terms of presentations)

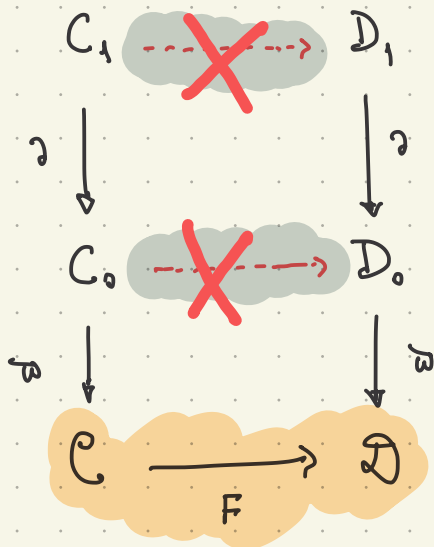
• We need: morphisms of crossed modules



Equivariant

$$f_1(x \cdot a) = f_0(x) \cdot f_1(a)$$

• However, given $F: \mathcal{C} \rightarrow \mathcal{D}$

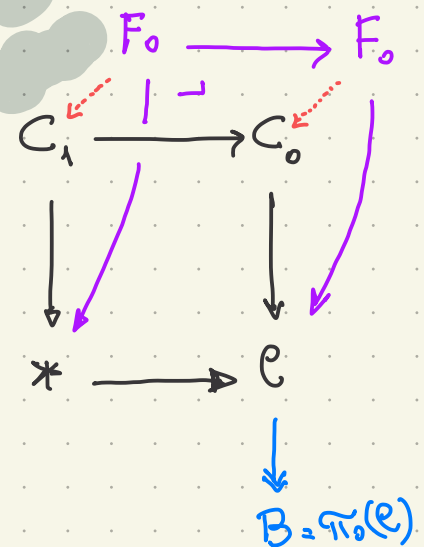
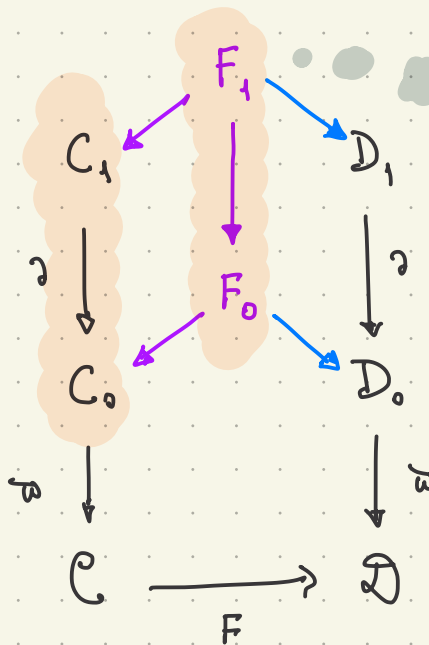


Get a weak equivalence

Resolve

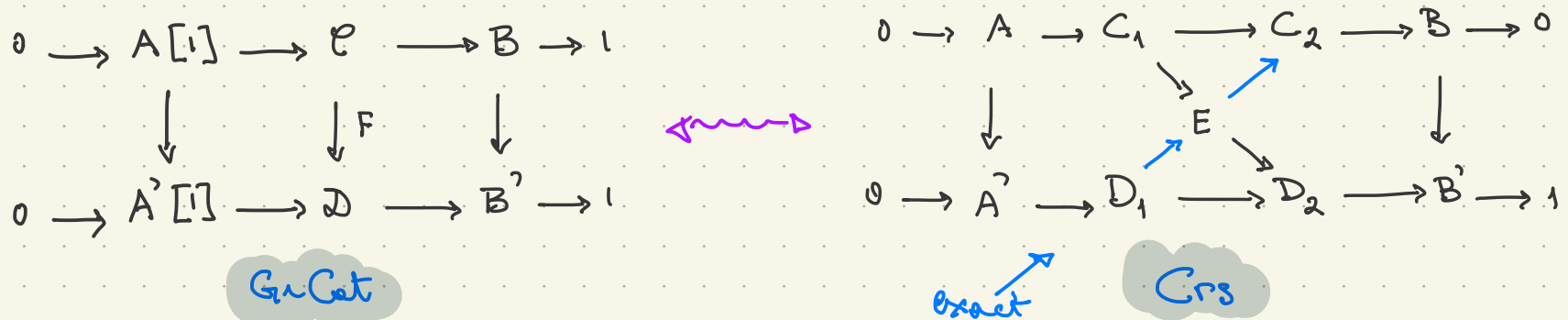
$C_1 \rightarrow C_0$

(as before)



Question: What do Categorical Groups form?

