AMS Fall Southeastern

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Categorical Groups, their Morphisms, and Higher Algebraic Structures

Ettore Aldrovandi

Florida State University

PLAN

1. General Def's of Categorical Groups & morphisms
2. Crossed Modules & Presentations
3. Postnikov Invariants
4. Classification / Homotopy Category
5. Commutative Structures
6. An Application
7. Categorical Rings

Categorical Groups

Definition of a categorical group $\mathcal{C}$. Axioms:

- $\mathcal{C}$ is a (small) groupoid (Standing assumption)
- group-like monoidal structure:

1. $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ multiplication
2. $I : * \longrightarrow \mathcal{C}$ unit object
3. $(-)^{-1} : \mathcal{C} \longrightarrow \mathcal{C}$ inversion

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "{(Id, (-)^{-1})}"] \arrow[dr, "I"'] & \mathcal{C} \times \mathcal{C} \arrow[d] & \mathcal{C} \arrow[l, "{((-)^{-1}, Id)}"'] \arrow[dl, "I"] \\
& *
\end{tikzcd}
$$

$$X \otimes X^{-1} = I = X^{-1} \otimes X$$

associativity

$$
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "{(\otimes, Id)}"] \arrow[d, "{Id, \otimes}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C}
\end{tikzcd}
$$

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "{(I, id)}"] \arrow[dr, equal, "Id"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] & \mathcal{C} \arrow[l, "{(Id, I)}"'] \arrow[dl, equal, "Id"] \\
& \mathcal{C}
\end{tikzcd}
$$

$$I \otimes X = X = X \otimes I$$

STRICT

Categorical Groups

Definition of a categorical group $\mathcal{C}$. Axioms:

- $\mathcal{C}$ is a (small) groupoid (Standing assumption)
- group-like monoidal structure:

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2. $I : * \longrightarrow \mathcal{C}$ unit object
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associativity

$$
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "{(\otimes, \mathrm{Id})}"] \arrow[d, "{\mathrm{Id}, \otimes}"'] \arrow[dr, phantom, "\alpha"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C}
\end{tikzcd}
$$

$(X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z)$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "{(I, \mathrm{id})}"] \arrow[dr, equal, "\mathrm{Id}"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] & \mathcal{C} \arrow[l, "{(\mathrm{Id}, I)}"'] \arrow[dl, equal, "\mathrm{Id}"] \\
& \mathcal{C} &
\end{tikzcd}
$$

$I \otimes X \xrightarrow{\simeq} X \xleftarrow{\simeq} X \otimes I$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "{(\mathrm{Id}, (-)^{-1})}"] \arrow[dr, "I"'] & \mathcal{C} \times \mathcal{C} \arrow[d] & \mathcal{C} \arrow[l, "{((-)^{-1}, \mathrm{Id})}"'] \arrow[dl, "I"] \\
& * &
\end{tikzcd}
$$

$X \otimes X^{-1} \cong I \cong X^{-1} \otimes X$

+ Additional constraints

For Example:

associativity

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "(\otimes, \mathrm{Id})"] \arrow[d, "\mathrm{Id}, \otimes"'] & \mathcal{C}\times\mathcal{C} \arrow[d, "\otimes"] \arrow[dl, phantom, "\overset{\alpha}{\swarrow}"] \\
\mathcal{C}\times\mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C}
\end{tikzcd}$$

$$(X\otimes Y)\otimes Z \xrightarrow{\simeq} X\otimes(Y\otimes Z)$$

Famous **Maclane's Pentagon:**

$$\begin{tikzcd}
 & (X\otimes(Y\otimes Z))\otimes W \arrow[dr] & \\
((X\otimes Y)\otimes Z)\otimes W \arrow[ur] \arrow[d] & \circlearrowright & X\otimes((Y\otimes Z)\otimes W) \arrow[d] \\
(X\otimes Y)\otimes(Z\otimes W) \arrow[rr] & & X\otimes(Y\otimes(Z\otimes W))
\end{tikzcd}$$

