Stacks of Categorical Rings and Their Morphisms

Ettore Aldrovandi

Florida State University

Octoberfest 2015 Annual Meeting Ottawa, October 31-November 1

- (Strictly) Picard Categorical Rings ⇝ (Regular) Ann-Categories (Hanoi)
- Crossed Bimodules (Presentations)
- Morphisms (Spans ⟷ Butterflies)
- What do categorical rings form?
- Taxonomy
  - Classification
  - Non Regular Cat. Rings

arXiv 1501.07592, TAC 30 ('15)

arXiv 1501.04664

Categorical Rings

▷ $$\begin{tikzcd} \mathcal{C} \arrow[d, "p"] \\ \mathcal{S} \end{tikzcd}$$

$\mathcal{C} \leftarrow$ Stack in groupoids

$\mathcal{S} \leftarrow$ Site

morphisms of fibered categories $/\mathcal{S}$

▷ Two monoidal structures $\oplus, \otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

- $(\mathcal{C}, \oplus, 0_{\mathcal{C}})$ is group-like symmetric (= Picard)
- $(\mathcal{C}, \otimes, \oplus, 0_{\mathcal{C}}, I_{\mathcal{C}})$ distributive

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

Bimonoidal with respect to $\oplus$

Objects X, Y, Z, W : $(X \oplus Y) \otimes Z \simeq (X \otimes Z) \oplus (Y \otimes Z)$

$X \otimes (Z \oplus W) \simeq (X \otimes Z) \oplus (X \otimes W)$

Categorical Rings

- $$\begin{array}{l} \mathcal{C} \longleftarrow \text{Stack in groupoids} \\ \downarrow p \\ \mathcal{S} \longleftarrow \text{Site} \end{array}$$

morphisms of fibered categories $/\mathcal{S}$

- Two monoidal structures $\oplus, \otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

  - $(\mathcal{C}, \oplus, 0_{\mathcal{C}})$ is group-like symmetric $\Leftarrow$ More about this in a moment
  - $(\mathcal{C}, \otimes, \oplus, 0_{\mathcal{C}}, I_{\mathcal{C}})$ distributive

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

Bimonoidal with respect to $\oplus$

Objects X, Y, Z, W : $(X \oplus Y) \otimes Z \simeq (X \otimes Z) \oplus (Y \otimes Z)$

$X \otimes (Z \oplus W) \simeq (X \otimes Z) \oplus (X \otimes W)$

About Group-like / Categorical Groups

$(\mathscr{C}, \oplus, 0_{\mathscr{C}})$ admits a presentation $\underbrace{C_1 \xrightarrow{\partial} C_0}_{\text{crossed module}} \longrightarrow \mathscr{C}$

- Presheaf $/\mathcal{S}$: $U \rightsquigarrow \Gamma_U : C_0(U) \rtimes C_1(U) \rightrightarrows C_0(U)$

- Equivalence : $\underbrace{[C_0 \rtimes C_1 \rightrightarrows C_0]^{\sim}}_{\text{associated stack}} \xrightarrow{\sim} \mathscr{C}$

$$\boxed{\begin{tikzcd} \Gamma \arrow[rr, "(\cdot)^{\sim}"] \arrow[dr] & & \mathscr{C} \arrow[dl] \\ & \mathcal{S} & \end{tikzcd}}$$

(Folk Thm, E.A. – B. Noohi : 09)

About Group-like / Categorical Groups

$(\mathcal{C}, \oplus, 0_{\mathcal{C}})$ admits a presentation $\underbrace{C_1 \xrightarrow{\partial} C_0}_{\text{Stable crossed module}} \longrightarrow \mathcal{C}$

– Condurché
– Joyal - Street

- Presheaf $/ \mathcal{S}$: $U \rightsquigarrow \Gamma_U : C_0(U) \rtimes C_1(U) \rightrightarrows C_0(U)$
- Equivalence : $[C_0 \rtimes C_1 \rightrightarrows C_0]^{\sim} \xrightarrow{\sim} \mathcal{C}$
  associated stack
- $\{\ ,\ \} : C_0 \times C_0 \longrightarrow C_1$ anti-symmetric

