Mapping spaces of Group-Like Stacks

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INTRODUCTION

Describe the hom space $\mathrm{HOM}(\mathcal{H}, \mathcal{G})$
where $\mathcal{H}, \mathcal{G}$ are group-like stacks (over some site $\mathcal{S}$, like $\underline{\mathrm{Top}}$)

WHY?

- (Short) exact sequences
- (Nonabelian) cohomology
- deg ≥ 3 cohomology / Postnikov invariants

* Group-like stacks are "easy," they essentially are 2-step complexes:

$$\mathcal{G} \simeq \left[G_1 \xrightarrow{\partial} G_0 \right]^{\sim}$$

G_0, G_1 are (sheaves of) groups

* Morphisms $\mathcal{H} \longrightarrow \mathcal{G}$ between them are difficult:

NOT morphisms of complexes, thus $\mathbb{R}\mathrm{Hom}(H_\bullet, G_\bullet)$

REFERENCES

WITH BEHRANG NOOHI

Butterflies I: Advances in Math. 221 (2009), 687–773

Butterflies II, Advances in Math. 225 (2010), 922–976

Butterflies III, ...

Butterflies IV,

WITH A. EMIN TATAR

Cohomology and coherence, final stages

STACKS

"Fix a base site $\mathcal{S}$ w/ a Grothendieck Topology ..."

(1) $\mathcal{S}$ = Open sets in an ordinary topological space / scheme S

(2) $\mathcal{S}$ is a category like <u>Top</u>, so you have <u>objects</u> : $T, \ldots$

<u>morphisms</u> : $T' \longrightarrow T$

<u>open covers</u> : $\coprod_{i \in I} U_i \longrightarrow T$

A stack $\mathcal{X}$ over $\mathcal{S}$ is, roughly, a sheaf of categories $/\mathcal{S}$:

* $T \in Ob(\mathcal{S}) \rightsquigarrow \mathcal{X}_T$
* $U \xrightarrow{f} T \in Mor(\mathcal{S}) \rightsquigarrow f^* : \mathcal{X}_T \longrightarrow \mathcal{X}_U$, such that
* $(f \circ g)^* \simeq g^* \circ f^*$ (and other coherence conditions)

We need a gluing condition on objects and morphisms of $\mathcal{X}$:

$$\vdots\ U_2 \rightrightarrows U_1 \rightrightarrows U_0 \xrightarrow{f} U$$

$$\begin{matrix} x_0, x_1, x_2 & x_0, x_1 & x & y \end{matrix}$$

$$\begin{tikzcd} x_1 \arrow[rr] & & x_0 \\ & x_2 \arrow[ul] \arrow[ur] \arrow[u, phantom, "\wr"] & \end{tikzcd}$$

must come from $y \in Ob\, \mathcal{X}_U$

STACKS (Cont'd)

Favorite example: Let G be a group (quite often: sheaf of groups)

$BG: T \rightsquigarrow \{\text{Principal homogeneous } G\text{-spaces}/_T\}$

$U \xrightarrow{f} T \rightsquigarrow f^*: BG_T \longrightarrow BG_U$ pull back of PHS

Sometimes we write: $\mathrm{TORS}(G)$ in place of BG

↳ Stack of G-torsors

Rmk: $\mathrm{TORS}(G)$ makes sense even when $\mathcal{S}$ is $\operatorname{Spec} k$ (Galois Cohomology)

Other famous example: $\mathcal{M}_g$ – moduli stack of curves

$$T \rightsquigarrow \left\{ \begin{array}{c} X \\ \downarrow \\ T \end{array} \middle|\ \text{relative curve of genus } g \right\}$$

$$U \xrightarrow{f} T \rightsquigarrow \mathcal{M}_g(T) \longrightarrow \mathcal{M}_g(U)$$

$$X/_T \longmapsto X \times_T U/_U$$

Gr-Stacks = 2-Groups

"French" terminology   New Age

A gr-stack $\mathcal{G}$ is a stack over $\mathcal{S}$ with "a group law:"

* multiplication morphism $\mathcal{G} \times \mathcal{G} \xrightarrow{m} \mathcal{G}$
* Identity object: $I \in \mathrm{Ob}\,\mathcal{G}$
* Inverse functor: $\mathcal{G} \xrightarrow{*} \mathcal{G}$ $\quad X \rightsquigarrow X^*$

Such that the triple $(m, I, *)$ behaves exactly like the corresponding one for an ordinary group, except that we "replace identities with isomorphisms."