+ Other diagrams for other data

In general, $\mathcal{C}$ will always be assumed to be **non strict**

Morphisms

$\mathcal{C}, \mathcal{D}$ categorical groups. A **morphism** $\mathcal{C} \longrightarrow \mathcal{D}$ is a **monoidal functor** (F, λ) : 1. $F: \mathcal{C} \longrightarrow \mathcal{D}$ (underlying functor)

2. **Natural** $\lambda_{X,Y}: F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$

$$\begin{array}{ccc} F(X) \otimes F(Y) & \longrightarrow & F(X \otimes Y) \\ \downarrow & & \downarrow \\ F(X') \otimes F(Y') & \longrightarrow & F(X' \otimes Y') \end{array}$$

whenever $X \to X'$, $Y \to Y'$ in $\mathcal{C}$

such that, for all $X, Y, Z \in \mathrm{ob}\,\mathcal{C}$

$$\begin{array}{ccccc} (F(X) \otimes F(Y)) \otimes F(Z) & \longrightarrow & F(X \otimes Y) \otimes F(Z) & \longrightarrow & F((X \otimes Y) \otimes Z) \\ \downarrow & & & & \downarrow \\ F(X) \otimes (F(Y) \otimes F(Z)) & \longrightarrow & F(X) \otimes (F(Y \otimes Z)) & \longrightarrow & F(X \otimes (Y \otimes Z)) \end{array}$$

Plus:

1. $F(I_{\mathcal{C}}) \xrightarrow{\sim} I_{\mathcal{D}}$

2. $F(X^{-1}) \xrightarrow{\sim} F(X)^{-1}$

Compatibly with the rest of the diagrams.

Examples

- Let R be a comm. ring with 1. $\mathrm{Pic}(R)^{\mathbb{Z}}$ is the (Picard) groupoid of "graded R-lines"

(Non strict) ⤳

Objects : (n, L), $n \in \mathbb{Z}$ and L is an invertible R-module

Morphisms : $(n, L) \to (n', L')$ iff $n = n'$, $\alpha : L \xrightarrow{\cong} L'$ (iso. of R-modules)

Monoidal structure :
$$(n, L) \otimes (n', L') = (n+n', L \otimes_R L')$$

- **Crossed Modules** ⤳ **Associated Cat. Groups** (strict)

Notation: $[C_1 \to C_0]^{\sim}$

A crossed Module:

* $C_1 \xrightarrow{\partial} C_0$ (group homomorphism)
* Action: $C_0 \times C_1 \longrightarrow C_1$ $\quad (x, a) \longmapsto {}^{x}a$
* Axioms
  1. $\partial({}^{x}a) = x(\partial a)x^{-1}$
  2. ${}^{\partial a}b = a b a^{-1}$

$x \in C_0,\ a \in C_1$
$b \in C_1$

- From ∂ form an action groupoid

$$C_1 \times C_0 \overset{s}{\underset{t}{\rightrightarrows}} C_0 \qquad (a, x) \begin{array}{l} \mapsto x \\ \mapsto (\partial a)x \end{array}$$

- Monoidal structure (group like)

$$\begin{array}{c} (a, x) \\ x \end{array} \otimes \begin{array}{c} (b, y) \\ y \end{array} = \begin{array}{cl} (a\,{}^{x}b, xy) & C_1 \times C_0 \\ xy & C_0 \end{array}$$

1. Normal Subgroups $\quad N \trianglelefteq G \quad (N \hookrightarrow G)$
2. Central Extensions $\quad A \hookrightarrow E \twoheadrightarrow N$

Splice ⤳

$$\begin{tikzcd} E \arrow[rr, "\partial"] \arrow[dr, two heads] & & G \\ & N \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} St(R) \arrow[rr] \arrow[dr, two heads] & & GL(R) \\ & E(R) \arrow[ur, hook] & \end{tikzcd}$$

EVERY CAT. GROUP IS EQUIVALENT TO A STRICT ONE (COMING FROM A CROSSED MODULE)

(FOLK THEOREM — BAUES, BREEN, ..., NOOHI – E.A., ...)

