Cubes in Picard Groupoids, MacLane's Q-construction, determinants

Octoberfest 2022

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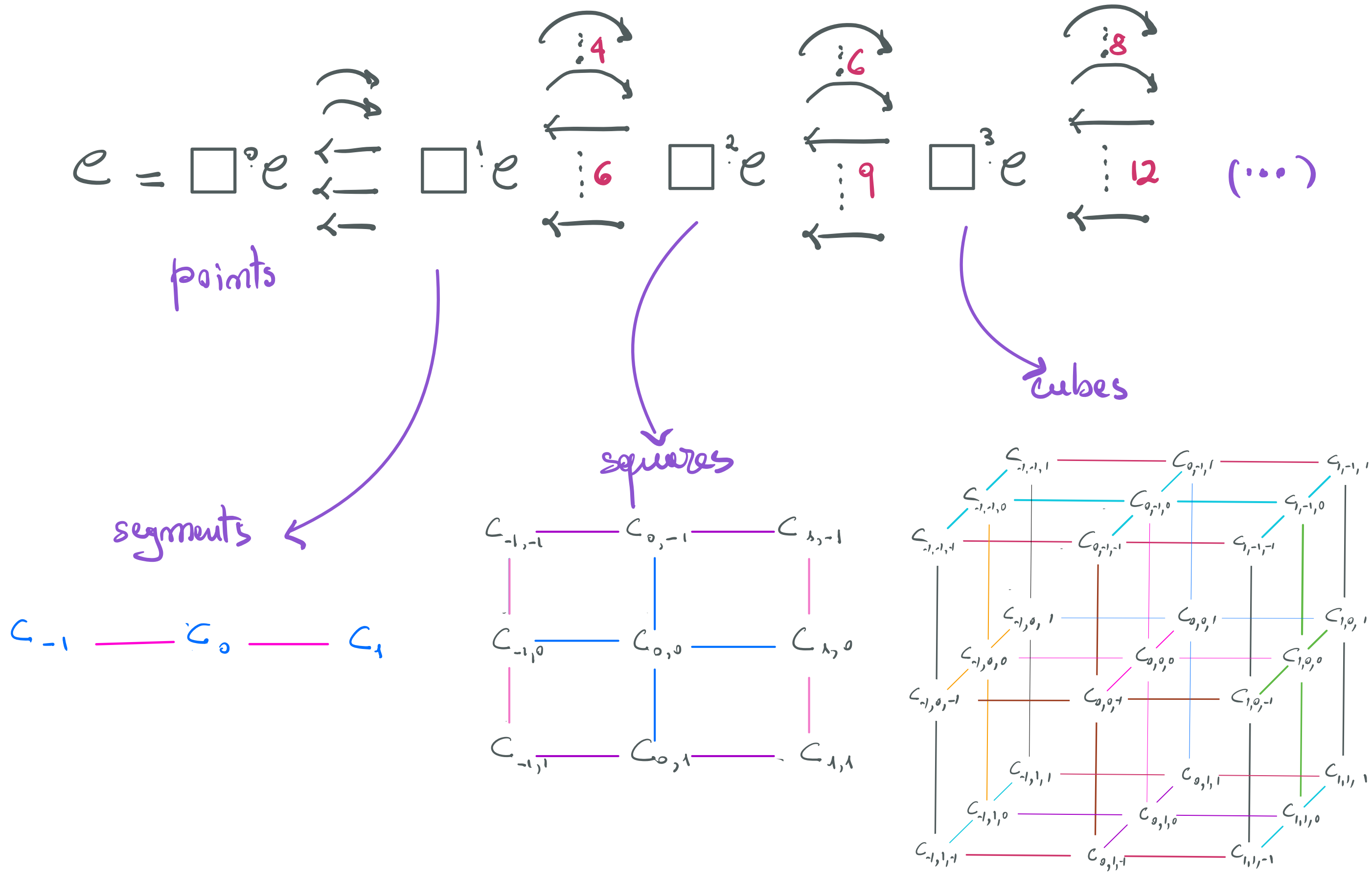
Picard Groupoid =. Symmetric monoidal
group-like
groupoid

= Symmetric Cat-Group

Diagrams of Picard Groupoids



* Shapes inspired by J.L. Loday's "Cubes of Fibrations"
  (Loday, McCarthy, J. Burgos-Gil, ...)

