Categorifying Algebraic Cycles and Intersection Theory

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Joint work with:

- Niranjan Ramachandran
  - *Cup products, the Heisenberg group, and codimension two algebraic cycles*
    Documenta Mathematica 21 (2016) 1313–1344
  - *Fiber integration of gerbes and Deligne line bundles*
    Homology, Homotopy and Applications, Volume 25 (2023) 21–51
- Niranjan Ramachandran & Maxime Ramzi
  - *Categorification of Chow Rings*

Plan of the talk

Introduction

A crash course on K-theory

The Gersten complex and conjecture

Categorification of cycles and higher torsors

Categorification of intersection theory

Codimension 2

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Categorification?

In mathematics, categorification is the process of replacing set-theoretic theorems with category-theoretic analogues. Categorification [...] replaces sets with categories, functions with functors, and equations with natural isomorphisms of functors satisfying additional properties.
Wikipedia (quoting Louis Crane)

Categorification is the process of promoting an algebraic object to one with more structure. [...] Lauda & Sussan, Notices AMS, January 2022.

Categorification, in the broad sense, refers to the realization of a mathematical object as the Grothendieck group of certain [higher] *category.*
Peng Shan, *Categorification and Applications*, ICM 2022

Motivation: Divisors and line bundles

- X smooth, proper variety over a field F. (More generally, separated smooth scheme of finite type over a field.)
- $Z^i(X)$ abelian group of algebraic cycles of codimension i on X.
- $\mathrm{CH}^i(X)$ Chow group of algebraic cycles codimension i modulo rational equivalence.

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- $Z^i(X)$ abelian group of algebraic cycles of codimension i on X.
- $\mathrm{CH}^i(X)$ Chow group of algebraic cycles codimension i modulo rational equivalence.
- For $i = 1$, $Z^1(X) =$ Divisors

$$
\begin{array}{ccc}
\mathrm{CH}^1(X) & \xrightarrow{\sim} & \mathrm{H}^1(X, \mathcal{O}_X^*) \\
\uparrow & & \uparrow \pi_0 \\
Z^1(X) & \longrightarrow & \textsc{Tors}(\mathcal{O}_X^*)
\end{array}
$$

$$D \longmapsto L_D$$

Divisorial correspondence

X connected, η generic point. Well-known exact sequence:

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \eta_* F(X)^* \xrightarrow{\text{Div}} \oplus_{x \in X^{(1)}} J_{x_*} \mathbb{Z} \longrightarrow 0$$

- Cycle $D = \sum n_x x$, with $n_x \in \mathbb{Z}$, determines (map of abelian sheaves)

$$\alpha_D : \mathbb{Z} \to \oplus_{x \in X^{(1)}} J_{x_*} \mathbb{Z}, \quad 1 \mapsto D = \sum n_x x.$$

- As sheaves of sets: $\alpha_D : * \longrightarrow \oplus_{x \in X^{(1)}} J_{x_*} \mathbb{Z}$
- The $\mathcal{O}_X^*$-torsor L_D is the pullback

$$\begin{array}{ccc} L_D & \longrightarrow & \eta_* F(X)^* \\ \downarrow & \lrcorner & \downarrow \text{Div} \\ * & \xrightarrow{\alpha_D} & \oplus_{x \in X^{(1)}} J_{x_*} \mathbb{Z} \end{array}$$

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  - Zariski open cover $\{U_i\}$ of X
  - $f_i \in F(X)^*$, $\operatorname{ord}_x(f) = n_x$ on $x|_{U_i}$.

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  - On $U_i \cap U_j$: $f_i = g_{ij} f_j$, $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$

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– On $U_i \cap U_j$: $f_i = g_{ij} f_j$, $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$
– $\{g_{ij}\}$ are the transition functions of L_D.

Higher codimension cycles?

