

# MAPPING SPACES OF GROUP-LIKE STACKS

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# INTRODUCTION

Describe the hom space  $\text{Hom}(\mathcal{H}, \mathcal{G})$

where  $\mathcal{H}, \mathcal{G}$  are group-like stacks (over some site  $\mathcal{I}$ , like  $\underline{\text{Top}}$ )

Why?

- (Short) exact sequences
- (Nonabelian) cohomology
- $\text{deg} \geq 3$  cohomology / Postnikov invariants

\* Group-like stacks are "easy," they essentially are 2-step complexes:

$$\mathcal{G} \simeq [\mathcal{G}_1 \xrightarrow{\partial} \mathcal{G}_0]^\sim \quad \mathcal{G}_0, \mathcal{G}_1 \text{ are (sheaves of) groups}$$

\* Morphisms  $\mathcal{H} \rightarrow \mathcal{G}$  between them are difficult:

**NOT** morphisms of complexes, thus  $\mathbb{R}\text{Hom}(\mathcal{H}, \mathcal{G})$

# REFERENCES

WITH BEHRANG NOOHI

Butterflies I: *Advances in Math.* 221 (2009), 687–773

Butterflies II, *Advances in Math.* 225 (2010), 922–976

Butterflies III, .....

Butterflies IV,

WITH A. EMIN TATAR

Cohomology and coherence, final stages

# STACKS

"Fix a base site  $\mathcal{J}$  w/ a Grothendieck Topology ..."

(1)  $\mathcal{J} =$  Open sets in an ordinary topological space / scheme  $S$

(2)  $\mathcal{J}$  is a category like  $\underline{\text{Top}}$ , so you have objects :  $T, \dots$

morphisms :  $T' \rightarrow T$

open covers :  $\coprod_{i \in I} U_i \rightarrow T$

A stack  $\mathcal{X}$  over  $\mathcal{J}$  is, roughly, a sheaf of categories /  $\mathcal{J}$  :

\*  $T \in \text{Ob}(\mathcal{J}) \rightsquigarrow \mathcal{X}_T$

\*  $U \xrightarrow{f} T \in \text{Mor}(\mathcal{J}) \rightsquigarrow f^* : \mathcal{X}_T \rightarrow \mathcal{X}_U$ , such that

\*  $(f \circ g)^* \simeq g^* \circ f^*$  (and other coherence conditions)

We need a gluing condition on objects and morphisms of  $\mathcal{X}$  :

$$\begin{array}{c} \rightarrow U_2 \rightrightarrows U_1 \rightrightarrows U_0 \xrightarrow{f} U \\ \vdots \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \quad x_0, \quad x_0 \quad x \quad y \\ \quad x_1, x_2 \quad x_1 \end{array}$$

$$\begin{array}{ccc} x_1 & \xrightarrow{f} & x_0 \\ & \searrow \quad \nearrow & \\ & x_2 & \end{array} \text{ must come from } y \in \text{Ob } \mathcal{X}_U$$

# STACKS (Cont'd)

Favorite example: Let  $G$  be a group (quite often: sheaf of groups)

$$BG : T \rightsquigarrow \{ \text{Principal homogeneous } G\text{-spaces}/T \}$$
$$U \xrightarrow{f} T \rightsquigarrow f^* : BG_T \longrightarrow BG_U \text{ pull back of PHS}$$


Sometimes we write:  $TORS(G)$  in place of  $BG$   
 $\hookrightarrow$  Stack of  $G$ -torsors

Rmk:  $TORS(G)$  makes sense even when  $\mathcal{J}$  is  $\text{Spec } k$  (Galois Cohomology)

Other famous example:  $\mathcal{M}_g$  - moduli stack of curves

$$T \rightsquigarrow \left\{ \begin{array}{c} X \\ \downarrow \\ T \end{array} \mid \text{relative curve of genus } g \right\}$$
$$U \xrightarrow{f} T \rightsquigarrow \mathcal{M}_g(T) \longrightarrow \mathcal{M}_g(U)$$
$$X/T \longmapsto X \times_T U/U$$

