Arithmetic Aspects of Liouville

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The Liouville Equation

$$\boxed{\frac{\partial^2 \varphi}{\partial z \, \partial \bar{z}} = \frac{1}{2} e^{\varphi}}$$

The Liouville Equation

$$\mathcal{U} \subset \mathbb{C} \quad , \quad \varphi \in C^{\infty}(\mathcal{U}, \mathbb{R})$$

$$\boxed{\frac{\partial^2 \varphi}{\partial z \, \partial \bar{z}} = \frac{1}{2} e^{\varphi}}$$

$z = x + \sqrt{-1}\, y$ complex coordinate

The Liouville Equation

Conformal metrics : $\rho = e^{\varphi} |dz|^2$
(over $U \subset \mathbb{C}$)

$$\frac{\partial^2 \varphi}{\partial z \, \partial \bar{z}} = \frac{1}{2} e^{\varphi} \iff K_\rho = -1$$

Scalar Curvature

$$K_\rho = -2 e^{-\varphi} \frac{\partial^2 \varphi}{\partial z \, \partial \bar{z}}$$

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$$f \in C^\infty(U, \mathbb{R})$$

$$\iff \boxed{\left.\frac{d}{dt}\right|_{t=0} S\left(e^{tf} \rho\right) = 0}$$

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$$S(\rho) = \frac{\sqrt{-1}}{2} \int_{\mathcal{U}} \partial\varphi \wedge \bar{\partial}\varphi + \frac{\sqrt{-1}}{2} \int_{\mathcal{U}} e^{\varphi} \, dz \wedge d\bar{z}$$

Invariance Properties

$$S(\rho) = \frac{\sqrt{-1}}{2}\int_{\mathcal{U}} \partial\varphi \wedge \bar{\partial}\varphi + \frac{\sqrt{-1}}{2}\int_{\mathcal{U}} e^{\varphi}\, dz \wedge d\bar{z}$$

▷ $$e^{\varphi}\, dz \wedge d\bar{z} = e^{\varphi'}\, dz' \wedge d\bar{z}', \qquad z' = f(z)$$

Area term: OK

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▷ $$\partial\varphi \wedge \bar{\partial}\varphi - \partial\varphi' \wedge \bar{\partial}\varphi' = d\,(\text{something})$$

Ill-defined on the nose.

The Liouville Functional (for real...)

X : Riemann Surface, $g = g(X) \geq 2$. ($X = \mathcal{X}_\infty$, where $\mathcal{X}$ is a smooth surface over an arithmetic ring A, e.g. $A = (\mathbb{C}, \{1, \sigma\}, F_\infty)$)

$\rho : T_X \longrightarrow \mathcal{E}_X^+$ (sheaf of positive real-valued functions)

$S : \mathcal{CM}_X \longrightarrow \mathbb{R}$

Conformal factors (relative to the fixed conformal structure)

$$S(\rho) = \check{S}(\rho) + \int_X \mathrm{vol}_\rho$$

Quadratic Term

Area Term

$\Downarrow$

The Liouville Functional (for real...)

$$S(\rho) = \check{S}(\rho) + \int_X \mathrm{vol}_\rho$$

▷ Cup square : $\check{S}(\rho) = [(T_X, \rho)] \cup [(T_X, \rho)]$

Hermitian Deligne Cohomology.

▷ Determinant of Cohomology.

▷ Hermitian holonomy of a 2-gerbe (... that never was...)

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  Hermitian Deligne Cohomology.

- Determinant of Cohomology.
- Hermitian holonomy of a 2-gerbe (... that never was...)
- Regularized Volume of N^3, where $\partial N^3 = X$
- Result of a transgression map (Conjectural, in part)

Metrized Line Bundles

X : algebraic variety / Complex manifold

$\mathcal{L}$: line bundle $\rightsquigarrow$ Invertible sheaf $\mathcal{U} \subset X \rightsquigarrow \mathcal{L}(\mathcal{U})$