$[C_1 \longrightarrow C_0]^{\sim} \xrightarrow{\sim} \mathcal{C}$ Presentation of $\mathcal{C}$

- If $\mathcal{C}$ is strict : $G_\bullet = N\mathcal{C}$, the nerve of $\mathcal{C}$, is a simplicial Group : $\Delta^{op} \xrightarrow{\text{Functor}} Grp$

  Take the Moore Complex $M_\bullet(G_\bullet)$: $M_m(G) = \bigcap_{0 < i \leq m} \ker(d_i)$

  $\partial = d_0$

  CROSSED MODULE $C_1 \longrightarrow C_0$

- General $\mathcal{C}$: (Several ways, one will come up later). For now:

  1. $\mathcal{C} \rightsquigarrow \Omega^{-1}\mathcal{C} = \mathbb{B}$ = bicategory w/ a single object $*$, $Hom_{\mathbb{B}}(*,*) = \mathcal{C}$

     $\leftarrow$ suspension

  2. $N\mathbb{B}$: geometric nerve (rigidified bicat. nerve) $X := N\mathbb{B} \in sSet_0$ Reduced Simplicial Set

  3. Loop functor : $G(X) \in sGrp$

  4. Continue as above

Question: What do Categorical Groups form?

- Understanding $\mathrm{GrCat} \subset \mathrm{Cat}$ (as a 2-Category)

$$\mathrm{GrCat} \downarrow \mathrm{Ho}(\mathrm{GrCat})$$

$\mathrm{Ho}(\mathrm{GrCat}) \leftsquigarrow$ $\mathcal{C}$'s up to weak equivalence

Homotopy theory point of view

relies upon

- Dévissage (Algebra)

"Exact"

$$\pi_1(\mathcal{C})[1] \longrightarrow \mathcal{C} \xrightarrow{\varpi} \pi_0(\mathcal{C})$$

$[A \to 1]^{\sim}$

$B = \pi_0(\mathcal{C}) = \mathrm{ob}\,\mathcal{C}/\sim$ iso

$X \sim Y$ iff $X \xrightarrow{\cong} Y$

$A = \pi_1(\mathcal{C}) = \mathrm{Aut}_{\mathcal{C}}(I)$ (Abelian)

with a presentation

$$0 \to \underset{\ker \partial}{\overset{\shortparallel}{A}} \to C_1 \xrightarrow{\partial} C_0 \to \underset{\mathrm{coker}\,\partial}{\overset{\shortparallel}{B}} \to 1$$

Exact on the nose

Analyze Twisting

- Classification of these sequences $\rightsquigarrow$ POSTNIKOV INVARIANT $k \in H^3(B, A)$

(HOÀNG XUÂN SÍNH – 1975)

$$\begin{array}{c} \mathcal{C} \\ \varpi \big\downarrow \big\uparrow \sigma \\ \pi_0(\mathcal{C}) = B \end{array}$$

Choose a lift

$\varpi \circ \sigma = id_B$

Equivalence Classes of Objects

$$\sigma(x) \otimes \sigma(y) \longrightarrow \sigma(xy)$$

$$\begin{tikzcd}
(\sigma(x)\sigma(y))\sigma(z) \arrow[r] \arrow[d, "\text{associator } \alpha"] & \cdots \arrow[r] & \sigma(xyz) \arrow[d, "k"] \\
\sigma(x)(\sigma(y)\sigma(z)) \arrow[r] & \cdots \arrow[r] & \sigma(xyz)
\end{tikzcd}$$