Rewrite: if $F: \mathcal{C} \rightarrow \mathcal{D}$, it induces $\pi_i(F): \pi_i(\mathcal{C}) \rightarrow \pi_i(\mathcal{D})$ $i=0,1$



Thm $[C_1 \rightarrow C_0] \sim \leftarrow C_1 \rightarrow C_0$

(Noohi, E.A.) is an equivalence of bicategories

Cor There is an equivalence of categories $\text{Ho}(\text{GrCat}) \xleftarrow{\sim} \text{Ho}(\text{Crs})$

Classification Thm

- \mathcal{C}, \mathcal{D} are equivalent: $F: \mathcal{C} \xrightarrow{\sim} \mathcal{D}$
- \mathcal{C}, \mathcal{D} have the same $k \in H^3(B, A)$
- The presentations are linked by a diagram with exact diagonals

(set $A' = A, B' = B$
in the diagrams)

PLUS:

- Long exact sequence in non abelian cohomology
- Extensions
- ...

COMMUTATIVE STRUCTURES & STABILITY

Return to **cat groups** in general. There are **different levels** of **commutativity**

- **Braiding**: commutativity isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$, which is natural

$$[(k, c)] \in H^4(K(B,2), A) \xrightarrow{\text{suspension}} H^3(K(B,1), A) = H^3(B, A) \ni k$$

Postnikov

Eilenberg-MacLane
cohomology for
Abelian Groups

- **Symmetric Braiding** or **Picard**:

$$X \otimes Y \xrightarrow{c_{X,Y}} Y \otimes X \xrightarrow{c_{Y,X}} X \otimes Y = \text{id} \Rightarrow I = II$$

$$[(k, c)] \in H^5(K(B,3), A) \xrightarrow{\text{Susp}} H^4(K(B,2), A) \xrightarrow{\text{Susp}} H^3(B, A) \ni k = 0!$$

5 - 3 = 2 < 3 STABLE!

$$\text{Hom}(B/{}_2B, A) \cong \text{Hom}(B, {}_2A)$$

Underlying Postnikov
invariant vanishes
 $\mathcal{C} \simeq K(B,0) \times K(A,1)$ as a
categorical group.

- **Strictly Commutative** or **Strictly Picard** $c_{X,X} = \text{id}$

$$[(k, c)] \in \text{Ext}^3(B, A) = 0$$

$\leadsto \mathcal{C}$ is trivial

COMMUTATIVE STRUCTURES & STABILITY FOR CROSSED MODULES

Let $C_1 \xrightarrow{\partial} C_0$ be a crossed module. A Braiding corresponds to:

$$\{ \cdot, \cdot \} : C_0 \times C_0 \rightarrow C_1 \quad \text{such that} \quad \partial \{x, y\} = [x, y] = x y x^{-1} y^{-1}$$

(Almost a) lift of the commutator map

$$y x \xrightarrow{\{x, y\}} x y$$

A Symmetric Braiding or Picard crossed module: $\{x, y\} = \{y, x\}^{-1}$

$(C_1 \xrightarrow{\partial} C_0, \{ \cdot, \cdot \})$ STABLE 2-MODULE class $[(C_0, C)] : \pi_0 C_0 \otimes \mathbb{Z}/2 \rightarrow \pi_1(C_0)$

$$x \otimes 1 \mapsto \{x, x\}$$

CAUTION: C is "commutative" and

A, B are abelian, however,

C_1, C_0 are only Nil₂-class: $[x, [y, z]] = 0$

• Strictly Commutative or Strictly Picard $C_{x,x} = \text{id}$ $(C_1 \rightarrow C_0, \{ \cdot, \cdot \})$ is a full lift