$\mathcal{O}_X(\mathcal{U})$-module

$\rho : \mathcal{L} \longrightarrow \mathcal{E}_{X,+}$ hermitian fiber metric

$s \longmapsto \rho(s)$ positive, smooth, $\mathbb{R}$-valued

$X = \bigcup_i \mathcal{U}_i$: $s_j = s_i\, g_{ij}$, $g_{ij}\, g_{jk} = g_{ik}$, $\rho(s_j) = \rho(s_i)\, |g_{ij}|^2$

$\widehat{\mathrm{Pic}}(X) = \{ [\mathcal{L}, \rho] \}$ Arithmetic Picard Grp

Lemma $\{ [(\mathcal{L}, \rho)] \in \widehat{H}^2_D(X, 1) \} = \mathbb{H}^2(X, \mathbb{Z} \to \mathcal{O}_X \to \mathcal{E}_X)$

$= \mathbb{H}^1(X, \mathcal{O}_X^* \longrightarrow \mathcal{E}_{X,+})$

Products (Arithmetic Intersection)

$$\hat{H}_D^2(X;1)\times\hat{H}_D^2(X;1)\xrightarrow{\cup}\hat{H}_D^4(X,2)$$

$$\widehat{\mathrm{Pic}}(X)\times\widehat{\mathrm{Pic}}(X) \qquad \mathbb{H}^2\left(X,\ \begin{array}{ccccc} \mathbb{Z} & \xrightarrow{2} & \mathcal{O}_X & \xrightarrow{d} & \Omega^1_X \\ & & \downarrow{\scriptstyle -\mathrm{Im}} & & \downarrow{\scriptstyle \mathrm{Im}} \\ & & \mathcal{E}^0_X & \xrightarrow{-d} & \mathcal{E}^1_X & \xrightarrow{-\mathrm{proj}} & \mathcal{E}^{1,1}_X \end{array}\right)$$

holomorphic

Smooth, Imaginary

$$(\mathcal{L},\rho),\ (\mathcal{M},\tau)\longmapsto \widehat{(\mathcal{L},\mathcal{M}]}$$

Hermitian 2-Gerbe

PRODUCTS (ARITHMETIC INTERSECTION)

$$\hat{H}^2_{\mathcal{D}}(X;1) \times \hat{H}^2_{\mathcal{D}}(X;1) \xrightarrow{\ \cup\ } \hat{H}^4_{\mathcal{D}}(X,2)$$

$$\widehat{\mathrm{Pic}}(X) \times \widehat{\mathrm{Pic}}(X) \qquad\qquad H^2\left(X, \begin{tikzcd} \mathbb{Z} \arrow[r, "2"] & \mathcal{O}_X \arrow[r, "d"] \arrow[d, "-\mathrm{Im}"] & \Omega^1_X \arrow[d, "\mathrm{Im}"] \\ & \mathcal{E}^0_X \arrow[r, "-d"] & \mathcal{E}^1_X \arrow[r, "-\mathrm{pr}\circ d"] & \mathcal{E}^{1,1}_X \end{tikzcd}\right)$$

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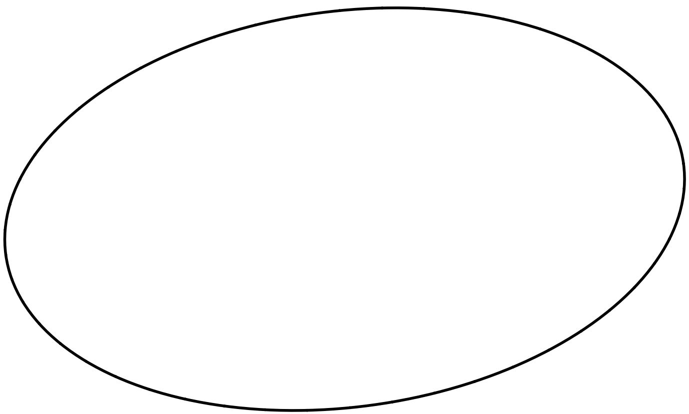
Hermitian 2-Gerbe