About Group-like / Categorical Groups

$(\mathcal{C}, \oplus, 0_{\mathcal{C}})$ admits a presentation $\underbrace{C_1 \xrightarrow{\partial} C_0}_{\text{Picard crossed module}} \longrightarrow \mathcal{C}$

- Presheaf$/_{\mathcal{S}}$: $U \rightsquigarrow \Gamma_U : C_0(U) \rtimes C_1(U) \rightrightarrows C_0(U)$
- Equivalence : $\underbrace{[C_0 \rtimes C_1 \rightrightarrows C_0]^{\sim}}_{\text{associated stack}} \xrightarrow{\sim} \mathcal{C}$
- $\{\ ,\ \} : C_0 \times C_0 \longrightarrow C_1$ Alternating : $\boxed{\{\ ,\ \} \circ \Delta_{C_0} = e_{C_1}}$

About Group-like / Categorical Groups

$(\mathcal{C}, \oplus, 0_{\mathcal{C}})$ admits a presentation $\underbrace{\overbrace{C_1 \xrightarrow{\partial} C_0}^{\text{Abelian Groups}}}_{\text{Picard crossed module}} \longrightarrow \mathcal{C}$

- Presheaf $/\mathcal{J}$: $U \rightsquigarrow \Gamma_U : \; C_0(U) \ltimes C_1(U) \rightrightarrows C_0(U)$
- Equivalence : $[C_0 \ltimes C_1 \rightrightarrows C_0]^{\sim} \xrightarrow{\simeq} \mathcal{C}$
  associated stack
- $\{\,,\,\} : C_0 \times C_0 \longrightarrow C_1$ ALTERNATING : $\boxed{\{\,,\,\} \circ \Delta_{C_0} = e_{C_1}}$

STANDING ASSUMPTION

(Back to) Categorical Rings

Definition $C_1 \xrightarrow{\partial} C_0$ is a crossed bimodule if

(i) C_0 is a ring (of $Sh(\mathcal{S}) = \mathcal{S}^{\sim}$), with 1, usually.

(ii) C_1 —— C_0-bimodule

(iii) PFEIFFER: $\forall c_1, c_1' \in C_1$ $\quad (\partial c_1)\, c_1' = c_1\, (\partial c_1')$

(Back to) Categorical Rings

Definition $C_1 \xrightarrow{\partial} C_0$ is a crossed bimodule if

(i) C_0 is a ring (of $Sh(\mathcal{S}) = \mathcal{S}^\sim$), with 1, usually.

(ii) C_1 —— C_0-bimodule

(iii) PFEIFFER: $\forall c_1, c_1' \in C_1$ $\quad (\partial c_1)\, c_1' = c_1\, (\partial c_1')$

Theorem (EA '15) $\mathcal{C}$: STRICTLY PICARD Categorical Ring of $\mathcal{S}^\sim$

$\exists$ Presentation $\boxed{C_1 \xrightarrow{\partial} C_0} \longrightarrow \mathcal{C}$

Crossed Bimodule of $\mathcal{S}$

Morphisms

Morphism $F : \mathcal{C} \longrightarrow \mathcal{D}$ of strictly Picard categorical rings

① $F : \mathcal{C} \longrightarrow \mathcal{D}$: morphism b/t underlying strictly Picard stacks

②
$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, "F \times F"'] \arrow[dr, phantom, "\lambda \swArrow"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D} \times \mathcal{D} \arrow[r, "\otimes"'] & \mathcal{D}
\end{tikzcd}$$
$+$ Conditions on λ $\Longrightarrow$

Morphisms

Morphism $F : \mathcal{C} \longrightarrow \mathcal{B}$ of strictly Picard categorical rings

① $F : \mathcal{C} \longrightarrow \mathcal{B}$: morphism b/t underlying strictly Picard stacks