* k-invariant of e

$$k \in \varinjlim_{m} H^{2+m}(\pi_0, \pi_1) \cong H^{2+3}(K(\pi_0, 3), \pi_1)$$

$$\cong H^2(Q_\bullet(\pi_0), \pi_1)$$

* $\square^\bullet e$ receives determinant functors from
  Exact, Waldhausen, Triangulated cats

Construction of $\square^n \mathcal{C}$

$n=1$: $\square^1 \mathcal{C} = S_2 \mathcal{C}$ ("S_2" for S-construction)

* x —— y —— z $\qquad x, y, z$

$\qquad$ + : $x + z \xrightarrow{\alpha} y$

*

$$\begin{array}{ccccc} x & — & y & — & z \\ & & \downarrow & & \\ x' & — & y' & — & z' \end{array}$$

$$\begin{tikzcd} x + z \arrow[r, "\alpha"] \arrow[d] & y \arrow[d] \\ x' + z' \arrow[r, "\alpha'"'] & y' \end{tikzcd}$$

* $F : \mathcal{C} \longrightarrow \mathcal{D} \implies S_2(F) : S_2\mathcal{C} \longrightarrow S_2\mathcal{D}$

$$F(x) + F(z) \longrightarrow F(x+z) \xrightarrow{F(\alpha)} F(y)$$

* $S_2\mathcal{C} \times S_2\mathcal{C} \simeq S_2(\mathcal{C} \times \mathcal{C}) \longrightarrow S_2\mathcal{C}$ **Monoidal structure ?**

$$\begin{array}{c} (\underset{\substack{\wr\\ \downarrow \\ u}}{x+y}) + (\underset{\substack{\wr\\ \downarrow \\ v}}{z+t}) \longrightarrow (x+z) + (y+t) \longrightarrow u+v \end{array}$$

COAS — Commutassociativity

In a Picard groupoid $\mathcal{C}$

$$c_{x,y,z,t} : (x+y)+(z+t) \longrightarrow (x+z)+(y+t)$$

is the composite:

$$\begin{tikzcd}
 & (x+y)+(z+t) \arrow[dl] \arrow[dr] & \\
((x+y)+z)+t \arrow[d] \arrow[rr, phantom, "\text{pentagon}"] & & x+(y+(z+t)) \arrow[d] \\
(x+(y+z))+t \arrow[rr] \arrow[d, "\text{braid}"'] \arrow[drr, phantom, "\text{nat.}"] & & x+((y+z)+t) \arrow[d, "\text{braid}"] \\
(x+(z+y))+t \arrow[rr] \arrow[d] & & x+((z+y)+t) \arrow[d] \\
((x+z)+y)+t \arrow[dr] \arrow[rr, phantom, "\text{pentagon}"] & & x+(z+(y+t)) \arrow[dl] \\
 & (x+z)+(y+t) &
\end{tikzcd}$$

$n = 2 : \quad S_2(S_2\mathcal{C}) = \square^2\mathcal{C}$

$*$

①

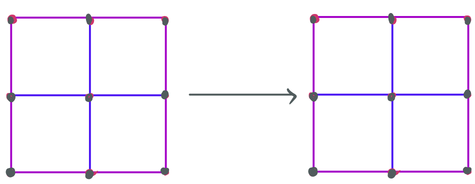
$$\begin{tikzcd}
C_{-1,-1} \arrow[r, dash] \arrow[d, dash] & C_{0,-1} \arrow[r, dash] \arrow[d, dash] & C_{1,-1} \arrow[d, dash] \\
C_{-1,0} \arrow[r, dash] \arrow[d, dash] & C_{0,0} \arrow[r, dash] \arrow[d, dash] & C_{1,0} \arrow[d, dash] \\
C_{-1,1} \arrow[r, dash] & C_{0,1} \arrow[r, dash] & C_{1,1}
\end{tikzcd}$$