of the commutator map: $\{x, x\} = 1$

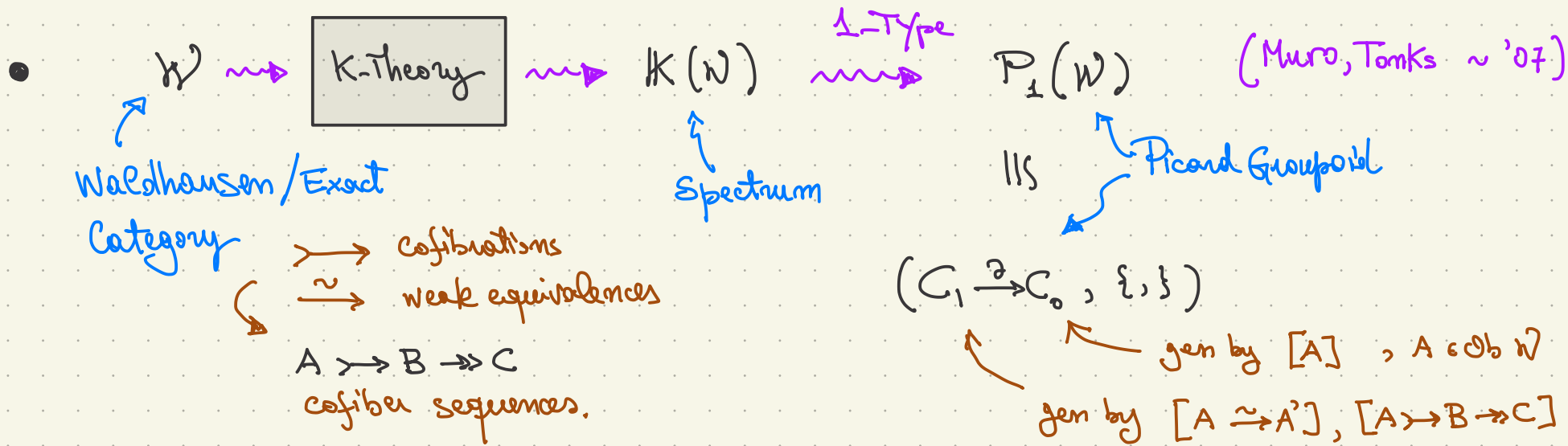
$$(C_1 \rightarrow C_0, \{ \cdot, \cdot \}) \simeq (Z_1 \xrightarrow{\partial} Z_0)$$

homomorphism
of abelian groups.

SKETCH OF APPLICATION

Determinants and $\mathbb{1}$ -Types

$\mathbb{1}$ -Type of the K-Theory Spectrum.



• \mathcal{W} 's are known to form a **Multicategory WALD**
 (Zakharovich, '15; Elmendorf-Mandell, '09)

• We (E.A., Y. Valdes) proved

Thm $\mathbb{P}_{\mathbb{1}}$ extends to a **multifunctor**

$$\text{WALD} \rightarrow \text{Pic}$$

\uparrow
 morphism of multicategories

Multicategory Picard groupoids

• Muro & Tomks: $\mathbb{P}_{\mathbb{1}}$ is multiplicative for blexact functors
 $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$
 of Waldhausen Categories.

HIGHER STRUCTURES ~ CATEGORICAL RINGS ~ 2-RINGS

- **PICARD** (Braided Symmetric) Groupoid $(\mathcal{R}, \oplus, 0)$ "underlying Abelian group of a ring"

- A biexact functor $\otimes : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ with respect to \oplus :

$$* (x \oplus y) \otimes z \xrightarrow{\sim} (x \otimes z) \oplus (y \otimes z)$$

$$* x \otimes (y \oplus z) \xrightarrow{\sim} (x \otimes y) \oplus (x \otimes z)$$

$$* (x \oplus y) \otimes (z \oplus w) \longrightarrow (x \otimes (z \oplus w)) \oplus (y \otimes (z \oplus w))$$

BIKONOIDAL

$$\downarrow$$

$$((x \oplus y) \otimes z) \oplus ((x \oplus y) \otimes w)$$

$$\downarrow$$

$$(x \otimes z) \oplus (y \otimes z) \oplus (x \otimes w) \oplus (y \otimes w) \longrightarrow ((x \otimes z) \oplus (x \otimes w)) \oplus ((y \otimes z) \oplus (y \otimes w))$$