GEOMETRY



X

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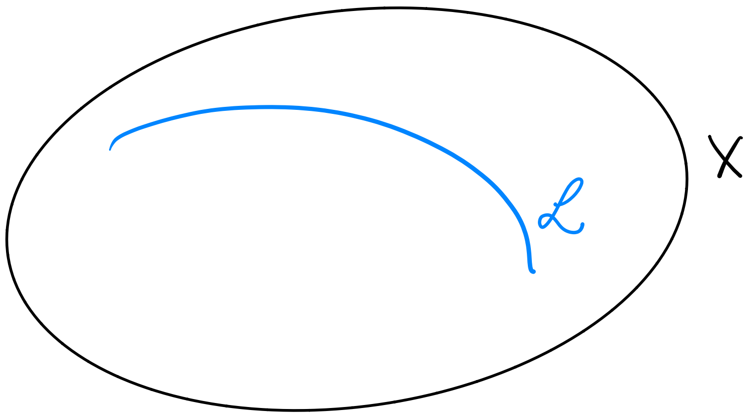

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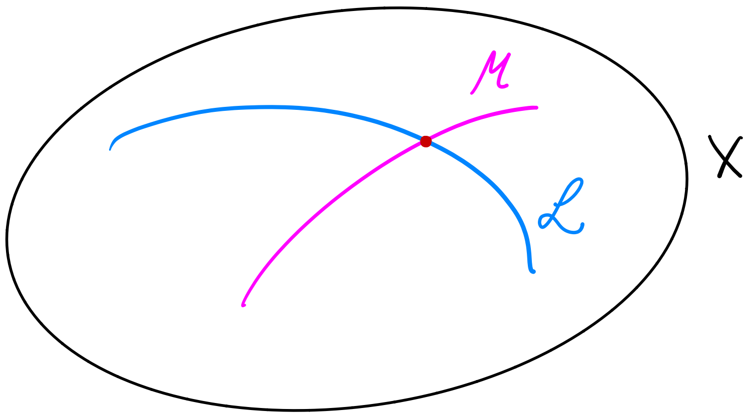
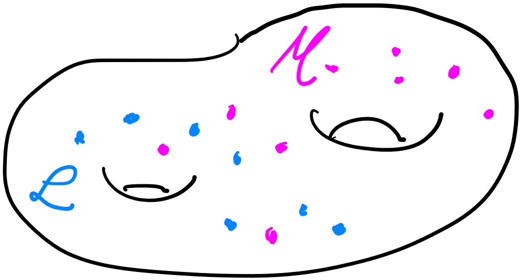
Hermitian 2-Gerbe

Geometry



BACK TO $\mathrm{Dim}_{\mathbb{C}} = 1$ X : Riemann Surface / Smooth Alg. Curve



NO INTERSECTION !

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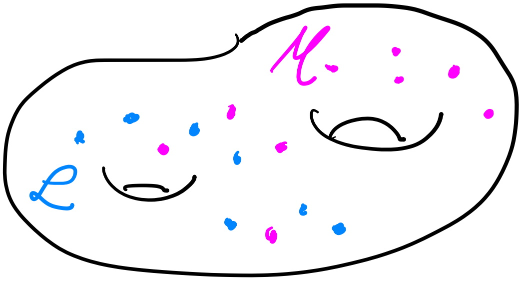

No Intersection!

Still ....

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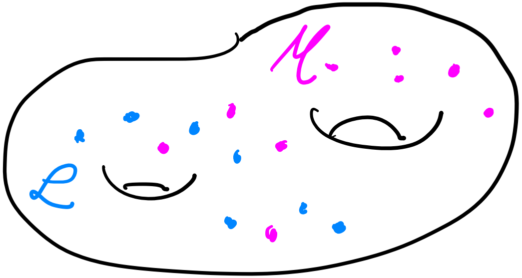

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$$\cdots \to H^2\left(X, \mathcal{E}^0_X \xrightarrow{-d} \mathcal{E}^1_X \xrightarrow{-\mathrm{prod}} \mathcal{E}^{1,1}_X\right) \to \hat{H}^4_{\mathcal{D}}(X,2) \to H^4_{\mathcal{D}}(X,\mathbb{Z}(2)) \oplus \mathcal{E}^{(2,2)}(X) \to \cdots$$

$$\simeq \quad \mathbb{R}\otimes 2\pi\sqrt{-1} \dashrightarrow [\mathcal{L},\mathcal{M}]^{\wedge} \longmapsto 0 \qquad 0$$