S. Bloch, K_2 *and algebraic cycles*, 1979

- X smooth, algebraic surface over a field F.
  That a point $p \in X$ can *locally* be defined by a pair of equations $f = 0$, $g = 0$ gives

$$\mathrm{CH}^2(X) \xrightarrow{\simeq} \mathrm{H}^2(X, \mathcal{K}_{2,X})$$

- In fact,

$$\begin{array}{ccccc} \mathrm{H}^1(X, \mathcal{O}_X^*) \times \mathrm{H}^1(X, \mathcal{O}_X^*) & \xrightarrow{\cup} & \mathrm{H}^2(X, \mathcal{O}_X^* \otimes_{\mathbb{Z}} \mathcal{O}_X^*) & \xrightarrow{\{,\}} & \mathrm{H}^2(X, \mathcal{K}_{2,X}) \\ (\{g_{ij}\}, \{h_{ij}\}) & \longmapsto & & & \{g_{ij}, h_{jk}\} \end{array}$$

- Lift $L_1 \oplus L_2 \oplus L_1^{-1} \otimes L_2^{-1}$ through the central extension

$$1 \to \mathcal{K}_{2,X} \to \mathrm{St}(\mathcal{K}_{2,X}) \to \mathrm{SL}(\mathcal{K}_{2,X}) \to 1\,, \qquad \mathrm{SL}(A) = \varinjlim_n \mathrm{SL}_n(A)$$

Recap

Coefficients

"You are doing K-theory."

In fact, $\mathcal{O}_X^* = \mathcal{K}_{1,X}$.

Codimension 2

$\mathrm{H}^2(X, \mathcal{K}_{2,X})$: gerbes banded by $\mathcal{K}_{2,X}$.

In general

Codimension i cycle $\alpha \rightsquigarrow$ Higher stack $\mathcal{C}_\alpha$

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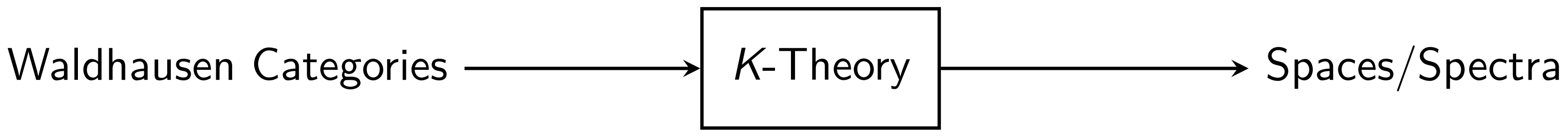
Algebraic K-theory



Waldhausen categories

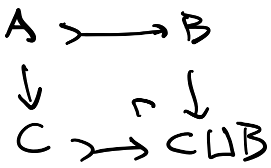
Waldhausen category

A Waldhausen category is a category $\mathcal{C}$ with:

- A zero object $*$
- A class of *cofibrations* (denoted $\rightarrowtail$) containing all isomorphisms
- A class of *weak equivalences* (denoted $\xrightarrow{\sim}$) containing all isomorphisms

Axioms:

1. $* \rightarrowtail A$ for all $A \in \mathcal{C}$
2. Pushouts of cofibrations exist and are cofibrations
3. Weak equivalences are closed under pushouts along cofibrations
4. Gluing axioms



Waldhausen categories II

Examples

- $\mathbf{Fin}_*$: Finite pointed sets
- ☞ $\mathbf{Mod}_R^{\mathrm{fg}}$: *Finitely generated projective modules over a ring R* ☜
- (Quillen) Exact categories
- Perfect complexes on schemes

Waldhausen categories II

Examples

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- (Quillen) Exact categories
- Perfect complexes on schemes

Pushouts and cofibers

Given a cofibration $A \rightarrowtail B$, the *cofiber* of $A \rightarrowtail B$ is the pushout

$$\begin{array}{ccc} A & \xrightarrow{\rightarrowtail} & B \\ \downarrow & & \downarrow \\ * \ \not{0} & \longrightarrow & B/A \end{array}$$

Waldhausen categories III: $S_\bullet$-construction

The $S_\bullet$*-construction* associates to a Waldhausen category $\mathcal{C}$ a simplicial category $wS_\bullet\mathcal{C}$.