Same diagram as in Mumford, Grothendieck

BIEXTENSIONS

MacLane's Commutator-associator

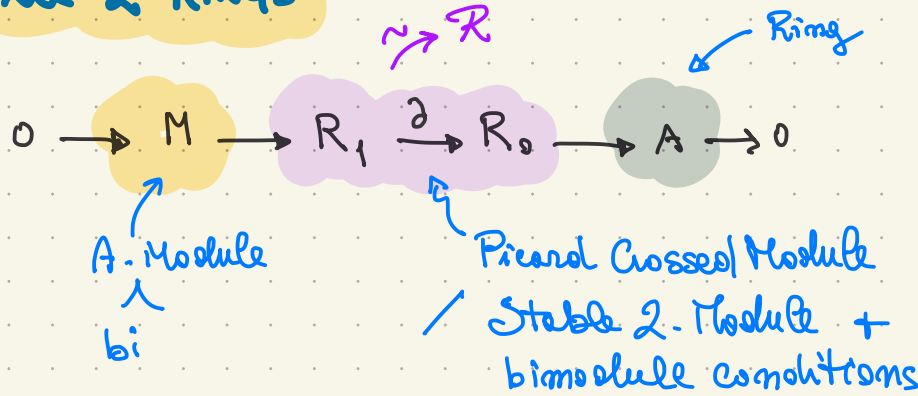
- * A host of other (less interesting) diagrams

CATEGORICAL RINGS ~ 2-RINGS

$$A = \pi_0(\mathcal{R}) = \text{Obj } \mathcal{R} / \sim$$

$$M = \pi_1(\mathcal{R}) = \text{Aut}_{\mathcal{R}}(0)$$

* Presentations



* Classification Theorems

$R_1 \rightarrow R_0$ (or \mathcal{R}) is Strictly Picard \rightarrow

$k \in H^3(A, M) \cong \mathbb{L}^2 \text{Der}(A, M)$
 André Quillen - Shukla Cohomology

$R_1 \rightarrow R_0$ (or \mathcal{R}) is Picard

(Barnes-Pirashvili, N.T. Quang, N.T. Quang-Hanh, E.A.)

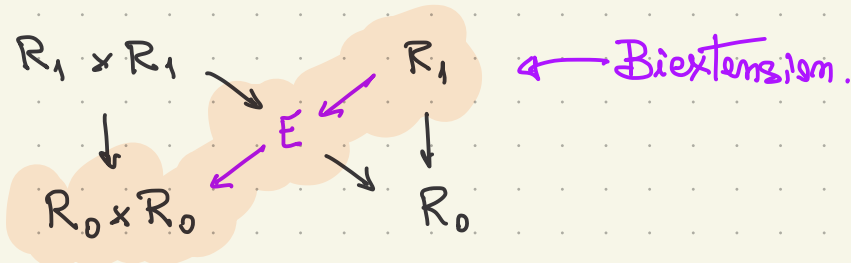
The difference is the k -invariant of the underlying Picard Groupoid

$k \in \text{ML}^3(A, M) \cong \text{THH}^3(A, M)$ Mochane Cohomology

Top. Hochschild
 (Barnes-Pirashvili, Sibblaze-Pirashvili E.A.)

(Almost Hochschild cohomology of A with coefficients in M)

* Thm (EA '17) The 2nd monoidal functor $\otimes : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$



Remark: $HH^2(A, M) =$ Iso. classes of "singular" extensions of
 A (k -algebra) by M , where M is an A -bimodule

Fix a common
ground ring k

$$0 \rightarrow M \rightarrow E \xrightarrow{P} A \rightarrow 0$$

$$\uparrow$$

$E, A \in k\text{-Alg}$

A -bimodule

such that

$$M^2 = 0$$

such that the underlying k -Mod. short exact
sequence is split.

Remark II

$HH^3(A, M)$

same for

$$0 \rightarrow M \rightarrow E_1 \rightarrow E_0 \rightarrow A \rightarrow 0$$

WHERE TO USE & TO FIND CATEGORICAL RINGS ~ 2-RINGS

- Usually, you have **2RINGS** not 2Rings : Example $\text{Vect}_k^{\text{fin}}$ with \oplus, \otimes

(Vect_k, \oplus) is monoidal, but it lacks an additive inverse.

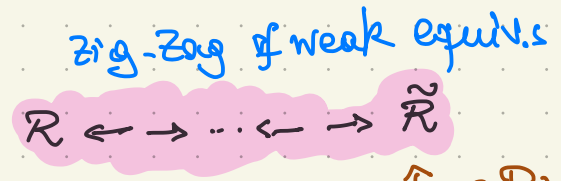
R. THOMASON "Beware of the phony multiplication..."