So, we get an evaluation to $H^2(X,\mathbb{R})\otimes 2\pi\sqrt{-1}$

(Further: extract a number by evaluating against $[X] = \int_X$)

Back to Liouville

Hyperbolic Metric

$$(\mathcal{L}, \rho) = (\mathcal{M}, \tau) = \left(T_X, e^{\varphi}|dz|^2\right)$$

$$\langle (T_X, \rho)\, U(T_X, \rho), [X] \rangle = \sum_{i \in I} \int_{\Delta_i^2} \frac{1}{2} \varphi_i \bar{\partial}\partial \varphi_i$$

Standard Quadratic Term

Back to Liouville

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Correction Terms

$$+ \sum_{\langle i,j\rangle \in I^2} \frac{1}{2}\int_{\Delta^1_{ij}} d^c \log|g_{ij}|\,\varphi_j - \log|g_{ij}|\, d^c\varphi_j$$

$$+ \sum_{\langle ijk\rangle \in I^3} \int_{\Delta^0_{ijk}} (\text{Bloch-Wigner dilogs})_{ijk}$$

Thm $\langle (T_X,\rho),(T_X,\rho)\rangle + \int_X \mathrm{vol}_\rho$ well defined; $\frac{d}{dt} S(e^{tf}\rho) = 0 \iff K_\rho = -1$

Remark: The Determinant of Cohomology X as before

$\mathcal{L}, \mathcal{M}$ with rational sections s, t with $(s) = D$, $(t) = E$

$\langle s,t \rangle$: $\mathbb{C}$-torsor ; Relations: $\langle fs, t \rangle = f(E) \langle s,t \rangle$, $\langle s, gt \rangle = g(D) \langle s,t \rangle$ Compatible by Weil Reciprocity.

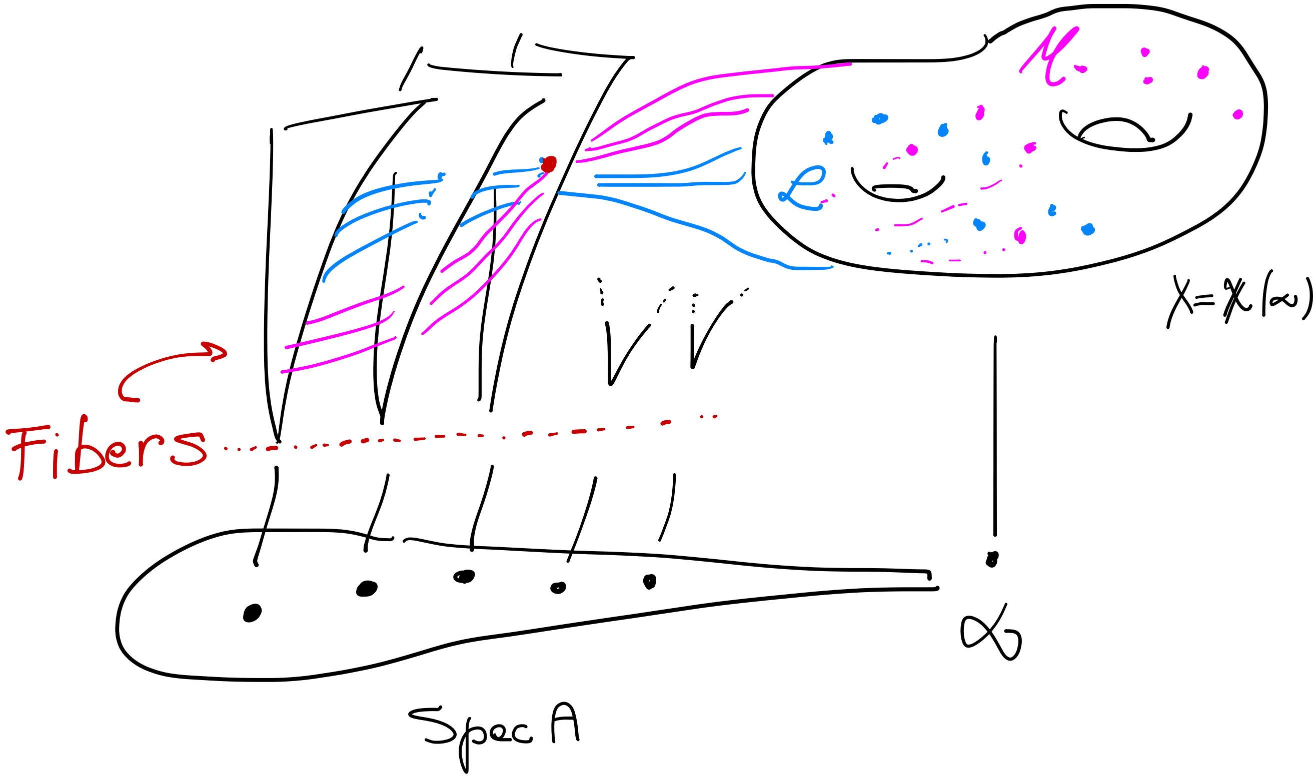
Metrized Line Bundles: $(\mathcal{L}, \rho), (\mathcal{M}, \tau)$ Denote the norms by $\|\cdot\|^2$.