② coherence (= data)

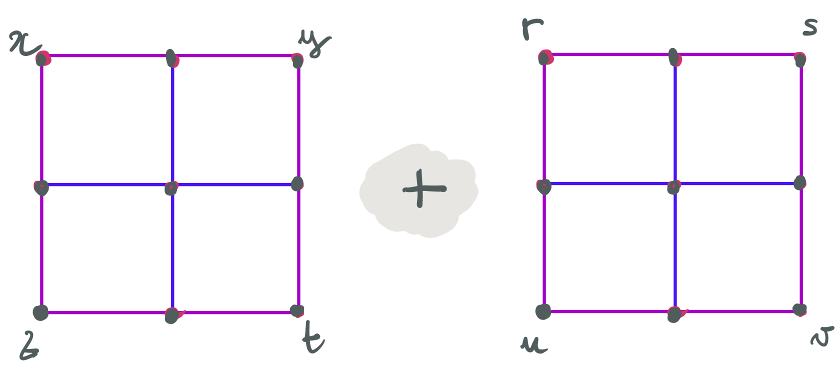
$$\begin{tikzcd}
(C_{-1,-1} + C_{-1,1}) + (C_{1,-1} + C_{1,1}) \arrow[r, "\text{coas}"] \arrow[d] & (C_{-1,-1} + C_{1,-1}) + (C_{-1,1} + C_{1,1}) \arrow[d] \\
C_{-1,0} + C_{1,0} \arrow[r] \arrow[ur, phantom, "2"] & C_{0,-1} + C_{0,1}
\end{tikzcd}$$

$$C_{-1,0} + C_{1,0} \longrightarrow C_{0,0} \longleftarrow C_{0,-1} + C_{0,1}$$

$*$



$*$ $\quad S_2^2\mathcal{C} \times S_2^2\mathcal{C} \simeq S_2^2(\mathcal{C}\times\mathcal{C}) \longrightarrow S_2^2\mathcal{C}$ Monoidal structure ?



The giant hexagon

(Balteanu, Fiedorowicz, Schwänzl, Vogt '03)

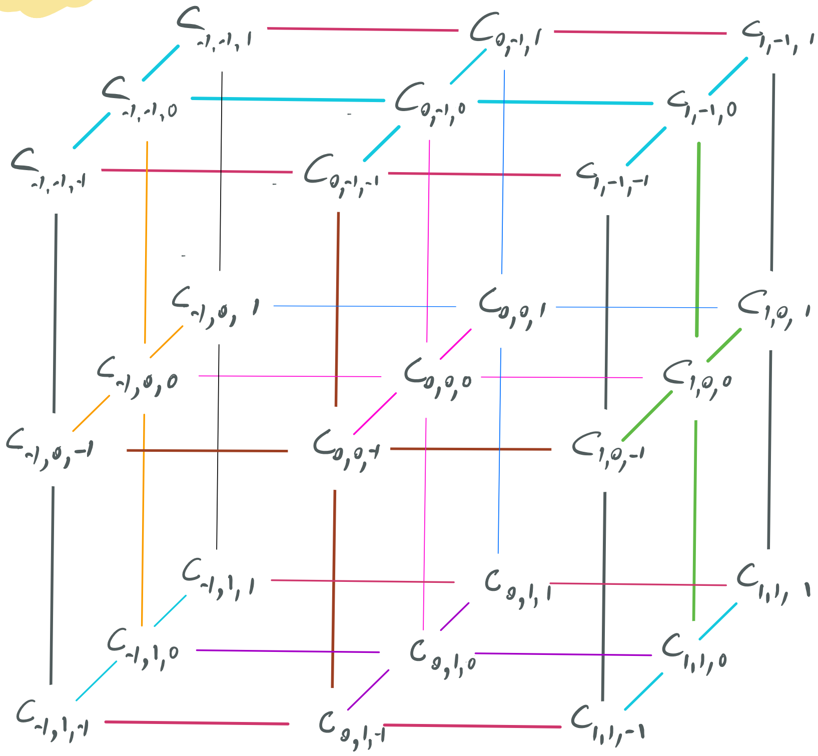
$$\begin{tikzcd}
((x+y)+(z+t)) + ((r+s)+(u+v)) \arrow[r, "c+c"] \arrow[d, "c"] & ((x+z)+(y+t)) + ((r+u)+(s+v)) \arrow[d, "c"] \\
((x+y)+(r+s)) + ((z+t)+(u+v)) \arrow[d, "c+c"] & ((x+z)+(r+u)) + ((y+t)+(s+v)) \arrow[d, "c+c"] \\
((x+r)+(y+s)) + ((z+u)+(t+v)) \arrow[r, "c"] & ((x+r)+(z+u)) + ((y+s)+(t+v))
\end{tikzcd}$$