- $wS_n\mathcal{C}$ has objects: diagrams

$$\begin{array}{ccccccccc}
* = A_{0,0} & \xrightarrow{\rightarrowtail} & A_{0,1} & \xrightarrow{\rightarrowtail} & A_{0,2} & \xrightarrow{\rightarrowtail} & \cdots & \xrightarrow{\rightarrowtail} & A_{0,n} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & * = A_{1,1} & \xrightarrow{\rightarrowtail} & A_{1,2} & \xrightarrow{\rightarrowtail} & \cdots & \xrightarrow{\rightarrowtail} & A_{1,n} \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & * = A_{2,2} & \xrightarrow{\rightarrowtail} & \cdots & \xrightarrow{\rightarrowtail} & A_{2,n} \\
 & & & & & & & & \vdots \\
 & & & & & & & & * = A_{n,n}
\end{array}$$

  such that each square is a pushout.
- Morphisms are levelwise weak equivalences.

K-Theory space/spectrum

- The *K-theory space* associated to a Waldhausen category $\mathcal{C}$ is

$$\mathrm{K}(\mathcal{C}) = \Omega|\operatorname{diag} N_{\bullet}(wS_{\bullet}\mathcal{C})| \overset{\text{def}}{=} \Omega|wS_{\bullet}\mathcal{C}|$$

- The *K-theory spectrum* is the sequence of spaces

$$\mathbf{K}(\mathcal{C}) = \Omega|wS_{\bullet}\mathcal{C}|,\ |wS_{\bullet}\mathcal{C}|,\ |wS_{\bullet}S_{\bullet}\mathcal{C}|,\ \dots,\ |wS_{\bullet}^{(n)}\mathcal{C}|,\ \dots$$

obtained by iterating the $S_{\bullet}$-construction, to form the multi-simplicial category $wS_{\bullet}^{(m)}\mathcal{C}$.

- The *K-groups* are the homotopy groups of the K-theory space/spectrum:

$$\mathrm{K}_i(\mathcal{C}) \overset{\text{def}}{=} \pi_i\mathbf{K}(\mathcal{C}) = \pi_i\mathrm{K}(\mathcal{C}), \quad i \geq 0.$$

K-Theory groups

Examples

- $\mathrm{K}_0(\mathcal{C})$ is the Grothendieck group of $\mathcal{C}$
- If R is a (nice commutative) ring, define $\mathrm{K}_i(R) = \mathrm{K}_i(\mathbf{Mod}_R^{\mathrm{fg}})$.
  - $\mathrm{K}_0(R) = \mathbb{Z}$
  - $\mathrm{K}_1(R) = R^*$

    (In general there is a map $\mathrm{K}_1(R) \longrightarrow R^*$)

K-Theory sheaves

Recall: X smooth, proper variety over a field F.
(More generally, separated smooth scheme of finite type over a field.)

Definition

- The *K-theory sheaf* $\mathcal{K}_{i,X}$ is the Zariski sheaf associated to the presheaf

$$U \longmapsto \mathrm{K}_i(\mathcal{O}_X^*(U))$$

- We can consider the *sheaf of spectra* $\mathbf{K}_X$ associated to the Zariski presheaf

$$U \longmapsto \mathbf{K}(\mathcal{O}_X^*(U))$$

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Definition

For $i \geq 0$, the Gersten Complex is the complex of sheaves on X

$$\mathcal{G}er_{i,X} : \eta_* K_i(F(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} i_* K_{i-1}(k(x)) \longrightarrow \cdots \longrightarrow \bigoplus_{x \in X^{(i-1)}} i_* K_1(k(x)) \longrightarrow \bigoplus_{x \in X^{(i)}} i_* K_0(k(x))$$

$i : x \longrightarrow X$

$F(X)$ is the function field of X, $X^{(j)}$ is the set of points of codimension j in X, and $k(x)$ is the residue field at x.