# GR-STACKS = 2-GROUPS

"French" terminology 

New Age 

A gr-stack  $\mathcal{G}$  is a stack over  $\mathcal{I}$  with "a group law:"

\* multiplication morphism  $\mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G}$

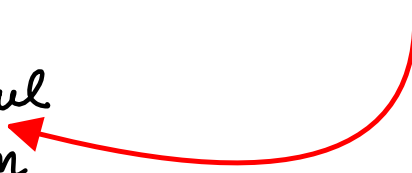
\* Identity object:  $I \in \text{Ob } \mathcal{G}$

\* Inverse functor:  $\mathcal{G} \xrightarrow{*} \mathcal{G} \quad X \mapsto X^*$

Such that the triple  $(m, I, *)$  behaves exactly like the corresponding one for an ordinary group, *except* that we "replace identities with isomorphisms."

One must be careful

$I$  and  $*$  are non

canonical: there are better ways 

# GR-STACKS (Cont'd)

Examples are given by Automorphism Stacks:

$\mathcal{E}q(\mathcal{X})$ : self-equivalences of the stack  $\mathcal{X}$

\*  $\mathcal{E}q(BG)$ : equivalences  $BG \begin{matrix} \xrightarrow{\mu} \\ \alpha \Downarrow \\ \xrightarrow{\nu} \end{matrix} BG$   $P \mapsto u(P)$   
 $\alpha_P: u(P) \mapsto v(P)$

Morita Theory for torsors:  $u \rightsquigarrow E_u$  - **bi**torsor such that

$$u(P) = P \wedge^G E_u$$

$$u \xrightarrow{\alpha} v \rightsquigarrow E_u \xrightarrow{\alpha} E_v$$

\* **VARIANT**: If  $\varrho: G \rightarrow H$  is a homomorphism, then  $\varrho_*: BG \rightarrow BH$  **morphism of stacks**

$\mathcal{E}q_{BH}(BG)$ : only consider self-equivalences  $u: BG \rightarrow BG$

$$\begin{array}{ccc} BG & \xrightarrow{u} & BG \\ \varrho_* \searrow & & \swarrow \varrho_* \\ & BH & \end{array}$$

# STACKS & CATEGORIES

Another way to look at categories: **SIMPLICIAL SETS**

$\mathcal{C}$  category  $\rightsquigarrow$   $N_{\bullet}\mathcal{C}$  nerve

2-coskeletal simplicial set

(modulo taking the geometric realization to define  $\pi_i$ )  $N_{\bullet}\mathcal{C}$  is a 2-type.

Moral (but can be made rigorous):

Stacks can be viewed as simplicial (pre)sheaves which are 2-truncated.



# GR-STACKS & CROSSED MODULES

For a gr-category  $\mathcal{G}$ , the nerve  $N\mathcal{G}$  is a 2-skeletal simplicial object w/ group law up to homotopy ( $\mathcal{G}_\infty$ -space?)

If  $N\mathcal{G}$  has a **rigid** group law, then it is a **simplicial group**

By taking into account the 2-truncation,  $N\mathcal{G}$  arises from a **crossed module**  $G_1 \xrightarrow{\partial} G_0$ , i.e.

$$N\mathcal{G}_0 = G_0, \quad N\mathcal{G}_i = \underbrace{G_1 \times \dots \times G_1}_{i\text{-times}} \times G_0$$

## CROSSED MODULE:

Action  $G_1 \times G_0 \rightarrow G_1$ ,  $(g, x) \mapsto g^x$

i)  $\partial(g^x) = x^{-1}(\partial g)x$

ii)  $\partial g^{\partial g} = g^{-1} \partial g$

## EXAMPLES:

1)  $G \rightarrow \text{Aut}(G)$

2)  $N \hookrightarrow G$  inclusion of a normal subgroup

3)  $(F, *) \rightarrow (E, *) \rightarrow (B, *)$  fibration

$$\pi_1(F, *) \xrightarrow{\partial} \pi_1(E, *)$$

What if  $\mathcal{G}$  is a gr-stack? 

# GR-STACKS & CROSSED MODULES



Let  $\mathcal{G}$  be a gr-stack over  $\mathcal{S}$ . The situation is analogous to that of a single category—Apparently well known, but no detailed proof.

