Lunedì 24 Giugno 2013

Seminario di Teoria delle Categorie

Università degli Studi di Milano

9.30 - 10:15 - **Semidirect products of topological semi-abelian algebras**

Andrea Montoli - *Universidade de Coimbra*

M.M.Clementino and F.Borceux [1] proved that all topological models of a semi-abelian variety admit semidirect products in the categorical sense introduced by D. Bourn and G. Janelidze [2]. Using some techniques developed in [3], we can show that, in any semi-abelian variety, the semidirect product of two objects X and B always appears as a subset of a certain cartesian product built using X and B. This allows us to give an explicit description of the topology of the semidirect product in topological semi-abelian algebras. *joint work with M.M. Clementino and L. Sousa*

[1] M.M.Clementino and F.Borceux, Topological semi-abelian algebras, Adv. in Math. 190 (2005).
[2] D.Bourn and G. Janelidze, Protomodularity, descent, and semidirect products, Th. Appl. Categories 4 (1998).
[3] J.Gray and N.Martins-Ferreira, On algebraic and more general categories whose split epimorphisms have underlying product projections, preprint arXiv:1208.2032.

10:15 - 11:00 - **Compact preordered spaces and the stable units property**

Joao Xarex - *Universidade de Aveiro*

We will identify the semi-left-exact (also called admissible, in the sense of categorical Galois theory) subreflections into Priestley spaces of Nachbin's (pre)ordered compact (Hausdoff) spaces. In order to do so we need the simplification given in [4] to the pullback preservation conditions in the definition of a semi-left-exact reflection(see [3]). Then we generalize the proofs in [1, 5.6, 5.7]; in particular, we work with an appropriate notion of connected component, and present a non-symmetrical generalization of entourage. Furthermore, we will show that these admissible subreflections necessarily have the stronger property of stable units, and characterize monotone maps in such cases. It will be argued that these results provide good ground for the project of extending the classical monotone-light factorization of compact Hausdorff spaces via Stone spaces (itself an extension of Eilenberg's factorization for metric spaces; see [2]) to non-trivial (pre)ordered spaces.

[1] Borceux, F., Janelidze, G.,Galois theories, Cambridge University Press, 2001.
[2] Carboni, A., Janelidze, G., Kelly, G. M., Pare', R.,On localization and stabilization for factorization systems, App. Cat. Struct. 5 (1997) 1-58.
[3] Cassidy, C., Hebert, M., Kelly, G. M.,Reflective subcategories, localizations and factorization systems, J. Austral. Math. Soc. 38A (1985) 287-329.
[4] Joao J. Xarez, Generalising connected components, J. Pure Appl. Algebra 216 (2012) 1823-1826.
[5] Nachbin, L.,Topology and Order, Von Nostrand, Princeton, N. J., 1965.

11:00 - 11:45 **Butterflies and morphisms of monoidal and bimonoidal stacks**

Ettore Aldrovandi - *Florida State University*

Morphisms between homotopy types in low degrees can be efficiently computed by way of special diagrams called Butterflies, owing to their shape. In a geometric context, butterflies describe morphisms between stacks equipped with monoidal or, in a new development, bimonoidal structures. I plan to survey the main points and some of the applications of the theory in the former case first, and then to discuss the latter case of stacks which are categorical rings (ring-like, for want of a better name). Ring-like stacks ought to be considered as akin to truncated cotangent complexes, and connections to Shukla, MacLane cohomology of rings, and more generally functor cohomology, can be found.

Tutti gli interessati sono calorosamente invitati a partecipare.

Sandra Mantovani, Beppe Metere

NOTES

- Connected homotopy 2-types : $\pi_i = 0$, $i \neq 1,2$
- Fibrant objects → simplicial groups w/ π_i trivial except $i=1,2$.
- Crossed Modules (via Moore Complex)

First describe (briefly) what happens over a point.

- Morphisms b/t homotopy types

$$\mathrm{Hom}_{Ho(\mathcal{C})}(C,D) \simeq \mathrm{Hom}_{\mathcal{C}}(QC,D)$$

$$\begin{tikzcd} & QC \arrow[dl, two heads, "\sim"'] \arrow[dr] & \\ C & & D \end{tikzcd}$$

Cofibrant replacement

$\overset{\sim}{\twoheadleftarrow}$: acyclic fibration.