Using the Postnikov Invariant to find a presentation of $\mathcal{C}$

Start:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\varpi"] \\ \mathcal{B} = \pi_0(\mathcal{C}) \end{tikzcd}
\quad \xrightarrow{\text{resolve } \mathcal{B}} \quad
\begin{tikzcd} & \mathcal{C} \arrow[d, "\varpi"] \\ F_0 \arrow[r, two heads] & \mathcal{B} \end{tikzcd}
\quad \xrightarrow{\text{pullback}} \quad
\begin{tikzcd} \mathcal{C} \times_{\mathcal{B}} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "\varpi"] \\ F_0 \arrow[r, two heads] & \mathcal{B} \end{tikzcd}$$

(free)

$\pi_1(\mathcal{C} \times_{\mathcal{B}} F) = \pi_1(\mathcal{C}) = A$

Postnikov invt's

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{B}} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "\varpi"] \\ F_0 \arrow[r, two heads] & \mathcal{B} \end{tikzcd}$$

$k_{p.b.} \in H^3(F, A) = 0$ (F is free)

$k_{\mathcal{C}} \in H^3(\mathcal{B}, A)$

Section

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{B}} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "\varpi"] \\ F_0 \arrow[r, two heads] \arrow[ur, "\sigma"] & \mathcal{B} \end{tikzcd}$$

True Monoidal functor

pullback $*$ along σ

Crossed module

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, "\partial"] \arrow[dr, phantom, "\simeq \text{ h.p.b.}"] & * \arrow[d] \\ F_0 \arrow[r, "\sigma"] \arrow[dr] & \mathcal{C} \arrow[d, "\varpi"] \\ & \mathcal{B} \end{tikzcd}$$

$$\begin{tikzcd} {[F_1 \to F_0]^\sim} \arrow[d, "\simeq"] \\ \mathcal{C} \end{tikzcd}$$

$\cdot \to \cdot$ exact up to homotopy

Question: What do Categorical Groups form? II

- How to describe morphisms $F : \mathcal{C} \longrightarrow \mathcal{D}$ (say, in terms of presentations)

- We need: morphisms of crossed modules

$$\begin{tikzcd} C_1 \arrow[r, "f_1"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowright"] & D_1 \arrow[d, "\partial"] \\ C_0 \arrow[r, "f_0"'] & D_0 \end{tikzcd} \quad + \quad \text{Equivariant} \quad f_1({}^{x}a) = {}^{f_0(x)} f_1(a)$$

- However, given $F : \mathcal{C} \longrightarrow \mathcal{D}$

$$\begin{tikzcd} C_1 \arrow[r, dashed, "\times"] \arrow[d, "\partial"'] & D_1 \arrow[d, "\partial"] \\ C_0 \arrow[r, dashed, "\times"] \arrow[d, "\varpi"'] & D_0 \arrow[d, "\varpi"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

Get a weak equivalence

Resolve $C_1 \to C_0$

(as before)

$$\begin{tikzcd} & F_1 \arrow[dl] \arrow[dr] \arrow[dd] & \\ C_1 \arrow[dd, "\partial"'] & & D_1 \arrow[dd, "\partial"] \\ & F_0 \arrow[dl] \arrow[dr] & \\ C_0 \arrow[d, "\varpi"'] & & D_0 \arrow[d, "\varpi"] \\ \mathcal{C} \arrow[rr, "F"'] & & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d, dashed] \arrow[dd, bend left] & F_0 \arrow[d, dashed] \arrow[dd, bend left] \\ C_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C_0 \arrow[d] \\ * \arrow[r] & \mathcal{C} \arrow[d] \\ & B = \pi_0(\mathcal{C}) \end{tikzcd}$$

Question: What do Categorical Groups form?