The $\mathbb{C}$-torsor $\langle s,t \rangle$ is metrized with norm given by

$$\exp\left\{ \frac{1}{2\pi\sqrt{-1}} \int_X \partial\bar\partial \log \|s\|^2 \log \|t\|^2 + \log \|s\|^2 (E) + \log \|t\|^2 (D) \right\}$$

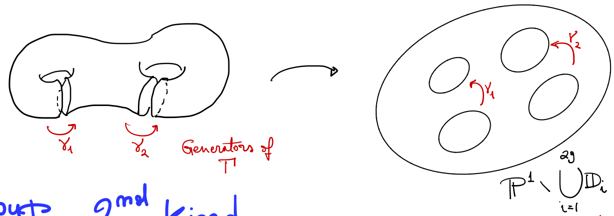
Thm Up to a factor $= \check{S}_{Liouv}(\rho)$ (The quadratic part)

Pictorial "Explanation"



One More Thing ... Use an étale covering $\mathcal{U} \xrightarrow{\Gamma} X$

▷ Schottky Uniformization



Fundamental domain

$\Gamma \subset PSL_2(\mathbb{C})$ (discrete sbgrp)

▷ In general: Γ kleinian group, 2^{nd} kind

Domain of discontinuity of Γ

▷ $X = \partial N^3$, $N^3 \cong \Omega/\Gamma$, $\mathrm{Vol}(N^3) = \infty$, $\check{S}(\rho) = \mathrm{reg\,vol}(N^3)$ (Well known)

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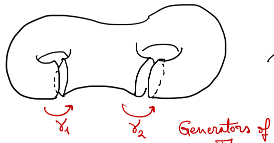

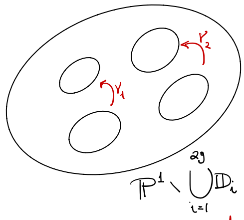

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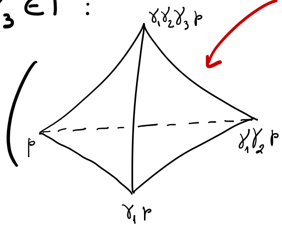
- ▷ Compute $\check{S}(\rho)$ as before using $\Omega \xrightarrow{\Gamma} X$:

$\forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma$:

$\check{S}(\rho)$ exists $\iff$ $\left[HVol \left(\right. \right.$  $\left. \left. \right) \right] = 0$ in $H^3(B\Gamma, \mathbb{R})$

Hyperbolic tetrahedron in H^3 (= interior of $\mathbb{P}^1$)

ONE MORE THING ...

One is tempted to formulate the conjecture:

$\check{S}(p)$ is the transgression of the hyperbolic volume class

$$HF \longrightarrow B\Gamma \longrightarrow B\,PSL_2(\mathbb{C})$$

homotopy fiber

A FINAL WORD ...


happy birthday