Usually, you can't apply the Grothendieck completion to the underlying monoidal category $(\mathcal{R}, \oplus, 0)$ without harm.

unsolved

BAAS, DUNDAS, ROGNES, ... (~ '11)

Ring Category:



2Ring

Do k-Theory!

Tensor TRIANGULATED CATEGORY (BALMER)

* \mathcal{T} : triangulated category w/ $\Sigma: \mathcal{T} \rightarrow \mathcal{T}$ and small sums

* $\otimes: \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$
Symmetric Monoidal & biexact

* $(\Sigma X) \otimes Y \cong \Sigma (X \otimes Y)$

Two constructions (among others):

- Picard Groupoid of \otimes -invertible objects
- Determinant functors yield 2-Rings

WORKSHOP OCTOBER 11, 2020

Group extensions \leadsto Motivation: why Crossed Modules?

$$1 \longrightarrow K \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

short exact sequence
group homomorphisms

$$\ker p = \text{Im } i$$

Also consider the conj. action of E onto itself including an action

$$E \times K \longrightarrow K$$

$$(x, k) \longmapsto x i(k) x^{-1}$$

$$p(x i(k) x^{-1}) =$$

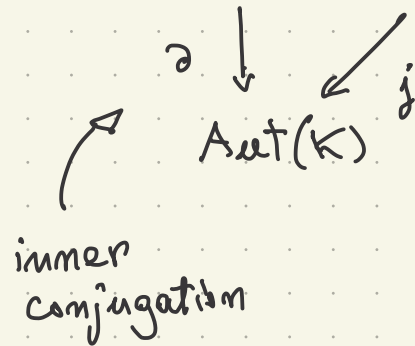
$$= p(x) p i(k) p(x)^{-1}$$

$$= 1$$

$$\leadsto j: E \longrightarrow \text{Aut}(K)$$

Rewrite :

$$1 \rightarrow K \xrightarrow{i} E \rightarrow G \rightarrow 1$$



Write:

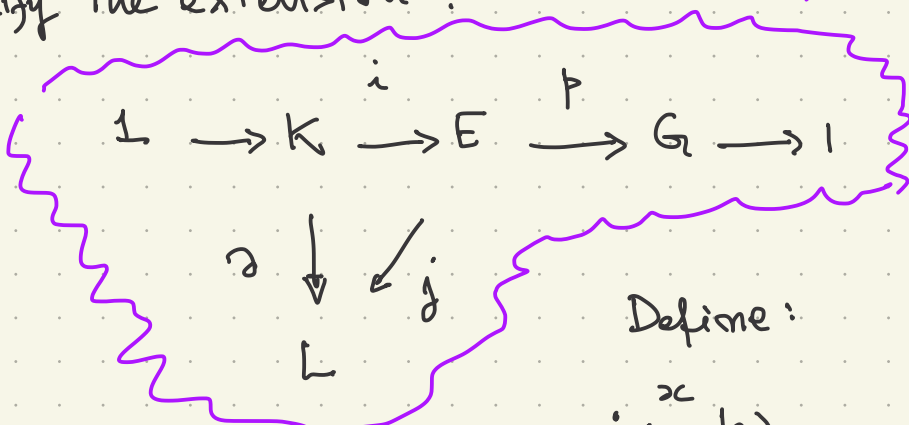
$${}^x k = x i(k) x^{-1} \in K$$

$$\begin{array}{c} \uparrow \\ \downarrow \\ j(x)(k) \end{array}$$

$x \in E$
 $k \in K$

Modify the extension :

Extension of G by a
Crossed
Module



Define:

$$i({}^x k) = x i(k) x^{-1} \in \text{im}(i) \subseteq E$$

$$\partial = j \circ i$$

$$\textcircled{1} \quad \partial({}^x k) = j i({}^x k) = j(x) j i(k) j(x)^{-1} \quad j(x) \in L$$