Standard model Category construction

- By virtue of a factorization of $Q \longrightarrow C \times D$ we get a **butterfly**

$$\begin{tikzcd} Q \arrow[rr] \arrow[dr, "\wr"'] & & C \times D \\ & E \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} C_2 \arrow[dd] \arrow[dr] & & D_1 \arrow[dl] \arrow[dd] \\ & E_1 \arrow[dl] \arrow[dr] & \\ C_1 & & D_1 \end{tikzcd} , \quad E_2 = C_2 \times D_2$$

- Crossed modules ⇒ Categorical groups
- More generally : Site $\mathcal{S}$, with topology ⇝ (Pre)Sheaves of crossed modules
- Crossed modules ⇒ Group-like Stacks : $G_2 \to G_1$ ⇝ Groupoid $\Gamma : G_1 \ltimes G_2 \rightrightarrows G_1$

⇝ Associated Stack $\mathcal{G} = \Gamma^{\sim}$

- Recalculation of the Butterfly w/ fibered product
- Main Theorem : $\underline{\mathcal{H}om}(\mathcal{G},\mathcal{H}) \simeq \underline{\mathcal{B}}(\mathcal{G},\mathcal{H})$ ← Equivalence of Stacks over $\mathcal{S}$.

⇒ XMod → $\mathcal{S}$ is a 2-stack (Bicategories as fibers!)

NOTES (CONT.)

* Discussion (Brief) of applications:

Terminal object of $Sh(\mathcal{S})$

$\boxed{i \leq 1}$

Nonabelian cohomology $H^i(e, \mathcal{G}) = \mathrm{Hom}_{Ho}(e, \Omega^{-i}\mathcal{G})$

simplicial model

loop/suspension

$$= \operatorname*{colim}_{[X \to e]} [X, \Omega^{-i}\mathcal{G}] \longleftarrow \text{[Verdier]}$$

Hom. classes of hypercovers

Short exact sequence $\mathcal{K} \to \mathcal{H} \xrightarrow{p} \mathcal{G}$ p: NOT assumed to be a fibration

Long exact sequence $\cdots \to H^0(e, \mathcal{G}) \to H^1(e, \mathcal{K}) \to H^1(e, \mathcal{H}) \to H^1(e, \mathcal{G})$

In particular, w/ $\pi_2(\mathcal{G})[1] \to \mathcal{G} \to \pi_1(\mathcal{G})$:

Sheaf of connected components of $\mathcal{G}$

$$\cdots \to H^0(e, \pi_1\mathcal{G}) \to H^2(e, \pi_2\mathcal{G}) \to H^1(e, \mathcal{G}) \to H^1(e, \pi_1\mathcal{G}) \to H^3(e, \pi_2\mathcal{G})$$

w/ $e = \pi_1\mathcal{G}$: Group Cohomology

Postnikov Invariant

NOTES (CONT. 2)

- Translation for Algebras : k — commutative ring.
  $\mathcal{C} = k\text{-Alg}$ ⇝ not necessarily comm. k-Algebras.
  ($k = \mathbb{Z}$: $k\text{-Alg} =$ Rings)
  $s\mathcal{C} = sk\text{-Alg}$ ⇝ simplicial k-Algebras

- Simplicial k-Algebra : $R_\bullet : \overset{\to}{\underset{\to}{\vdots}}\, R_2 \substack{\to\\ \to\\ \to} R_1 \rightrightarrows R_0$ ⇝ Moore Complex : $\cdots \to NR_1 \xrightarrow{d_0} NR_0 = R_0$

- Crossed Bimodule : $\tau_{\leq 1} NR_\bullet$: $NR_1 / \mathrm{Im}\, NR_2 \xrightarrow{\partial} R_0$

  Axioms : $M \xrightarrow{\partial} R$ ⟵ morphism of R bimodules

  k-Alg

  R-Bimodule

  $(\partial m_1) m_2 = m_1 (\partial m_2)$: Pfeiffer identity

- Groupoid (Picard) : $\Gamma : R \oplus M \rightrightarrows R \quad (r,m) \begin{smallmatrix} \mapsto r \\ \mapsto r + \partial m \end{smallmatrix}$

  CATEGORICAL RING : $(r,m)(r',m') = (rr', rm' + mr' + m\,\partial m')$ ⟵ multiplication on Morphisms

  Compatible w/ target morphism

- Makes sense over a site $\mathcal{S}$: (Pre)Sheaves of Crossed Bimodules

- Associated Stack : $\mathcal{R} = \Gamma^{\sim} = [M \xrightarrow{\partial} R]^{\sim}$ Ring-like Stack : Picard Stack w/ $\oplus$

  $\otimes : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ 2^{nd} monoidal structure

NOTES (CONT. 3)

* XBiMod : category of crossed bimodules

Model Structure (from $s(k\text{-Alg})$) : w.e. are q-isos, fib. levelwise epis.