The Gersten conjecture

The Gersten complex $\mathcal{G}er_{i,X}$ is a *(flasque) resolution of the sheaf* $\mathcal{K}_{i,X}$.

The Gersten complex I

Definition

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The Gersten conjecture

The Gersten complex $\mathcal{G}er_{i,X}$ is a *(flasque) resolution of the sheaf* $\mathcal{K}_{i,X}$.

As a result:

$$0 \longrightarrow \mathcal{K}_{i,X} \longrightarrow K_i(F(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} K_{i-1}(k(x)) \longrightarrow \cdots \longrightarrow \bigoplus_{x \in X^{(i-1)}} K_1(k(x)) \longrightarrow \bigoplus_{x \in X^{(i)}} K_0(k(x)) \longrightarrow 0$$

is exact.

The Gersten complex II

Remarks

- $\oplus_{x \in X^{(i)}} K_0(k(x)) \cong \oplus_{x \in X^{(i)}} \mathbb{Z}$ is the sheaf associated to the presheaf $U \mapsto Z^i(U)$.

The Gersten complex II

Remarks

- $\bigoplus_{x\in X^{(i)}} K_0(k(x)) \cong \bigoplus_{x\in X^{(i)}} \mathbb{Z}$ is the sheaf associated to the presheaf $U \mapsto Z^i(U)$.
- $\bigoplus_{x\in X^{(i-1)}} K_1(k(x)) \longrightarrow \bigoplus_{x\in X^{(i)}} K_0(k(x))$ is the "ord" map:

$\| \; k(x)^*$

$$\bigoplus_{x\in X^{(i-1)}} k(x)^* \longrightarrow \bigoplus_{x\in X^{(i)}} \mathbb{Z}, \quad f \mapsto \sum_x \mathrm{ord}_x(f)x$$

The Gersten complex II

Remarks

- $\bigoplus_{x \in X^{(i)}} K_0(k(x)) \cong \bigoplus_{x \in X^{(i)}} \mathbb{Z}$ is the sheaf associated to the presheaf $U \mapsto Z^i(U)$.
- $\bigoplus_{x \in X^{(i-1)}} K_1(k(x)) \longrightarrow \bigoplus_{x \in X^{(i)}} K_0(k(x))$ is the "ord" map:
- $\bigoplus_{x \in X^{(i-2)}} K_2(k(x)) \longrightarrow \bigoplus_{x \in X^{(i-1)}} K_1(k(x))$ is given by tame symbols

$$T_x\{f, g\} = (-1)^{\mathrm{ord}_x(f)\,\mathrm{ord}_x(g)} \left(\frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}} \mod x \right) \in k(x)^*$$

By Matsumoto's theorem, $K_2(F)$ is generated by symbols $\{f, g\}$ satisfying bilinearity, anti-symmetry, and the *Steinberg relation* $\{f, 1-f\} = 1$.

Bloch-Quillen formula

Theorem (Bloch, Quillen)

If X is a smooth, proper variety over a field F, then

$$\mathrm{CH}^i(X) \xrightarrow{\simeq} \mathrm{H}^i(X, \mathcal{K}_{i,X})$$

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Remark

The part of the Gersten complex

$$\mathscr{Ger}_{i,X}\colon \cdots \longrightarrow \bigoplus_{x \in X^{(i-1)}} (k(x))^* \xrightarrow{\text{ord}} \bigoplus_{x \in X^{(i)}} \mathbb{Z} \longrightarrow 0$$

gives the cycles modulo rational equivalence.

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Animas and higher stacks

Following Česnavičius–Scholze, Lurie...

- The ∞-category $\mathcal{A}b$ of Abelian Animas of Animated Abelian Groups is the ∞-category obtained from the category of simplicial abelian groups by inverting weak equivalences.
- $\mathcal{A}b \simeq \mathcal{D}^{\leq 0}(\mathbb{Z})$, the connective part of the ∞-derived category of $\mathbb{Z}$.
- $Sh(X, \mathcal{A}b)$ is the ∞-category of Zariski sheaves of anima on X with values in $\mathcal{A}b$.