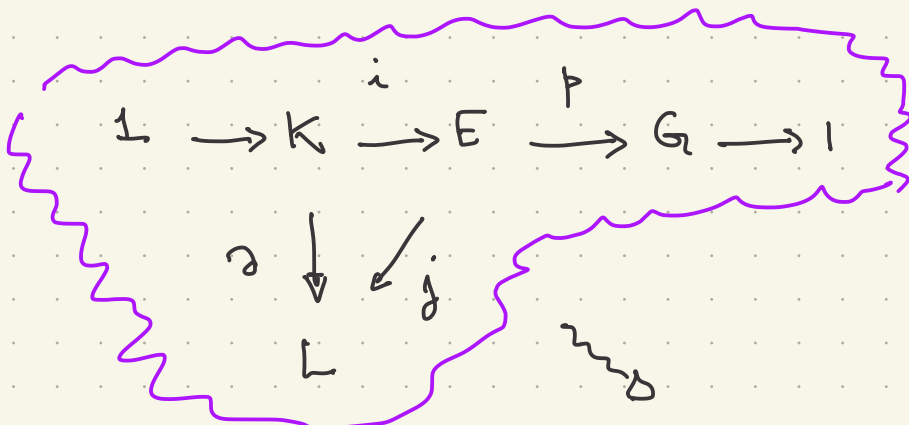
$$= j(x) \partial k j(x)^{-1}$$

$$\textcircled{2} \quad (*)$$

If $x = i(k')$

$$\begin{aligned} \partial k' &= j(i(k')) = i(k') i(k) i(k')^{-1} \\ &= i(k' k k'^{-1}) \end{aligned}$$

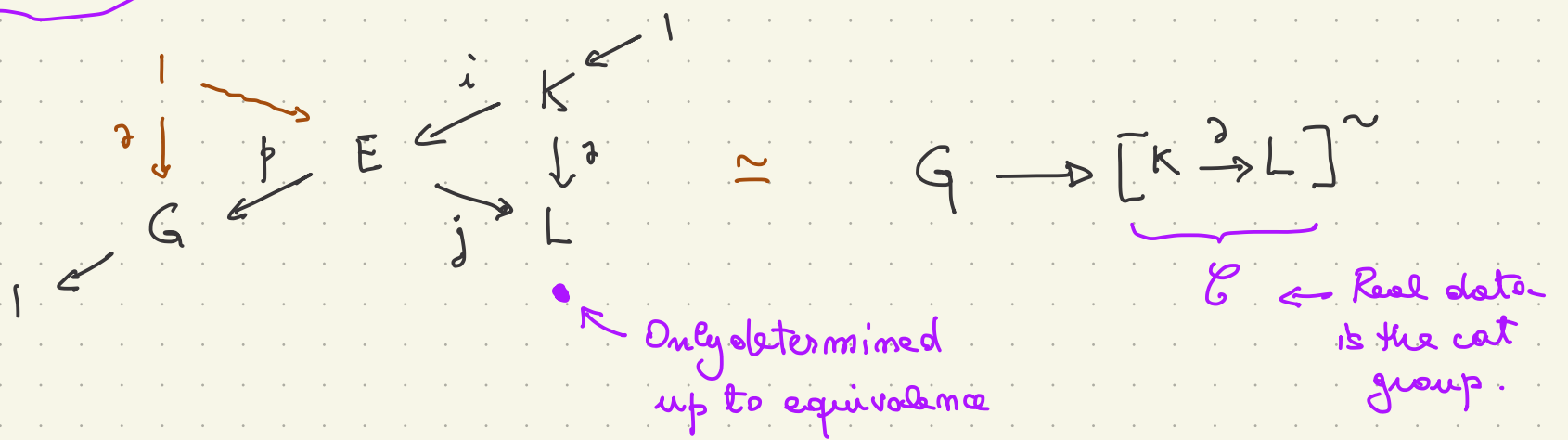
(*)



Crossed Modules \rightarrow Grp Cat

$$C_1 \xrightarrow{\alpha} C_0 \quad 1 \rightarrow [C_1 \times C_0 \rightrightarrows C_0]$$

$$\cong [C_1 \xrightarrow{\alpha} C_0]^\sim$$



Remark: Char Class of the extension

lives in $H_{gr}^2(G, "?") \leftarrow$ non abelian cohomology

homotopy classes.

$$H_{gr}^1(G, K \rightarrow L) \cong [K(G, 1), \mathcal{B}]$$

\Downarrow

Nerve of \mathcal{B} + suspension.