Proposition The giant hexagon commutes iff $\mathcal{C}$ is symmetric monoidal.

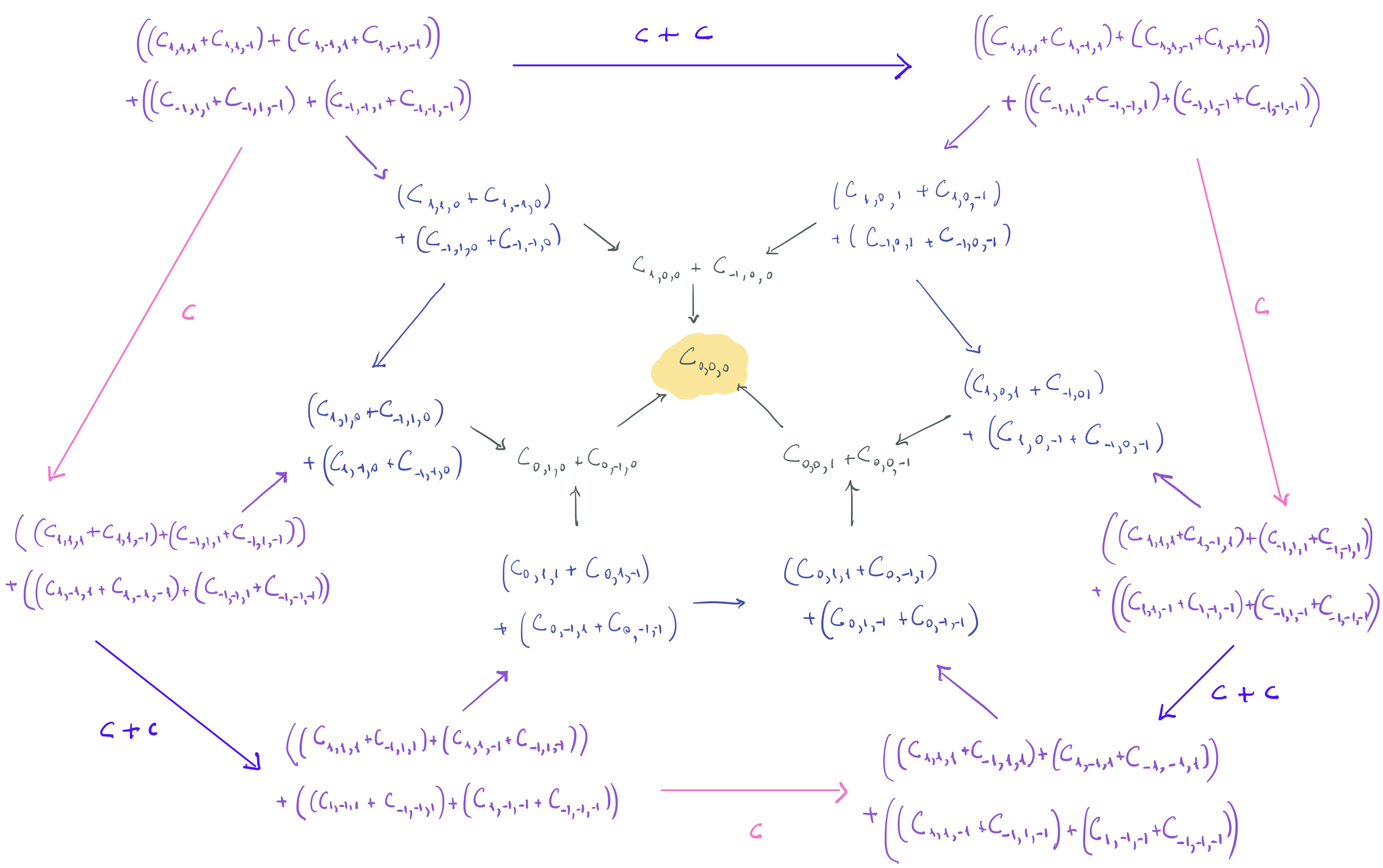
∎

$n = 3 \quad S_2^{(3)}\mathcal{C}$:

(1)



(2) Coherence data :



We can define

$\square^n \mathcal{C} := S_2^{(n)} \mathcal{C}$, $n \geq 0$ (No other conditions are needed)

In fact

Thm (Corollary of Joyal–Street, Balteanu–Fiedorowicz–Schwänzl–Vogt, ...)

The diagram $\square^{\cdot} \mathcal{C}$ exists $\iff$ $\mathcal{C}$ is a symmetric monoidal category

Proof n-fold monoidal category

$\square_1 = \square_2 = \dots = \square_n = +$

§ The Complex : Q- construction

$\square^{\bullet}\mathcal{C}$: $\qquad \partial_i^j : \square^n\mathcal{C} \longrightarrow \square^{n-1}\mathcal{C} \qquad j=-1,0,1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,\dots,n$

$$\partial_1^0 \begin{array}{ccccc} x' & - & y' & - & z' \\ | & & | & & | \\ x & - & y & - & z \\ | & & | & & | \\ x'' & - & y'' & - & z'' \end{array} = \begin{array}{c} y' \\ | \\ y \\ | \\ y'' \end{array}$$

$s_i^j : \square^{n-1}\mathcal{C} \longrightarrow \square^n\mathcal{C} \qquad j=0,1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,\dots,n$

$$s_1^0(x - y - z) = \begin{array}{ccccc} x & - & y & - & z \\ | & & | & & | \\ x & - & y & - & z \\ | & & | & & | \\ 0 & - & 0 & - & 0 \end{array}$$

SLABS

$F : \mathcal{P}ic \longrightarrow \mathcal{A}$

(2-) cat of Picard Groupoids

abelian category

$F(\mathcal{C}) = \mathbb{Z}[\mathrm{Ob}\,\mathcal{C}]$

$$Q'_\bullet(F;\mathcal{C}) : \cdots \longrightarrow Q'_n(F;\mathcal{C}) \xrightarrow{\partial} Q'_{n-1}(F;\mathcal{C}) \longrightarrow \cdots \longrightarrow Q'_0(F;\mathcal{C})$$