Rewrite: if $F : \mathcal{C} \longrightarrow \mathcal{D}$, it induces $\pi_i(F) : \pi_i(\mathcal{C}) \longrightarrow \pi_i(\mathcal{D})$ $i = 0, 1$

$$
\begin{tikzcd}
0 \arrow[r] & A[1] \arrow[r] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d, "F"] & B \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & A'[1] \arrow[r] & \mathcal{D} \arrow[r] & B' \arrow[r] & 1
\end{tikzcd}
$$

GrCat

$$\leftrightsquigarrow$$

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[dd] & C_1 \arrow[rr] \arrow[dr] & & C_2 \arrow[r] & B \arrow[r] \arrow[dd] & 0 \\
& & & E \arrow[ur] \arrow[dr] & & & \\
0 \arrow[r] & A' \arrow[r] & D_1 \arrow[rr] \arrow[ur] & & D_2 \arrow[r] & B' \arrow[r] & 1
\end{tikzcd}
$$

exact

Crs

Thm (Noohi, E.A.) $[C_1 \to C_0]^{\sim} \longleftarrow C_1 \to C_0$ is an equivalence of bicategories

Cor There is an equivalence of categories $Ho(GrCat) \xleftarrow{\sim} Ho(Crs)$

Classification Thm

- $\mathcal{C}, \mathcal{D}$ are equivalent : $F : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$
- $\mathcal{C}, \mathcal{D}$ have the same $k \in H^3(B, A)$
- The presentations are linked by a diagram with exact diagonals

(Set $A' = A$, $B' = B$ in the diagrams)

PLUS:

- Long exact sequence in non abelian Cohomology
- Extensions
- ...

Commutative Structures & Stability

Return to cat groups in general. There are different levels of commutativity

- **Braiding**: commutativity isomorphism $c_{X,Y} : X\otimes Y \xrightarrow{\cong} Y\otimes X$, which is natural and MacLane's hexagons commute

$$
\begin{tikzcd}
& (YX)Z \arrow[r, "\alpha"] & Y(XZ) \arrow[dr, "c"] & \\
(XY)Z \arrow[ur, "c"] \arrow[dr, "\alpha"'] & \arrow[r, phantom, "\mathrm{I}"] & & Y(ZX) \\
& X(YZ) \arrow[r, "c"'] & (YZ)X \arrow[ur, "\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X(ZY) \arrow[r, "\alpha^{-1}"] & (XZ)Y \arrow[dr, "c"] & \\
X(YZ) \arrow[ur, "c"] \arrow[dr, "\alpha^{-1}"'] & \arrow[r, phantom, "\mathrm{II}"] & & (ZX)Y \\
& (XY)Z \arrow[r, "c"'] & Z(XY) \arrow[ur, "\alpha^{-1}"'] &
\end{tikzcd}
$$

- **Symmetric Braiding** or **Picard**: $X\otimes Y \xrightarrow{c_{X,Y}} Y\otimes X \xrightarrow{c_{Y,X}} X\otimes Y = id$ $\Rightarrow \mathrm{I} = \mathrm{II}$

  $\hookrightarrow c_{X,X}^2 = id$

  Example $\mathrm{Pic}(R)^{\mathbb{Z}}$: Graded lines $(n, L) \in \mathbb{Z} \times \mathrm{Pic}(R)$

  $$(n,L)\otimes(n',L') = (n+n', L\otimes L') \xrightarrow{c} (n'+n, L'\otimes L) = (n',L')\otimes(n,L) \qquad L\otimes L' \xrightarrow{\cong} L'\otimes L$$

  $$u\otimes v \longmapsto (-1)^{nn'} v\otimes u$$

  NOT STRICT

- **Strictly Commutative** or **Strictly Picard** $\quad c_{X,X} = id$

COMMUTATIVE STRUCTURES & STABILITY

Return to cat groups in general. There are different levels of commutativity

- Braiding: commutativity isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$, which is natural

Hex I + II

$$[(k, c)] \in H^4(K(B,2), A) \xrightarrow{\text{suspension}} H^3(K(B,1), A) = H^3(B, A) \ni k$$

Postnikov

Eilenberg MacLane cohomology for Abelian Groups

- Symmetric Braiding or Picard: $X \otimes Y \xrightarrow{c_{X,Y}} Y \otimes X \xrightarrow{c_{Y,X}} X \otimes Y = id$ $\Rightarrow$ I = II

Hex I = II

$5 - 3 = 2 < 3$ STABLE!

$$[(k, c)] \in H^5(K(B,3), A) \xrightarrow{\text{Susp}} H^4(K(B,2), A) \xrightarrow{\text{Susp}} H^3(B, A) \ni k = 0!$$

$$\text{IIS}$$

$$\mathrm{Hom}(B/_{2B}, A) \cong \mathrm{Hom}(B, {}_2A)$$

Underlying Postnikov invariant vanishes

$\mathcal{C} \simeq K(B,0) \times K(A,1)$ as a categorical group.