Span (in $Ch_+(\tilde{S}_{ab})$)

$$\begin{tikzcd}
 & E_1 \arrow[dl] \arrow[d] \arrow[dr] & \\
C_1 \arrow[d] & E \arrow[dl] \arrow[dr] & B_1 \arrow[d] \\
C_0 & & B_0
\end{tikzcd}$$

(Deligne, SGA 4, XVIII)

Morphisms

Morphism $F : \mathcal{C} \longrightarrow \mathcal{D}$ of strictly Picard categorical rings

① $F : \mathcal{C} \longrightarrow \mathcal{B}$: morphism b/t underlying strictly Picard stacks

Butterfly in $\mathrm{Ch}_+(\widetilde{\mathcal{S}_{ab}})$

exact

$$
\begin{tikzcd}
C_1 \arrow[dd] \arrow[dr] & & B_1 \arrow[dl] \arrow[dd] \\
& E \arrow[dl] \arrow[dr] & \\
C_0 & & B_0
\end{tikzcd}
$$

Morphisms

Morphism $F : \mathcal{C} \longrightarrow \mathcal{D}$ of strictly Picard categorical rings

① $F : \mathcal{C} \longrightarrow \mathcal{B}$: morphism b/t underlying strictly Picard stacks

Butterfly in $\mathrm{Ch}_+(\widetilde{\mathcal{S}_{ab}})$

Not

exact

$$\begin{tikzcd}
C_1 \arrow[d] \arrow[dr, "k"] & & B_1 \arrow[dl, "i"'] \arrow[d] \\
C_0 & E \arrow[l, "p"] \arrow[r, "j"'] & B_0
\end{tikzcd}$$

Morphisms

Morphism $F: \mathcal{C} \longrightarrow \mathcal{D}$ of strictly Picard categorical rings

① $F: \mathcal{C} \longrightarrow \mathcal{D}$: morphism b/t underlying strictly Picard stacks

②
$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, "F \times F"'] \arrow[dr, phantom, "\swarrow\lambda"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D} \times \mathcal{D} \arrow[r, "\otimes"'] & \mathcal{D}
\end{tikzcd}$$
$+$ Conditions on $\lambda \Longrightarrow$

Morphisms

We need the $\bowtie$ between crossed bimodules

Definition

Crossed Bimodules

$$\begin{tikzcd}
C_1 \arrow[d] \arrow[dr, "k"] & & B_1 \arrow[d] \arrow[dl, "i"'] \\
 & E \arrow[dl, "p"] \arrow[dr, "j"'] & \\
C_0 & & B_0
\end{tikzcd}$$

Ring Extension

1. $E \xrightarrow{p} C_0$ ring homo.
2. $B_1 \xrightarrow{i} E$ (bilateral) ideal
3. $B_1^2 \neq 0$, in general

$\rightsquigarrow$ So non singular

Morphisms

We need the $\bowtie$ between crossed bimodules

Definition

Still not exact

Just: $j \circ k = 0$

① $E \xrightarrow{j} C_0$ ring homo.

$$\begin{tikzcd}
C_1 \arrow[dd] \arrow[dr, "k"] & & B_1 \arrow[dd] \arrow[dl, "i"'] \\
& E \arrow[dl, "p"'] \arrow[dr, "j"] & \\
C_0 & & B_0
\end{tikzcd}$$

Ring Extension

① $E \xrightarrow{p} C_0$ ring homo.

② $B_1 \xrightarrow{i} E$ (bilateral) ideal

③ $B_1^2 \neq 0$, in general

$\rightsquigarrow$ So non singular

Morphisms

We need the $\bowtie$ between crossed bimodules

Definition

$$\text{(i)} \quad e\, i(b_1) = i\big(j(e)\, b_1\big)$$

$$\text{(ii)} \quad i(b_1)\, e = i\big(b_1\, j(e)\big)$$

Still not exact

Just: $j \circ k = 0$

① $E \xrightarrow{j} C_0$ ring homo.

$$\begin{tikzcd}
C_1 \arrow[dd] \arrow[dr, "k"] & & B_1 \arrow[dl, "i"'] \arrow[dd] \\
& E \arrow[dl, "p"'] \arrow[dr, "j"] & \\
C_0 & & B_0
\end{tikzcd}$$

Ring Extension

① $E \xrightarrow{p} C_0$ ring homo.