$$Q'_n(F;\mathcal{C}) := F(\square^n \mathcal{C})$$

$$\partial = \sum_{i=1}^{n} \sum_{j=-1}^{1} (-1)^{i+j+1} F(\partial_i^j)$$

$$Q_\bullet(F;\mathcal{C}) = Q'_\bullet(F;\mathcal{C}) \,/\, \text{slabs}$$

Examples

$$* \quad \partial \begin{pmatrix} x' & \text{—} & y' & \text{—} & z' \\ | & & | & & | \\ x & \text{—} & y & \text{—} & z \\ | & & | & & | \\ x'' & \text{—} & y'' & \text{—} & z'' \end{pmatrix} = -\begin{pmatrix} x' \\ | \\ x \\ | \\ x'' \end{pmatrix} + \begin{pmatrix} y' \\ \\ y \\ \\ y'' \end{pmatrix} - \begin{pmatrix} z' \\ \\ z \\ \\ z'' \end{pmatrix}$$

$$+ (x' - y' - z') - (x - y - z)$$

$$- (x'' - y'' - z'')$$

$$\partial^2 = 0$$

$*$ If $\mathcal{C} = \mathcal{B}$: Abelian group

$$\partial \begin{pmatrix} x' & \text{—} & y' & \text{—} & z' \\ | & & | & & | \\ x & \text{—} & y & \text{—} & z \\ | & & | & & | \\ x'' & \text{—} & y'' & \text{—} & z'' \end{pmatrix} = -\begin{pmatrix} x' \\ \\ x'' \end{pmatrix} + \begin{pmatrix} x' + z' \\ \\ x'' + z'' \end{pmatrix} - \begin{pmatrix} z' \\ \\ z'' \end{pmatrix}$$

$$+ \ldots$$

* $\mathcal{C} = \mathcal{B}$: Abelian group

$$H_\bullet(Q_\bullet(B)) = \varinjlim_n H_{n+\bullet}(K(B,n))$$

$$H^\bullet(Q_\bullet(B), A) \cong H^{\bullet+n}(K(B,n), A), \quad n > \bullet$$

The k-invariant

$\mathcal{C}$: Picard Groupoid

$B = \pi_0 \mathcal{C}$

$A = \pi_1 \mathcal{C} = \mathrm{Aut}_{\mathcal{C}}(0)$

$k \in H^2_{st}(B, A)$

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] \\ B \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

$x, y \in B$ $\qquad \sigma(x) + \sigma(y) \xrightarrow{\cong} \sigma(x+y)$

$\rightsquigarrow$

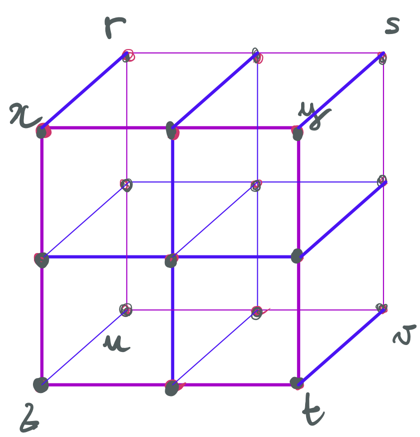
$\left(\sigma(x) \text{ --- } \sigma(x+y) \text{ --- } \sigma(y) \right) \in Q_1(\mathcal{C})$

Question $\quad x, y, z, t \in B$

$$\left(\qquad \qquad \right) \in Q_2(\mathcal{C})$$

$x, y, z, t \in B = \pi_0 \mathcal{C}$

$$
\begin{tikzcd}
\big(\sigma(x)+\sigma(y)\big)+\big(\sigma(z)+\sigma(t)\big) \arrow[r, "\alpha_{x,y}+\alpha_{z,t}"] \arrow[ddd, "c_{\sigma(x),\dots,\sigma(t)}"'] & \sigma(x+y)+\sigma(z+t) \arrow[d, "\alpha_{x+y,z+t}"] \\
& \sigma(x+y+z+t) \arrow[d, "\alpha\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \pi_1\mathcal{C}"] \\
& \sigma(x+z+y+t) \\
\big(\sigma(x)+\sigma(z)\big)+\big(\sigma(y)+\sigma(t)\big) \arrow[r, "\alpha_{x,z}+\alpha_{y,t}"'] & \sigma(x+z)+\sigma(y+t) \arrow[u, "\alpha_{x+z,y+t}"']
\end{tikzcd}
$$