- Strictly Commutative or Strictly Picard $c_{X,X} = id$

$[(k, c)] \in \mathrm{Ext}^2(B, A) = 0$

$\rightsquigarrow \mathcal{C}$ is trivial

COMMUTATIVE STRUCTURES & STABILITY FOR CROSSED MODULES

Let $C_1 \xrightarrow{\partial} C_0$ be a crossed module. A Braiding corresponds to:

$\{\cdot,\cdot\} : C_0 \times C_0 \longrightarrow C_1$ such that $\partial\{x,y\} = [x,y] = xyx^{-1}y^{-1}$

(Almost a) lift of the commutator map $\quad yx \xrightarrow{\{x,y\}} xy$

A Symmetric Braiding or Picard crossed module: $\{x,y\} = \{y,x\}^{-1}$

$(C_1 \xrightarrow{\partial} C_0, \{,\})$ STABLE 2-MODULE Class $[\langle c_0, c \rangle] : \pi_0 C_\bullet \otimes \mathbb{Z}/2 \longrightarrow \pi_1(C_\bullet)$

$x \otimes 1 \longmapsto \{x,x\}$

CAUTION: $\mathcal{C}$ is "commutative" and

A, B are abelian, however,

C_1, C_0 are only Nil_2-class: $[x,[y,z]] = 0$

- Strictly Commutative or Strictly Picard $c_{X,X} = id$ $\quad (C_1 \to C_0, \{,\})$ is a full lift of the commutator map: $\{x,x\} = 1$

$(C_1 \to C_0, \{,\}) \simeq (Z_1 \xrightarrow{\partial} Z_0)$

homomorphism of abelian Groups.

SKETCH OF APPLICATION

Determinants and 1-Types

1-Type of the K-Theory Spectrum.

- $\mathcal{W} \rightsquigarrow \boxed{\text{K-Theory}} \rightsquigarrow \mathbb{K}(\mathcal{W}) \overset{\text{1-Type}}{\rightsquigarrow} P_1(\mathcal{W})$ (Muro, Tonks ~ '07)

  Waldhausen/Exact Category

  $\rightarrowtail$ cofibrations

  $\xrightarrow{\sim}$ weak equivalences

  $A \rightarrowtail B \twoheadrightarrow C$ cofiber sequences.

  Spectrum

  Picard Groupoid

  $P_1(\mathcal{W}) \cong (C_1 \xrightarrow{\partial} C_0, \{,\})$

  C_0: gen by $[A]$, $A \in \mathrm{Ob}\,\mathcal{W}$

  C_1: gen by $[A \xrightarrow{\sim} A']$, $[A \rightarrowtail B \twoheadrightarrow C]$ + lots of relations.

- $\mathcal{W}$'s are known to form a Multicategory **WALD** (Zakharevich, '15; Elmendorf-Mandell, '09)

- We (E.A., Y. Valdes) proved

  Thm P_1 extends to a multifunctor

  $$\mathbf{WALD} \longrightarrow \mathbf{PIC}$$

  Multi category Picard groupoids

  morphism of multicategories

- Muro & Tonks: P_1 is multiplicative for biexact functors $F: \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ of Waldhausen Categories.