② $B_1 \xrightarrow{i} E$ (bilateral) ideal

③ $B_1^2 \neq 0$, in general

$\rightsquigarrow$ So non singular

Morphisms

We need the $\bowtie$ between crossed bimodules

Definition

(i) $e\, i(b_1) = i\left(j(e)\, b_1\right)$

(ii) $i(b_1)\, e = i\left(b_1\, j(e)\right)$

(iii) $e\, k(c_1) = k\left(p(e)\, c_1\right)$

(iv) $k(c_1)\, e = k\left(c_1\, p(e)\right)$

Still not exact

Just: $j \circ k = 0$

① $E \xrightarrow{j} C_0$ ring homo.

$$\begin{tikzcd}
C_1 \arrow[dd] \arrow[dr, "k"] & & B_1 \arrow[dl, "i"'] \arrow[dd] \\
& E \arrow[dl, "p"'] \arrow[dr, "j"] & \\
C_0 & & B_0
\end{tikzcd}$$

Ring Extension

① $E \xrightarrow{p} C_0$ ring homo.

② $B_1 \xrightarrow{i} E$ (bilateral) ideal

③ $B_1^2 \neq 0$, in general

$\rightsquigarrow$ So non singular

Morphisms

▷ morphisms $F: \mathcal{C} \to \mathcal{B}$ form a groupoid $\underline{\mathrm{Hom}}(\mathcal{C},\mathcal{B})$: $\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow \theta"] & \mathcal{B} \end{tikzcd}$

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, "F \times F"'] \arrow[dr, phantom, "\swarrow \lambda^F"] & \mathcal{C} \arrow[d, "F"] \arrow[d, bend left=60, "G"] \\ \mathcal{B} \times \mathcal{B} \arrow[r, "\otimes"'] & \mathcal{B} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, bend right=50, "F \times F"'] \arrow[d, bend left=50, "G \times G"] \arrow[d, phantom, "\Rightarrow"] \arrow[dr, phantom, "\swarrow \lambda^G"] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{B} \times \mathcal{B} \arrow[r, "\otimes"'] & \mathcal{B} \end{tikzcd}$$

Morphisms

▷ morphisms $F : \mathcal{C} \to \mathcal{B}$ form a groupoid $\underline{\mathrm{Hom}}(\mathcal{C},\mathcal{B})$: $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\theta \Downarrow"] & \mathcal{B} \end{tikzcd}$

$$
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, "F \times F"'] \arrow[dr, phantom, "\swarrow \lambda^F"] & \mathcal{C} \arrow[d, "F"'] \arrow[d, bend left=60, "G"] \arrow[d, phantom, shift left=3, "\Rightarrow"] \\
\mathcal{B} \times \mathcal{B} \arrow[r, "\otimes"'] & \mathcal{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, bend right=50, "F \times F"'] \arrow[d, bend left=50, "G \times G"] \arrow[d, phantom, "\Rightarrow"] \arrow[dr, phantom, "\swarrow \lambda^G"] & \mathcal{C} \arrow[d, "G"] \\
\mathcal{B} \times \mathcal{B} \arrow[r, "\otimes"'] & \mathcal{B}
\end{tikzcd}
$$

▷ So do butterflies: $\mathcal{S}p(C_\bullet, B_\bullet)$

$$
\begin{tikzcd}
C_1 \arrow[dddd] \arrow[ddr, "k"'] \arrow[dr, "k'"] & & B_1 \arrow[dddd] \arrow[ddl, "i"] \arrow[dl, "i'"'] \\
& E' \arrow[dddl, "p'"'] \arrow[dddr, "j'"] & \\
& E \arrow[u] \arrow[ddl, "p"] \arrow[ddr, "j"'] & \\
& & \\
C_0 & & B_0
\end{tikzcd}
$$

Morphisms

Theorem (EA, TAC 30 (2015))