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Remarks

- Concretely, an object of $\mathcal{A}b$ is a simplicial abelian group $A_\bullet$ or a connective chain complex of abelian groups A_*.
- Dold-Kan correspondence:

$$\mathrm{Ch}_{\geq 0}(\mathrm{Ab}) \simeq s\mathrm{Ab}$$

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  but *invert weak equivalences.*

Classifying map of a cycle?

Write the Bloch-Quillen formula as

$$\mathrm{CH}^i(X) \cong \mathrm{H}^i(X, \mathcal{K}_{i,X}) \cong \mathrm{H}^0(X, \mathrm{B}^i\mathcal{K}_{i,X}) \cong \pi_0 \Gamma(X, \mathrm{B}^i\mathcal{K}_{i,X})$$

Problem/Question

Up to equivalence, a cycle $\alpha \in Z^i(X)$ should correspond to a "classifying" map

$$f_\alpha \colon * \longrightarrow \mathrm{B}^i\mathcal{K}_{i,X}$$

in $Sh(X, \mathcal{A}b)$. By pullback, we get the "$\mathrm{B}^{i-1}\mathcal{K}_{i,X}$ torsor" $\mathcal{C}_\alpha$ corresponding to α:

$$\begin{array}{ccc} \mathcal{C}_\alpha & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{f_\alpha} & \mathrm{B}^i\mathcal{K}_{i,X} \end{array}$$

$(E^i \mathcal{K}_{i,X})$

Bloch-Quillen-Gersten revisited

Write the Gersten resolution as a morphism

$$
\begin{array}{ccccccccc}
\mathcal{K}_{i,X} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
K_i(F(X)) & \longrightarrow & \oplus_{x\in X^{(1)}} K_{i-1}(k(x)) & \longrightarrow & \cdots & \longrightarrow & \oplus_{x\in X^{(i-1)}} K_1(k(x)) & \longrightarrow & \oplus_{x\in X^{(i)}} K_0(k(x))
\end{array}
$$

Bloch-Quillen-Gersten revisited

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\end{array}$$

Then $\mathcal{Ger}_{i,X}$ (or rather the anima it determines) is a model for $\mathrm{B}^i\mathcal{K}_{i,X}$ in $Sh(X, \mathcal{A}b)$.

Degrees are now *homological*, so $\mathcal{K}_{i,X}$ is in degree i. We should really write $\mathcal{Ger}_{i,X}[i]$.

Higher torsors attached to cycles

- Cycle $\alpha = \sum_x n_x x \in Z^i(X)$ determines

$$f_\alpha \colon \mathbb{Z}[0] \longrightarrow \mathcal{G}er_{i,X}[i]$$

sending $1 \in \mathbb{Z}$ to the vector $\{n_x\} \in \oplus_{x \in X^{(i)}} K_0(k(x))$.

- In $Sh(X, \mathcal{A}ni)$, we have

$$f_\alpha \colon * \longrightarrow \oplus_{x \in X^{(i)}} K_0(k(x)) \longrightarrow \mathcal{G}er_{i,X}[i] \simeq \mathrm{B}^i \mathcal{K}_{i,X}$$

- Pullback gives the $\mathrm{B}^{i-1}\mathcal{K}_{i,X}$-torsor $\mathcal{C}_\alpha$ attached to α:

$$\begin{array}{ccc} \mathcal{C}_\alpha & \longrightarrow & * \\ \big\downarrow & \lrcorner & \big\downarrow \\ * & \xrightarrow[f_\alpha]{} & \mathrm{B}^i \mathcal{K}_{i,X} \end{array}$$

Higher torsors attached to cycles II

- Rewrite the equivalence $\mathcal{G}er_{i,X}[i] \simeq \mathrm{B}^i\mathcal{K}_{i,X}$ as an extension in $Sh(X,\mathcal{A}b)$:

$$\mathrm{B}^{i-1}\mathcal{K}_{i,X} \longrightarrow \tau_{\geq 1}\mathcal{G}er_{i,X}[i] \longrightarrow \oplus_{x\in X^{(i)}} K_0(k(x))$$