**PROPOSITION** (E.A. — B. NOOHI) There exists a (sheaf of) crossed modules  $G_1 \xrightarrow{\partial} G_0$  over  $\mathcal{S}$ , and a morphism (of gr-stacks)  $\pi: G_0 \rightarrow \mathcal{G}$  such that

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{\pi} \mathcal{G}$$

is exact at  $G_0$ , and essentially surjective at  $\mathcal{G}$ .

- \* "Essentially surjective:" every object of  $\mathcal{G}$  is locally isomorphic to a point of  $G_0$
- \* The sequence is a **presentation** of  $\mathcal{G}$  (**NOT** unique!)  
We write:  $\mathcal{G} \simeq [G_1 \xrightarrow{\partial} G_0]$ , when a presentation has been chosen.
- \* From now on, we'll always choose a presentation

## MISC. FACTS RE. $\mathcal{G}$

Assume  $\mathcal{G} \simeq [G_1 \xrightarrow{\vartheta} G_0]$ , for a crossed module  $G_1 \xrightarrow{\vartheta} G_0$ .

\* DELIGNE:  $\mathcal{G} \simeq \text{Tors}(G_1, G_0)$  (We should write  $B(G_1, G_0)$ ):  
principal homogeneous  $G_1$ -spaces  $P$  whose extension  $P \wedge^{G_1} G_0 \simeq G_0$

\* BREEN:  $\mathcal{G}$  is part of an exact sequence

$$\mathcal{G} \rightarrow BG_1 \rightarrow BG_0$$

\* In fact:

$$(1) \quad \mathcal{G} \simeq \text{Eq}_{BG_0}(BG_1)$$

$$(2) \quad \text{In particular, } \text{Eq}(BG) \simeq [G \rightarrow \text{Aut}(G)]$$

## THE PROBLEM WITH MORPHISMS

If  $\mathcal{H} \simeq [H_1 \xrightarrow{\partial} H_0]$  and  $\mathcal{G} \simeq [G_1 \xrightarrow{\partial} G_0]$ , and  $F: \mathcal{H} \rightarrow \mathcal{G}$  is a *morphism* of gr-stacks:

$$\lambda_{X,Y}: F(X \times Y) \rightarrow F(X) \times F(Y) ; X, Y \in \text{Ob}(\mathcal{H})$$

Then surely  $F$  corresponds to:

$$\begin{array}{ccc} H_1 & \longrightarrow & G_1 \\ \partial \downarrow & & \downarrow \partial \\ H_0 & \longrightarrow & G_0 \end{array} \quad ? \quad \text{NO!}$$

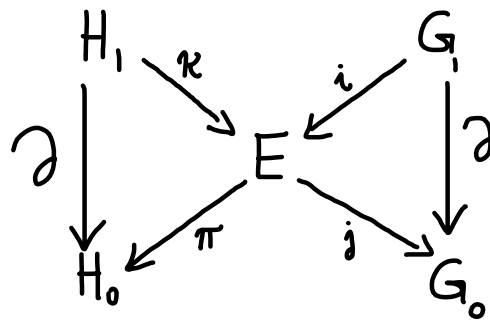
Same problem with describing morphisms in derived categories

By the same token, a natural transformation  $\alpha: F_1 \Rightarrow F_2: \mathcal{H} \rightarrow \mathcal{G}$  does not induce, or come from, a chain homotopy of complexes.

# BUTTERFLIES

Let  $H_1 \xrightarrow{\partial} H_0$  and  $G_1 \xrightarrow{\partial} G_0$  be two crossed modules. B. Noohi defines:

A **butterfly** from  $H_0$  to  $G_0$  is a diagram of groups:



(i) The NE-SW is a group extension

(ii) The NW-SE is a complex

(iii) Compatibilities:

$$i(g^j(e)) = e^{-1} i(g) e$$

$$\kappa(h^{\pi(e)}) = e^{-1} \kappa(h) e$$

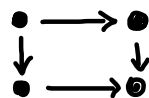
A **morphism of butterflies** is a group isomorphism

$$\alpha : E \xrightarrow{\cong} E'$$

compatible with the structures of the diagram above.