$\mathcal{C}, \mathcal{B}$: Strictly Picard Cat. Rings of $\mathcal{S}^\sim$

$$\left.\begin{array}{l} C_1 \longrightarrow C_0 \longrightarrow \mathcal{C} \\ B_1 \longrightarrow B_0 \longrightarrow \mathcal{B} \end{array}\right\} \text{presentations}$$

There is an equivalence of groupoids

$$\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{B}) \xrightarrow{\sim} \underline{\mathrm{Sp}}(C_\bullet, B_\bullet)$$

Morphisms

Theorem (EA, TAC $\underline{30}$ (2015))

$\mathcal{C}, \mathcal{B}$: Strictly Picard Cat. Rings of $\mathcal{S}^{\sim}$

$$\left.\begin{array}{l} C_1 \longrightarrow C_0 \longrightarrow \mathcal{C} \\ B_1 \longrightarrow B_0 \longrightarrow \mathcal{B} \end{array}\right\rangle \text{presentations}$$

There is an equivalence of groupoids

$$\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{B}) \xrightarrow{\sim} \underline{\mathrm{Sp}}(C_\bullet, B_\bullet)$$

Proof

$$\begin{tikzcd}
C_1 \arrow[d] & & B_1 \arrow[d] \\
C_0 \arrow[d] & & B_0 \arrow[d] \\
\mathcal{C} \arrow[rr, "F"'] & & \mathcal{B}
\end{tikzcd}$$

Morphisms

Theorem (EA, TAC 30 (2015))

$\mathcal{C}, \mathcal{B}$: Strictly Picard Cat. Rings of $\mathcal{S}^\sim$

$$\left.\begin{array}{l} C_1 \longrightarrow C_0 \longrightarrow \mathcal{C} \\ B_1 \longrightarrow B_0 \longrightarrow \mathcal{B} \end{array}\right\} \text{presentations}$$

There is an equivalence of groupoids

$$\underline{Hom}(\mathcal{C}, \mathcal{B}) \xrightarrow{\sim} \underline{Sp}(C_\bullet, B_\bullet)$$

Proof

$$\begin{tikzcd}
C_1 \arrow[d] & & B_1 \arrow[d] \\
C_0 \arrow[d, "\pi_{\mathcal{C}}"'] \arrow[drr] & C_0 \times_{\mathcal{B}} B_0 \arrow[l] \arrow[r] & B_0 \arrow[d, "\pi_{\mathcal{B}}"] \\
\mathcal{C} \arrow[rr, "F"'] & & \mathcal{B}
\end{tikzcd}$$

Stack fiber product

(c_0, f, b_0),

$$\pi(c_0) \xrightarrow{f} \pi(b_0) \text{ in } \mathcal{B}$$

Morphisms

Theorem (EA, TAC $\underline{30}$ (2015))

$\mathcal{C}, \mathcal{B}$: Strictly Picard Cat. Rings of $\mathcal{S}^\sim$

$$\left.\begin{array}{l} C_1 \longrightarrow C_0 \longrightarrow \mathcal{C} \\ B_1 \longrightarrow B_0 \longrightarrow \mathcal{B} \end{array}\right\} \text{presentations}$$

There is an equivalence of groupoids

$$\underline{Hom}(\mathcal{C}, \mathcal{B}) \xrightarrow{\sim} \underline{Sp}(C_\bullet, B_\bullet)$$

Proof

Exact cat. groups

$\Downarrow$

$$\begin{tikzcd}
C_1 \arrow[d] \arrow[dr] & & B_1 \arrow[d] \arrow[dl] \\
& C_0 \times_{\mathcal{B}} B_0 \arrow[dl] \arrow[dr] & \\
C_0 \arrow[d, "\pi_{\mathcal{C}}"'] \arrow[drr] & & B_0 \arrow[d, "\pi_{\mathcal{B}}"] \\
\mathcal{C} \arrow[rr, "F"'] & & \mathcal{B}
\end{tikzcd}$$

Morphisms

Theorem (EA, TAC $\underline{30}$ (2015))