- $\mathcal{C}_\alpha$ is the fiber of $\tau_{\geq 1}\mathcal{G}er_{i,X}[i]$ over the point $\alpha\colon * \to \oplus_{x\in X^{(i)}} K_0(k(x))$:

$$\begin{array}{ccccc}
\mathrm{B}^{i-1}\mathcal{K}_{i,X} & \longrightarrow & \tau_{\geq 1}\mathcal{G}er_{i,X}[i] & \longrightarrow & \oplus_{x\in X^{(i)}} K_0(k(x)) \\
 & & \uparrow & \lrcorner & \uparrow{\scriptstyle \alpha} \\
 & & \mathcal{C}_\alpha & \longrightarrow & *
\end{array}$$

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Intersection of cycles

Bloch-Quillen gives is compatible with intersection products:

$$\mathrm{CH}^i(X) \times \mathrm{CH}^j(X) \longrightarrow \mathrm{CH}^{i+j}(X)$$

Same as

Product in K-Theory

$$\mathrm{H}^i(X, \mathcal{K}_{i,X}) \times \mathrm{H}^j(X, \mathcal{K}_{j,X}) \xrightarrow{\cup} \mathrm{H}^{i+j}(X, \mathcal{K}_{i,X} \otimes \mathcal{K}_{j,X}) \xrightarrow{(*)} \mathrm{H}^{i+j}(X, \mathcal{K}_{i+j,X})$$

where the last map is induced by the product on K-theory spectra [Grayson]

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where the last map is induced by the product on K-theory spectra [Grayson]

Remark

There is no map at the cycle level:

$$Z^i(X) \times Z^j(X) \longrightarrow Z^{i+j}(X), \quad (\alpha, \beta) \longmapsto \alpha \cdot \beta$$

Intersection of cycles II

There is a map at the level of classifying maps of cycles:

$$* \xrightarrow{\alpha\wedge\beta} \mathrm{B}^i\mathcal{K}_{i,X} \wedge \mathrm{B}^j\mathcal{K}_{j,X} \longrightarrow \mathrm{B}^{i+j}(\mathcal{K}_{i,X} \otimes \mathcal{K}_{j,X}) \longrightarrow \mathrm{B}^{i+j}\mathcal{K}_{i+j,X}$$

Let $\mathcal{C}_{\alpha\wedge\beta}$ be the resulting $\mathrm{B}^{i+j-1}\mathcal{K}_{i+j,X}$-torsor.

Intersection of cycles II

There is a map at the level of classifying maps of cycles:

$$* \xrightarrow{\alpha\wedge\beta} \mathrm{B}^i\mathcal{K}_{i,X} \wedge \mathrm{B}^j\mathcal{K}_{j,X} \longrightarrow \mathrm{B}^{i+j}(\mathcal{K}_{i,X} \otimes \mathcal{K}_{j,X}) \longrightarrow \mathrm{B}^{i+j}\mathcal{K}_{i+j,X}$$

Let $\mathcal{C}_{\alpha\wedge\beta}$ be the resulting $\mathrm{B}^{i+j-1}\mathcal{K}_{i+j,X}$-torsor.

Questions

- Can we describe $\mathcal{C}_{\alpha\wedge\beta}$ in terms of $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$?
- Is $\mathcal{C}_{\alpha\wedge\beta}$ equivalent to $\mathcal{C}_{\alpha\cdot\beta}$?

Introduction

A crash course on K-theory

The Gersten complex and conjecture

Categorification of cycles and higher torsors

Categorification of intersection theory

Codimension 2

$\mathcal{K}_2$-gerbes attached to codimension 2 cycles

In codimension 2 we have the four-term exact sequence:

$$0 \longrightarrow \mathcal{K}_{2,X} \longrightarrow K_2(F(X)) \xrightarrow{\text{Tame}} \bigoplus_{x \in X^{(1)}} K_1(k(x)) \xrightarrow{\text{ord}} \bigoplus_{x \in X^{(2)}} K_0(k(x)) \longrightarrow 0$$

If α is a codimension 2 cycle, the corresponding $\mathrm{B}^1\mathcal{K}_{2,X}$-torsor $\mathcal{C}_\alpha$ is a $\mathcal{K}_{2,X}$-gerbe.