Higher Structures ⇝ Categorical Rings ⇝ 2-Rings

- Picard (Braided Symmetric) Groupoid $(\mathcal{R}, \oplus, 0)$ "Underlying Abelian group of a ring"
- A biexact functor $\otimes : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ with respect to $\oplus$:

Bimonoidal

* $(x \oplus y) \otimes z \xrightarrow{\sim} (x \otimes z) \oplus (y \otimes z)$
* $x \otimes (y \oplus z) \xrightarrow{\sim} (x \otimes y) \oplus (x \otimes z)$
*

$$\begin{tikzcd}
(x \oplus y) \otimes (z \oplus w) \arrow[r] \arrow[d] & (x \otimes (z \oplus w)) \oplus (y \otimes (z \oplus w)) \arrow[dd] \\
((x \oplus y) \otimes z) \oplus ((x \oplus y) \otimes w) \arrow[d] \arrow[ur, phantom, "\circlearrowright"] & \\
((x \otimes z) \oplus (y \otimes z)) \oplus ((x \otimes w) \oplus (y \otimes w)) \arrow[r] & ((x \otimes z) \oplus (x \otimes w)) \oplus ((y \otimes z) \oplus (y \otimes w))
\end{tikzcd}$$

Same diagram as in Mumford, Grothendieck | Biextensions

MacLane's Commuto-associator

* A host of other (less interesting) diagrams

CATEGORICAL RINGS ⇝ 2-RINGS

$A = \pi_0(\mathcal{R}) = \mathrm{Obj}\,\mathcal{R}/\sim$

$M = \pi_1(\mathcal{R}) = \mathrm{Aut}_{\mathcal{R}}(0_{\mathcal{R}})$

* Presentations

$$0 \to M \to R_1 \xrightarrow{\partial} R_0 \to A \to 0$$

$\rightsquigarrow \mathcal{R}$

Ring

A-Module (bi)

Picard Crossed Module / Stable 2-Module + bimodule conditions

* Classification Theorems

→ $R_1 \to R_0$ (or $\mathcal{R}$) is Strictly Picard → $k \in H^3(A, M) \cong \mathbb{L}^2\mathrm{Der}(A, M)$ André-Quillen-Shukla Cohomology

(Baues-Pirashvili, N.T Quang, N.T. Quang-Hanh, E. A.)

(Almost Hochschild cohomology of A with coefficients in M)

→ $R_1 \to R_0$ (or $\mathcal{R}$) is Picard

The difference is the k-invariant of the underlying Picard Groupoid

$k \in ML^3(A, M) \cong \underbrace{THH^3}_{\text{Top. Hochschild}}(A, M)$ Maclane Cohomology

(Baues-Pirashvili, Jibladze-Pirashvili E. A.)

* **Thm** (EA '17) The 2nd monoidal functor $\otimes : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$

$$\begin{tikzcd}
R_1 \times R_1 \arrow[d] \arrow[dr] & & R_1 \arrow[dl] \arrow[d] \\
& E \arrow[dl] \arrow[dr] & \\
R_0 \times R_0 & & R_0
\end{tikzcd}$$

← Biextension.

Remark: $HH^2(A,M)$ = Iso. classes of "singular" extensions of A (k-algebra) by M, where M is an A-bimodule

Fix a comm. ground ring k

$$0 \to M \to E \xrightarrow{p} A \to 0$$

$E, A \in k\text{-Alg}$

A-bimodule such that $M^2 = 0$

such that the underlying k-Mod. short exact sequence is split.

Remark II $HH^3(A,M)$ same for $0 \to M \to E_1 \to E_0 \to A \to 0$

WHERE TO USE & TO FIND CATEGORICAL RINGS ⇝ 2-RINGS

- Usually, you have 2RIGS not 2Rings : Example Vect_k^{fin} with $\oplus, \otimes$

  $(\mathrm{Vect}_k, \oplus)$ is monoidal, but it lacks an additive inverse

R. THOMASON "Beware of the phony multiplication..."

Usually, you can't apply the Grothendieck completion to the underlying monoidal category $(\mathcal{R}, \oplus, 0)$ without harm.

unsolved...

BAAS, DUNDAS, ROGNES, ... (~'11) $\mathcal{R}$ rig Category : $\mathcal{R} \leftarrow \rightarrow \cdots \leftarrow \rightarrow \tilde{\mathcal{R}}$

Zig-Zag of weak equiv.s

2Ring

Do K-Theory !