* $f : C \to D$ in XBiMod :

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, bend right] & & D \\ & E \arrow[ur] & \end{tikzcd}$$

$E_0 = C_0 \overset{C_1}{\ltimes} D_1$, $E_1 = D_1$

Factorization — universal push-out

* $\mathrm{Hom}_{Ho}(A, B) \simeq \mathrm{Hom}_{XBiMod}(Q, B)$

$$\begin{tikzcd} A & Q \arrow[l, "\sim"'] \arrow[r] & B \end{tikzcd}$$

Cofibrant

$$\begin{tikzcd} B \arrow[r, squiggly] & Q \arrow[rr] \arrow[dr] & & A \times B \\ & & E \arrow[ur] & \end{tikzcd}$$

universal push out

$$\begin{tikzcd} A_1 \times B_1 \arrow[d] \arrow[r] & E_0 \arrow[dl] \\ A_0 \times B_0 & \end{tikzcd} \rightsquigarrow \begin{tikzcd} A_1 \arrow[dr, "k"] \arrow[dd] & & B_1 \arrow[dl, "i"'] \arrow[dd] \\ & E_0 \arrow[dl, "\pi"'] \arrow[dr, "j"] & \\ A_0 & & B_0 \end{tikzcd}$$

E_0 : k-Algebra

$i(x\,j(e)) = i(x)\,e$, ...

\+ other axioms

* Over a site $\mathcal{S}$, we have ring-like stacks $\mathcal{A} \to \mathcal{S}$, $\mathcal{B} \to \mathcal{S}$, with $\mathcal{A} \simeq A_\cdot^{\sim}$, $\mathcal{B} \simeq B_\cdot^{\sim}$, and we have an alternative construction (as before)

$$\begin{tikzcd} A_1 \arrow[d] & & B_1 \arrow[d] \\ A_0 \arrow[d] \arrow[drr] & E_0 \arrow[l] \arrow[r] & B_0 \arrow[d] \\ \mathcal{A} \arrow[rr] & & \mathcal{B} \end{tikzcd}$$

, $E_0 = A_0 \times_{\mathcal{B}} B_0$

NOTES (CONT. 4)

* **Main Theorem** There is an equivalence of groupoids $\mathcal{H}om(\mathcal{A},\mathcal{B}) \simeq \mathbb{B}(\mathcal{A},\mathcal{B})$

Groupoid of Butterfly diagrams

Cohomology

Ring-like stack $\mathcal{R} \to \mathcal{S}$, w/ Presentation : $\mathcal{R} \simeq [R_1 \xrightarrow{\partial} R_0]^{\sim}$

$\mathbb{R} : R_0 \oplus R_1 \rightrightarrows R_0$: $\mathbb{R}$ groupoid — Strict categorical ring

[Baas Dundas Rognes / Osórno] $M_n(\mathbb{R})$: $n \times n$ matrices w/ entries in $\mathbb{R}$

strictness ⇒ definition not ambiguous.

$\mathcal{M} = M_n(\mathbb{R})$: monoidal w/ matrix product

$\mathbb{M} := B\mathcal{M} \equiv \mathcal{M}[1]$ bicategory w/ one object

(NonAbelian) Cohomology

$$H^i(e, GL_n(\mathcal{R})) = \underset{[X \to e]}{\operatorname{colim}} \, [X, N\, B\mathcal{M}]$$

$\mathrm{End}(\mathcal{R}^n)$ maybe this is better wrt invertibility questions

hypercovers

Nerve for bicategories BDR / Lack-Paoli / Carrasco-Cegarra-Garzón

NOTES (CONT. 5)

Example Computation with Čech coverings: $\coprod_i \mathcal{U}_i \longrightarrow \mathcal{U}$ cover of $\mathcal{U} \in \mathrm{Ob}\,\mathcal{S}$

A 1-cocycle has the form: $r_{ij} : \mathcal{U}_i \times_{\mathcal{U}} \mathcal{U}_j \longrightarrow R_0$, $m_{ijk} : \mathcal{U}_i \times_{\mathcal{U}} \mathcal{U}_j \times_{\mathcal{U}} \mathcal{U}_k \longrightarrow R_1$

$$r_{ij}\, r_{jk} = r_{ik} + \partial m_{ijk}$$

$$r_{ij}\, m_{jkl} - m_{ikl} + m_{ijl} - m_{ijk}\, r_{kl} = 0$$

Long exact sequence

Connected components: sheaf of rings

$$M = \pi_1(\mathcal{R})[1] \longrightarrow \mathcal{R} \longrightarrow \pi_0(\mathcal{R}) = A$$

"Ideals" in $\mathcal{R}$

$$\cdots \to H^2(e, M) \to H^1(e, \mathcal{R}) \longrightarrow H^1(e, A) \to H^3(e, M)$$

$[r_{ij}][r_{jk}] = [r_{ik}]$

Shukla Cohomology