A butterfly is **split** if the extension in it is trivial:  $E = H_0 \rtimes G_1$

In this case, we get an honest morphism:



## WEAK MORPHISMS

Butterflies from  $H_1 \xrightarrow{\varrho} H_0$  to  $G_1 \xrightarrow{\varrho} G_0$  form a category  $\mathcal{B}(H., G.)$  (in fact a groupoid).

**THEOREM** (E.A.—B. Noohi) Let the gr-stacks  $\mathcal{H}$  and  $\mathcal{G}$  have presentations  $\mathcal{H} \simeq [H_1 \xrightarrow{\varrho} H_0]$ ,  $\mathcal{G} \simeq [G_1 \xrightarrow{\varrho} G_0]$ . There is an equivalence of categories:

$$\text{Hom}(\mathcal{H}, \mathcal{G}) \simeq \mathcal{B}(H., G.)$$

Both members can be sheafified (= stackified) and the equivalence is an equivalence of stacks:

$$\underline{\text{Hom}}(\mathcal{H}, \mathcal{G}) \simeq \underline{\mathcal{B}}(H., G.)$$

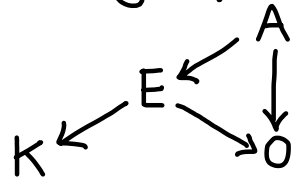
**Proof** :-> One direction assigns to  $F: \mathcal{H} \rightarrow \mathcal{G}$  the (stack) fibered product

$$E = H_0 \times_{\mathcal{G}} G_0$$

the other, assigns to  $E$  the morphism which sends an object  $Q$  of  $\mathcal{H}$  to  $\underline{\text{Hom}}_{H_0}(Q, E) \in \text{Ob}(\mathcal{G})$  ▣

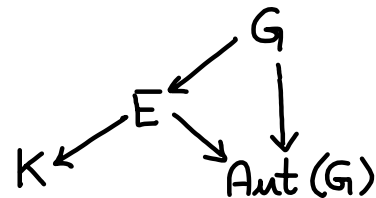
## FACTS AND EXAMPLES

If  $0 \rightarrow A \rightarrow E \rightarrow K \rightarrow 1$  is a group extension w/ abelian kernel, we get:



hence a morphism  $K \rightarrow \text{TORS}(A) (\cong BA)$

More generally, for an extension  $1 \rightarrow G \xrightarrow{i} E \xrightarrow{\pi} K \rightarrow 1$  ( $G$  not necessarily abelian) we get

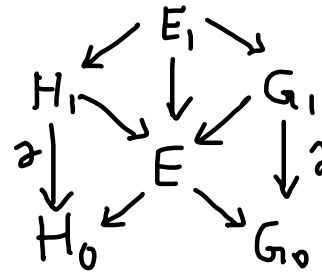


and so,  $K \rightarrow \text{Eq}(BG)$

(Schreier, Dedekker, Grothendieck, Breen)

# FACTS AND EXAMPLES II

A butterfly  $\begin{array}{ccc} H_1 & & G_1 \\ & \searrow & \swarrow \\ & E & \\ & \swarrow & \searrow \\ H_0 & & G_0 \end{array}$  can be completed to



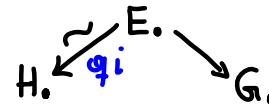
where  $E_1 = H_1 \times G_1 \rightarrow E$  is a crossed module. One proves:

$$\mathcal{H} \simeq [H_1 \xrightarrow{\alpha} H_0] \simeq [E_1 \rightarrow E]$$

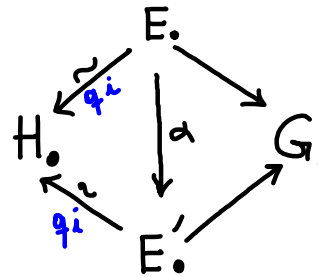
$$E_1 \rightarrow E \simeq H_1 \rightarrow H_0$$

quasi-isomorphism

Thus there is a fraction of crossed modules:



A morphism of butterflies  $\alpha: E \rightarrow E'$  yields



Non abelian derived category  $\mathcal{D}(XMod)$

(Abelian case: Deligne, SGA 4, XVIII)



## APPLICATIONS I

DEFINITION (L. BREEN) A **short exact** sequence of gr-stacks is a sequence of morphisms

$$\mathcal{K} \xrightarrow{i} \mathcal{H}_0 \xrightarrow{p} \mathcal{G}, \quad p \circ i \simeq 1$$

such that:

- i)  $\mathcal{K} \simeq \text{Homotopy Kernel}(p)$
- ii)  $p$  essentially surjective
- iii) Suitable notion of exactness at  $\mathcal{H}_0$
- iv)  **$p$  is a fibration (!)**

**Reason for iv)** The short exact sequence ought to yield a long exact sequence of (nonabelian) cohomology objects.

PROPOSITION (E.A.-B. NOONJ) The fibration hypothesis can be dispensed with.

Proof: Given  $p: \mathcal{H}_0 \rightarrow \mathcal{G}$  essentially surjective, there is a butterfly and, hence, a fraction

$$\begin{array}{ccc} & \mathcal{E}_0 & \\ \swarrow \tilde{\sim} & & \searrow \\ \mathcal{H}_0 & & \mathcal{G}_0 \end{array}$$

which gives an equivalent morphism  $p': \mathcal{E} \rightarrow \mathcal{G}$  which **is** a fibration



## APPLICATIONS II

The fibration replacement follows from

THEOREM (E.A.-B.N.) Let  $F: \mathcal{H} \rightarrow \mathcal{G}$  be a morphism of gr-stacks,  $\mathcal{K} \simeq \text{HKer}(F)$

There is a long exact sequence of 2-stacks:

$$0 \rightarrow \pi_1(\mathcal{K}) \rightarrow \pi_1(\mathcal{H}) \rightarrow \pi_1(\mathcal{G}) \rightarrow \mathcal{K} \rightarrow \mathcal{H} \xrightarrow{F} \mathcal{G} \xrightarrow{\Delta} \mathcal{F} \rightarrow \text{TORS}(\mathcal{H}) \xrightarrow{F_*} \text{TORS}(\mathcal{G})$$



\* Analog of "turning triangles", but unfortunately it terminates

\*  $\mathcal{F}$  is the "homotopy fiber" of  $F_*: \text{TORS}(\mathcal{H}) \rightarrow \text{TORS}(\mathcal{G})$

2-stack associated to the complex  $H_1 \rightarrow E \rightarrow G_0$ , the nonexact diagonal of the butterfly.

\* The fraction  $\begin{array}{ccc} & E & \\ \swarrow & & \searrow \\ H & & G \end{array}$  provides an explicit description of  $F_*$ :

$$\begin{array}{ccc} \mathcal{X}' := \mathcal{X} \times_{\text{TORS}(H_0)} \text{TORS}(E) & & \\ \mathcal{X} \rightsquigarrow & & \rightsquigarrow \\ & & F_*(\mathcal{X}') \end{array}$$

Sufficiently explicit  
to provide cocycles.

provides the way to obtain maps in nonabelian cohomology

## APPLICATIONS III

$$\text{Nonabelian cohomology: } H^i(*, \mathcal{G}) = \begin{cases} H^0(*, \pi_1(\mathcal{G})), & i = -1 \\ \pi_0(\mathcal{G}(*)), & i = 0 \\ \pi_0(\text{TORS}(\mathcal{G})(*)), & i = 1 \end{cases}$$

THEOREM (LOTS OF PEOPLE) Let  $F: \mathcal{H} \rightarrow \mathcal{G}$  be a morphism of gr-stacks as above.

There are functorial maps

$$F_*^i : H^i(*, \mathcal{H}) \longrightarrow H^i(*, \mathcal{G})$$

which are induced by

$$\pi_1(F) : \pi_1(\mathcal{H}) \longrightarrow \pi_1(\mathcal{G})$$

$$F : \mathcal{H} \longrightarrow \mathcal{G}$$

$$F_* : \text{TORS}(\mathcal{H}) \longrightarrow \text{TORS}(\mathcal{G})$$

found in the long exact sequence.