- Tensor TRIANGULATED CATEGORY (BALMER)

  * $\mathcal{T}$: triangulated category w/ $\Sigma : \mathcal{T} \to \mathcal{T}$ and small sums
  * $\otimes : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$

    Symmetric Monoidal & biexact
  * $(\Sigma X) \otimes Y \cong \Sigma (X \otimes Y)$

Two constructions (among others) :

- Picard Groupoid of $\otimes$-invertible objects
- Determinant functors yield 2-Rings

WORKSHOP OCTOBER 11, 2020

Group extensions $\rightsquigarrow$ Motivation: why Crossed Modules?

$$1 \longrightarrow K \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

short exact sequence
group homomorphisms
$\ker p = \operatorname{Im} i$

Also consider the conj. action of E onto itself inducing an action

$$E \times K \longrightarrow K$$
$$(x, k) \longmapsto x\, i(k)\, x^{-1}$$

$p(x\, i(k) x^{-1}) =$
$= p(x)\, pi(k)\, p(x)^{-1}$
$= 1$

$\rightsquigarrow \quad j : E \longrightarrow \operatorname{Aut}(K)$

Rewrite:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, "i"] \arrow[d, "\partial"'] & E \arrow[r] \arrow[dl, "j"] & G \arrow[r] & 1 \\ & \mathrm{Aut}(K) & & & \end{tikzcd}$$

inner conjugation

Write: $x \in E$, $k \in K$

$${}^{x}k = x\, i(k)\, x^{-1} \in K$$

$$j(x)(k)$$

If $x = i(k')$

$$\partial k' = j(i(k')) = i(k')\, i(k)\, i(k')^{-1}$$

$$= i(k' k k'^{-1}) \quad (*)$$

Modify the extension:

Extension of G by a Crossed Module

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, "i"] \arrow[d, "\partial"'] & E \arrow[r, "p"] \arrow[dl, "j"] & G \arrow[r] & 1 \\ & L & & & \end{tikzcd}$$

Define:

$$i({}^{x}k) = x\, i(k)\, x^{-1} \in \mathrm{im}(i) \leq E$$

$$\partial = j \circ i$$

1. $\partial({}^{x}k) = j\, i({}^{x}k) = j(x)\, j i(k)\, j(x)^{-1} \qquad j(x) \in L$

$$= j(x)\, \partial k\, j(x)^{-1}$$

2. $(*)$

$$
\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "i"] \arrow[d, "\partial"'] & E \arrow[r, "p"] \arrow[dl, "j"] & G \arrow[r] & 1 \\
 & L & & &
\end{tikzcd}
$$

$\rightsquigarrow$

$$
\begin{tikzcd}
 & 1 \arrow[d, "\partial"'] \arrow[rr] & & & & 1 \arrow[dl] \\
 & & & E \arrow[dll, "p"'] \arrow[dr, "j"'] & K \arrow[l, "i"'] \arrow[d, "\partial"] & \\
 & G \arrow[dl] & & & L & \\
1 & & & & &
\end{tikzcd}
\quad \simeq \quad G \longrightarrow \underbrace{[K \xrightarrow{\partial} L]}_{\mathcal{C}}^{\sim}
$$

Only determined up to equivalence

$\mathcal{C}$ ← Real data is the cat group.

Crossed modules $\longrightarrow$ Gr Cat

$$C_1 \xrightarrow{\partial} C_0 \longmapsto [C_1 \times C_0 \rightrightarrows C_0] = [C_1 \xrightarrow{\partial} C_0]^{\sim}$$

Remark: Char class of the extension lives in $H^2_{gr}(G, \text{"?"})$ ← non abelian cohomology

$$H^1_{gr}(G, K \to L) \cong [K(G,1), B\mathcal{C}]$$

homotopy classes.

$\rightsquigarrow$

Nerve of $\mathcal{C}$ + suspension.