$\mathcal{C}, \mathcal{B}$: Strictly Picard Cat. Rings of $\mathcal{S}^{\sim}$

$C_1 \longrightarrow C_0 \longrightarrow \mathcal{C}$
$B_1 \longrightarrow B_0 \longrightarrow \mathcal{B}$ $\Big\}$ presentations

There is an equivalence of groupoids

$$\underline{Hom}(\mathcal{C}, \mathcal{B}) \xrightarrow{\sim} \underline{Sp}(C_\bullet, B_\bullet)$$

Proof

Exact cat. groups $\Downarrow$

Exact $\Leftarrow$

$$\begin{tikzcd}
C_1 \arrow[dd] \arrow[dr] & & B_1 \arrow[dd] \arrow[dl] \\
& C_0 \times_{\mathcal{B}} B_0 \arrow[dl] \arrow[dr] & \\
C_0 \arrow[d, "\pi_{\mathcal{C}}"'] \arrow[drr] & & B_0 \arrow[d, "\pi_{\mathcal{B}}"] \\
\mathcal{C} \arrow[rr, "F"'] & & \mathcal{B}
\end{tikzcd}$$

Morphisms

Theorem (EA, TAC **30** (2015))

$\mathcal{C}, \mathcal{B}$: Strictly Picard Cat. Rings of $\mathcal{S}^{\sim}$

$$\left.\begin{array}{l} C_1 \longrightarrow C_0 \longrightarrow \mathcal{C} \\ B_1 \longrightarrow B_0 \longrightarrow \mathcal{B} \end{array}\right\} \text{presentations}$$

There is an equivalence of groupoids

$$\underline{Hom}(\mathcal{C}, \mathcal{B}) \xrightarrow{\sim} \underline{Sp}(C_\bullet, B_\bullet)$$

Proof

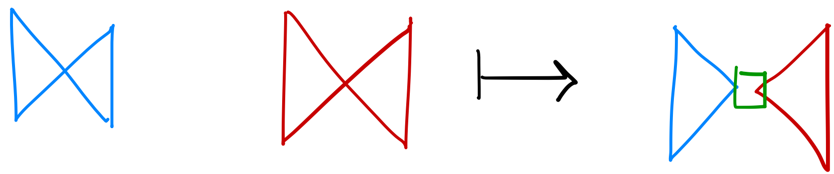
Not exact $\Rightarrow$

$$\begin{tikzcd}
C_1 \arrow[dr] \arrow[dd, "\partial"'] & & B_1 \arrow[dl] \arrow[dd] \\
& C_0 \times_{\mathcal{B}} B_0 \arrow[dl] \arrow[dr] & \\
C_0 \arrow[d, "\pi_{\mathcal{C}}"'] \arrow[drr, "F\circ\pi_{\mathcal{C}}"] & & B_0 \arrow[d, "\pi_{\mathcal{B}}"] \\
\mathcal{C} \arrow[rr, "F"'] & & \mathcal{B}
\end{tikzcd}$$

Upgraded Correspondence

- Bicategory $\underline{\mathrm{XMod}}(\mathcal{S})$
  - * Objects : $C_1 \to C_0$ crossed bimodules
  - * (Groupoids) Morphisms: $\underline{\mathrm{Sp}}(C_\bullet, B_\bullet)$ with composition

$$\underline{\mathrm{Sp}}(D_\bullet, C_\bullet) \times \underline{\mathrm{Sp}}(C_\bullet, B_\bullet) \longrightarrow \underline{\mathrm{Sp}}(D_\bullet, B_\bullet)$$