Theorem (E.A.–N. Ramachandran, 2016)

- The class of $\mathcal{C}_\alpha$ in $\mathrm{H}^2(X, \mathcal{K}_{2,X})$ corresponds to $\alpha \in \mathrm{CH}^2(X)$ via the Bloch-Quillen isomorphism.
- $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ are equivalent as $\mathcal{K}_{2,X}$-gerbes if and only if α and β are rationally equivalent.

$$0 \to \mathbb{Z}/2 \to D_8 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 0$$

The Heisenberg group is the central extension of abelian sheaves on X

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_{1,X} \otimes \mathcal{K}_{1,X} & \longrightarrow & H & \longrightarrow & \mathcal{K}_{1,X} \times \mathcal{K}_{1,X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K}_{2,X} & \longrightarrow & \hat{H} & \longrightarrow & \mathcal{K}_{1,X} \times \mathcal{K}_{1,X} & \longrightarrow & 0
\end{array}$$

$\{a, b'\} \in \mathcal{K}_{2,X}$

$((a,b),(a',b')) \mapsto a \otimes b'$

The classifying map is really the identity on $K_1 \otimes K_1$

The classifying map $\mathrm{B}\mathcal{K}_{1,X} \times \mathrm{B}\mathcal{K}_{1,X} \to \mathrm{B}^2\mathcal{K}_{2,X}$ corresponds to the gerbe of liftings of a $\mathcal{K}_{1,X} \times \mathcal{K}_{1,X}$-torsor to an H-torsor.

The cup product $\mathrm{H}^1(X, \mathcal{K}_{1,X}) \times \mathrm{H}^1(X, \mathcal{K}_{1,X}) \longrightarrow \mathrm{H}^2(X, \mathcal{K}_{2,X})$ arises from the Heisenberg group via the class of the extension.

The $\mathcal{K}_{2,X}$-gerbe of an intersection

Let D, E be divisors on X, with associated $\mathcal{O}_X^*$-torsors L, M.
The corresponding classifying maps f_D, f_E determine a $\mathcal{K}_{2,X}$-gerbe $\mathcal{C}_{D,E}$ via the pullback

$$* \longrightarrow \mathrm{B}\mathcal{K}_{1,X} \times \mathrm{B}\mathcal{K}_{1,X} \longrightarrow \mathrm{B}^2\mathcal{K}_{2,X}$$

of the class of the Heisenberg extension.

Theorem (E.A.–N. Ramachandran, 2016)

Let $\alpha = D \cdot E$ be the intersection cycle. There is a natural equivalence of $\mathcal{K}_{2,X}$-gerbes

$$\mathcal{C}_\alpha \xrightarrow{\sim} \mathcal{C}_{D,E}$$

Behavior under certain morphisms

Let $\pi\colon X \to S$ be a smooth projective morphism of relative dimension one.

Theorem (E.A.–N. Ramachandran, 2023)

- There exists a natural additive functor

$$\int_\pi : \mathrm{B}^2\mathcal{K}_{2,X} \longrightarrow \mathrm{B}^1\mathcal{K}_{1,S} \simeq \mathrm{TORS}(\mathcal{O}_S^*)$$

- Suppose that L, M are $\mathcal{O}_X^*$-torsors corresponding to divisors D, E on X. Let $\mathcal{C}_{D,E}$ be the $\mathcal{K}_{2,X}$-gerbe of liftings associated to L, M. Then

$$\int_\pi \mathcal{C}_{D,E} \cong \langle L, M\rangle$$

where $\langle L\rangle, M$ is the Deligne pairing of L and M.

$\langle L, M\rangle$

End

Thank you!