$\in Q_3(B)$

$$
\sigma\Big(\big((x+y)+(z+t)\big)+\big((r+s)+(u+v)\big)\Big) \xrightarrow{\alpha\begin{pmatrix} x & y \\ z & t \end{pmatrix}+\alpha\begin{pmatrix} r & s \\ u & v \end{pmatrix}} \sigma\Big(\big((x+z)+(y+t)\big)+\big((r+u)+(s+v)\big)\Big)
$$

$$
\begin{tikzcd}[column sep=small]
\cdots \arrow[d, "\alpha\begin{pmatrix} x+y & z+t \\ r+s & u+v \end{pmatrix}"] & \cdots \arrow[d, "\alpha\begin{pmatrix} x+z & y+t \\ r+u & s+v \end{pmatrix}"] \\
\sigma\Big(\big((x+y)+(r+s)\big)+\big((z+t)+(u+v)\big)\Big) \arrow[d, "\alpha\begin{pmatrix} x & y \\ r & s \end{pmatrix}+\alpha\begin{pmatrix} z & t \\ u & v \end{pmatrix}"] & \sigma\Big(\big((x+z)+(r+u)\big)+\big((y+t)+(s+v)\big)\Big) \arrow[d, "\alpha\begin{pmatrix} x & z \\ r & u \end{pmatrix}+\alpha\begin{pmatrix} y & t \\ s & v \end{pmatrix}"] \\
\sigma\Big(\big((x+r)+(y+s)\big)+\big((z+u)+(t+v)\big)\Big) \arrow[r, "\alpha\begin{pmatrix} x+r & y+s \\ z+u & t+v \end{pmatrix}"'] & \sigma\Big(\big((x+r)+(z+u)\big)+\big((y+s)+(t+v)\big)\Big)
\end{tikzcd}
$$

$$[\alpha] \in H^2(\underset{\pi_0}{\underset{\parallel}{Q.\langle B\rangle}}, \underset{\pi_1}{\underset{\parallel}{A}})$$

k-invariant

§ Determinants

$\mathcal{C}$: "category with a notion of exact sequence"

- Abelian
- Exact $\quad 0 \to X \to Y \to Z \to 0$
- Waldhausen $\quad X \rightarrowtail Y \twoheadrightarrow Z \quad$ (E)
- Triangulated $\quad X \to Y \to Z \to \Sigma X$

$\mathcal{P}$: Picard Groupoid

A determinant functor $\mathcal{C} \to \mathcal{P}$ consists of :

① det : $is(\mathcal{C}) \longrightarrow \mathcal{P} \qquad X \longmapsto [X]$

② Additivity data :

such that :

(a) Commutativity

$$X \to X \oplus Y \to Y \to \Sigma X$$

$$Y \to X \oplus Y \to X \to \Sigma X$$

(b) Associativity

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r] & Z \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & \Sigma X \\
 & W \arrow[r, equal] \arrow[d] & W \arrow[d] & \\
 & \Sigma Y \arrow[r] & \Sigma U &
\end{tikzcd}$$

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(for Triangulated Categories)

$$\begin{tikzcd}
([X]+[U])+[W] \arrow[r] \arrow[d] & [X]+([U]+[W]) \arrow[d] \\
[Y]+[W] & [X]+[V]
\end{tikzcd}$$

Thm $(\det, \bullet) : \mathcal{C} \to \mathcal{P} \iff D : \square^{\bullet}\mathcal{C} \to \square^{\bullet}\mathcal{P}$

($\bullet \leq 2$ for Δ-cat)

Compatible w/ $\otimes$-structures : $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$