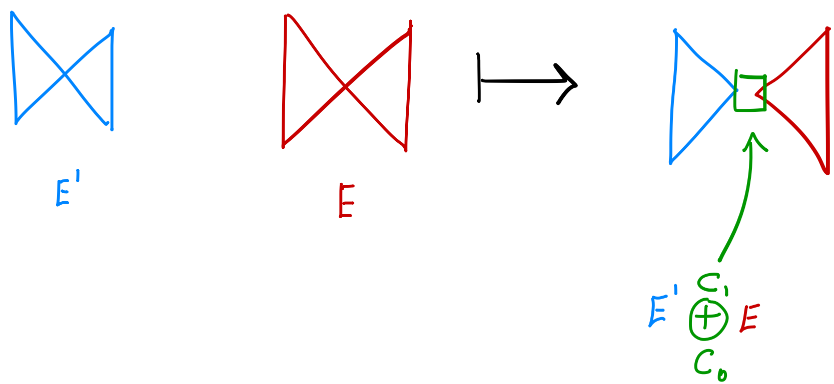


Upgraded Correspondence

▷ Bicategory $\underline{\mathrm{XMod}}(\mathcal{S})$

- Objects: $C_1 \to C_0$ crossed bimodules
- (Groupoids) Morphisms: $\underline{\mathrm{Sp}}(C_\bullet, B_\bullet)$ with composition

$$\underline{\mathrm{Sp}}(D_\bullet, C_\bullet) \times \underline{\mathrm{Sp}}(C_\bullet, B_\bullet) \longrightarrow \underline{\mathrm{Sp}}(D_\bullet, B_\bullet)$$



Upgraded Correspondence

- Bicategory $\underline{\underline{\mathrm{XMod}}}(\mathcal{S})$
- 2-category $\mathcal{P}ic(\mathcal{S})$: Strictly Picard Stacks$/\mathcal{S}$
  - has $\otimes : \mathcal{P}ic(\mathcal{S}) \times \mathcal{P}ic(\mathcal{S}) \longrightarrow \mathcal{P}ic(\mathcal{S})$ (Deligne, SGA 4, XVIII)
  - $2\mathcal{R}ing(\mathcal{S})$: 2monoid objects in $\mathcal{P}ic(\mathcal{S})$

Upgraded Correspondence

- Bicategory $\underline{\underline{\mathrm{XMod}}}(\mathcal{S})$
- 2-category $\mathcal{P}ic(\mathcal{S})$: Strictly Picard Stacks $/\mathcal{S}$
  - has $\otimes : \mathcal{P}ic(\mathcal{S}) \times \mathcal{P}ic(\mathcal{S}) \longrightarrow \mathcal{P}ic(\mathcal{S})$ (Deligne, SGA 4, XVIII)
  - $2\mathcal{R}ing(\mathcal{S})$: 2monoid objects in $\mathcal{P}ic(\mathcal{S})$

    = Our Strictly Picard Categorical rings

Upgraded Correspondence

- Bicategory $\underline{\underline{\mathrm{XMod}}}(\mathcal{S})$
- 2-category $\mathcal{P}\mathrm{ic}(\mathcal{S})$: Strictly Picard Stacks$/\mathcal{S}$
  - has $\otimes : \mathcal{P}\mathrm{ic}(\mathcal{S}) \times \mathcal{P}\mathrm{ic}(\mathcal{S}) \to \mathcal{P}\mathrm{ic}(\mathcal{S})$ (Deligne, SGA 4, XVIII)
  - $2\mathcal{R}\mathrm{ing}(\mathcal{S})$: 2monoid objects in $\mathcal{P}\mathrm{ic}(\mathcal{S})$
    = Our Strictly Picard Categorical rings

- **Theorem** (EA, ibid.) There is a biequivalence

$$\underline{\underline{\mathrm{XMod}}}(\mathcal{S}) \xrightarrow{\sim} 2\mathcal{R}\mathrm{ings}(\mathcal{S})$$

$$C_1 \to C_0 \longmapsto \left[C_0 \rtimes C_1 \rightrightarrows C_0\right]^{\sim}$$

Shukla, Barr-Beck, André-Quillen

Back to the crossed bimodule $C_1 \xrightarrow{\partial} C_0$

Crossed extension

$$0 \to M \to C_1 \to C_0 \to \Lambda \to 0$$

Λ-bimodule

Equivalence:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & C_1 \arrow[rr] \arrow[dr] & & C_0 \arrow[r] & \Lambda \arrow[r] \arrow[d, equal] & 0 \\
 & & & E \arrow[ur] \arrow[dr] & & & \\
0 \arrow[r] & M \arrow[r] & C_1' \arrow[rr] \arrow[ur] & & C_0' \arrow[r] & \Lambda \arrow[r] & 0
\end{tikzcd}$$

Shukla, Barr-Beck, André-Quillen

Back to the crossed bimodule $C_1 \xrightarrow{\partial} C_0$

Crossed extension

$$0 \to M \to C_1 \to C_0 \to \Lambda \to 0$$

Λ-bimodule

Equivalence:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & C_1 \arrow[rr] \arrow[dr] & & C_0 \arrow[r] & \Lambda \arrow[r] \arrow[d, equal] & 0 \\
& M & & E \arrow[ur] \arrow[dr] & & \Lambda & \\
0 \arrow[r] & M \arrow[r] & C_1' \arrow[rr] \arrow[ur] & & C_0' \arrow[r] & \Lambda \arrow[r] & 0
\end{tikzcd}$$

Well known: $\mathrm{XExt}(\Lambda, M) \cong SH^3(\Lambda, M)$ ⟵ Also: Cotriple, AQ

Depending only on $\mathcal{C} \simeq [C_0 \ltimes C_1 \rightrightarrows C_0]^\sim$

Non strict Picard = Non regular cat. rings

- $C_1 \xrightarrow{\partial} C_0 \longrightarrow \mathcal{C}$, where $\underbrace{C_1 \xrightarrow{\partial} C_0}_{\text{stable crossed module}}$

- $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ monoidal for $\oplus$ in both variables

- Theorem (EA '15) $(\mathcal{C}, \oplus, 0_{\mathcal{C}}, \otimes, I_{\mathcal{C}})$ equivalent to a biextension

$$\begin{tikzcd}
C_1 \times C_1 \arrow[d] \arrow[dr] & & C_1 \arrow[dl] \arrow[d] \\
C_0 \times C_0 & E_{\otimes} \arrow[l] \arrow[r] & C_0
\end{tikzcd}$$

Non strict Picard = Non regular cat. rings

▷ $$\begin{array}{c}
0 \longrightarrow M \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow \Lambda \longrightarrow 0 \\
\end{array}$$
M: Λ-bimodule, Λ: ring

▷ $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ monoidal for $\oplus$ in both variables

▷ **Theorem** (EA '15) $(\mathcal{C}, \oplus, 0_{\mathcal{C}}, \otimes, I_{\mathcal{C}})$ equivalent to a biextension

$$\begin{tikzcd}
C_1 \times C_1 \arrow[dr] \arrow[dd] & & C_1 \arrow[dl] \arrow[dd] \\
& E_{\otimes} \arrow[dl] \arrow[dr] & \\
C_0 \times C_0 & & C_0
\end{tikzcd}$$

$$(E_{\otimes} \times I) \wedge E_{\otimes} \overset{\sim}{\Rightarrow} (I \times E_{\otimes}) \wedge E_{\otimes}$$

Non strict Picard = Non regular cat. rings

▷ $$0 \longrightarrow M \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow \Lambda \longrightarrow 0$$

M: Λ-bimodule; Λ: ring

▷ $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ monoidal for $\oplus$ in both variables

▷ **Theorem** (EA '15) $(\mathcal{C}, \oplus, 0_{\mathcal{C}}, \otimes, I_{\mathcal{C}})$ equivalent to a biextension

$$\begin{tikzcd}
C_1 \times C_1 \arrow[dd] \arrow[dr] & & C_1 \arrow[dl] \arrow[dd] \\
& E_\otimes \arrow[dl] \arrow[dr] & \\
C_0 \times C_0 & & C_0
\end{tikzcd}$$

$$(E_\otimes \times I) \wedge E_\otimes \xrightarrow{\sim} (I \times E_\otimes) \wedge E_\otimes$$

$$\Downarrow$$

$$\boxed{[E_\otimes] \in HML^3(\Lambda, M)}$$

Mac Lane

THANK YOU !