One must be careful
I and $*$ are non
canonical : there are better ways

GR-STACKS (Cont'd)

Examples are given by Automorphism Stacks:

$\mathcal{E}q(\mathcal{X})$: self-equivalences of the stack $\mathcal{X}$

* $\mathcal{E}q(BG)$: equivalences
$$\begin{tikzcd} BG \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\alpha\Downarrow"] & BG \end{tikzcd} \qquad \begin{array}{l} P \mapsto u(P) \\ \alpha_P : u(P) \mapsto v(P) \end{array}$$

Morita Theory for torsors : $u \rightsquigarrow E_u$ – bitorsor such that
$$u(P) = P \overset{G}{\wedge} E_u$$
$$u \xrightarrow{\alpha} v \rightsquigarrow E_u \xrightarrow{\alpha} E_v$$

* VARIANT: If $\partial : G \longrightarrow H$ is a homomorphism, then $\partial_* : BG \longrightarrow BH$
morphism of stacks

$\mathcal{E}q_{BH}(BG)$: only consider self-equivalences $u : BG \longrightarrow BG$
$$\begin{tikzcd} BG \arrow[rr, "u"] \arrow[dr, "\partial_*"'] & & BG \arrow[dl, "\partial_*"] \\ & BH & \end{tikzcd}$$

Stacks & Categories

Another way to look at categories: SIMPLICIAL SETS

$\mathcal{C}$ category $\rightsquigarrow$ $N_\bullet\mathcal{C}$ nerve

2-coskeletal simplicial set

(modulo taking the geometric realization to define π_i) $N\mathcal{C}$ is a 2-type.

Moral (but can be made rigorous):

Stacks can be viewed as simplicial (pre)sheaves which are 2-truncated.

GR-STACKS & CROSSED MODULES

For a gr-category $\mathcal{G}$, the nerve $N\mathcal{G}$ is a 2-coskeletal simplicial object w/ group law up to homotopy ($\mathcal{G}_\infty$-space?)

If $N\mathcal{G}$ has a rigid group law, then it is a simplicial group

By taking into account the 2-truncation, $N\mathcal{G}$ arises from a crossed module $G_1 \xrightarrow{\partial} G_0$, i.e.

$$N\mathcal{G}_0 = G_0 \quad , \quad N\mathcal{G}_i = \underbrace{G_1 \times \cdots \times G_1}_{i\text{-times}} \times G_0$$

CROSSED MODULE:

Action $G_1 \times G_0 \longrightarrow G_1 \quad (g, x) \longmapsto g^x$

i) $\partial(g^x) = x^{-1}(\partial g)\, x$

ii) $'g^{\partial g} = g^{-1}\, 'g\, g$

EXAMPLES:

1) $G \to \mathrm{Aut}(G)$
2) $N \hookrightarrow G$ inclusion of a normal subgroup
3) $(F,*) \to (E,*) \to (B,*)$ fibration

$$\pi_1(F,*) \xrightarrow{\partial} \pi_1(E,*)$$

What if $\mathcal{G}$ is a gr-stack?

GR-STACKS & CROSSED MODULES

Let $\mathcal{G}$ be a gr-stack over $\mathcal{S}$. The situation is analogous to that of a single category—Apparently well known, but no detailed proof.

PROPOSITION (E.A.—B.NOOHI) There exists a (sheaf of) crossed modules $G_1 \xrightarrow{\partial} G_0$ over $\mathcal{S}$, and a morphism (of gr-stacks) $\pi : G_0 \longrightarrow \mathcal{G}$ such that

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{\pi} \mathcal{G}$$

is exact at G_0, and essentially surjective at $\mathcal{G}$.

* "Essentially surjective:" every object of of $\mathcal{G}$ is locally isomorphic to a point of G_0
* The sequence is a presentation of $\mathcal{G}$ (NOT unique!)
  We write: $\mathcal{G} \simeq [G_1 \xrightarrow{\partial} G_0]$, when a presentation has been chosen.

* From now on, we'll always choose a presentation

Misc. Facts re. $\mathcal{G}$

Assume $\mathcal{G} \simeq [G_1 \xrightarrow{\partial} G_0]$, for a crossed module $G_1 \xrightarrow{\partial} G_0$

* Deligne: $\mathcal{G} \simeq \mathrm{TORS}(G_1, G_0)$ (We should write $B(G_1, G_0)$):
  principal homogeneous G_1-spaces P whose extension $P \overset{G_1}{\wedge} G_0 \simeq G_0$
* Breen: $\mathcal{G}$ is part of an exact sequence
  $$\mathcal{G} \longrightarrow BG_1 \longrightarrow BG_0$$
* In fact:

  (1) $\mathcal{G} \simeq \mathcal{E}q_{BG_0}(BG_1)$

  (2) In particular, $\mathcal{E}q(BG) \simeq [G \longrightarrow \mathrm{Aut}(G)]$

THE PROBLEM WITH MORPHISMS

If $\mathcal{H} \simeq [H_1 \xrightarrow{\partial} H_0]$ and $\mathcal{G} \simeq [G_1 \xrightarrow{\partial} G_0]$, and $F : \mathcal{H} \longrightarrow \mathcal{G}$ is a morphism of gr-stacks:

$$\lambda_{X,Y} : F(XY) \longrightarrow F(X)\,F(Y) \ ; \ X, Y \in Ob(\mathcal{H})$$

Then surely F corresponds to:

$$\begin{tikzcd} H_1 \arrow[r] \arrow[d, "\partial"'] & G_1 \arrow[d, "\partial"] \\ H_0 \arrow[r] & G_0 \end{tikzcd} \quad ?$$

NO!

Same problem with describing morphisms in derived categories

By the same token, a natural transformation $\alpha : F_1 \Rightarrow F_2 : \mathcal{H} \longrightarrow \mathcal{G}$ does not induce, or come from, a chain homotopy of complexes.

BUTTERFLIES

Let $H_1 \xrightarrow{\partial} H_0$ and $G_1 \xrightarrow{\partial} G_0$ be two crossed modules. B. Noohi defines:

A butterfly from H_0 to G_0 is a diagram of groups:

$$\begin{tikzcd}
H_1 \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G_1 \arrow[dl, "i"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "\pi"] \arrow[dr, "j"'] & \\
H_0 & & G_0
\end{tikzcd}$$

(i) The NE-SW is a group extension

(ii) The NW-SE is a complex

(iii) Compatibilities:

$$i\left(g^{j(e)}\right) = e^{-1} i(g) e$$

$$\kappa\left(h^{\pi(e)}\right) = e^{-1} \kappa(h) e$$

A morphism of butterflies is a group isomorphism

$$\alpha : E \xrightarrow{\simeq} E'$$

compatible with the structures of the diagram above.

A butterfly is split if the extension in it is trivial: $E = H_0 \ltimes G_1$

In this case, we get an honest morphism:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

WEAK MORPHISMS

Butterflies from $H_1 \xrightarrow{\partial} H_0$ to $G_1 \xrightarrow{\partial} G_0$ form a category $B(H_\bullet, G_\bullet)$ (in fact a groupoid).

THEOREM (E.A.—B.NOOHI) Let the gr-stacks $\mathcal{H}$ and $\mathcal{G}$ have presentations $\mathcal{H} \simeq [H_1 \xrightarrow{\partial} H_0]$, $\mathcal{G} \simeq [G_1 \xrightarrow{\partial} G_0]$. There is an equivalence of categories:

$$\mathrm{HOM}(\mathcal{H}, \mathcal{G}) \simeq B(H_\bullet, G_\bullet)$$

Both members can be sheafified (= stackified) and the equivalence is an equivalence of stacks:

$$\underline{\mathrm{HOM}}(\mathcal{H}, \mathcal{G}) \simeq \underline{B}(H_\bullet, G_\bullet)$$

Proof :-) One direction assigns to $F: \mathcal{H} \longrightarrow \mathcal{G}$ the (stack) fibered product

$E = H_0 \times_{\mathcal{G}} G_0$

the other, assigns to E the morphism which sends an object Q of $\mathcal{H}$ to $\underline{\mathrm{Hom}}_{H_0}(Q, E) \in \mathrm{Ob}(\mathcal{G})$ ▨

FACTS AND EXAMPLES

If $0 \to A \to E \to K \to 1$ is a group extension w/ abelian kernel, we get:

$$\begin{tikzcd} & & A \arrow[dl] \arrow[d] \\ & E \arrow[dl] \arrow[r] & 0 \\ K & & \end{tikzcd}$$

hence a morphism $K \longrightarrow \mathrm{TORS}(A)$ $(\equiv BA)$

More generally, for an extension $1 \to G \xrightarrow{i} E \xrightarrow{\pi} K \to 1$ (G not necessarily abelian) we get

$$\begin{tikzcd} & & G \arrow[dl] \arrow[d] \\ & E \arrow[dl] \arrow[r] & \mathrm{Aut}(G) \\ K & & \end{tikzcd}$$

and so, $K \longrightarrow \mathcal{E}q(BG)$

(Schreier, Dedecker, Grothendieck, Breen)

FACTS AND EXAMPLES II

A butterfly

$$\begin{tikzcd} H_1 \arrow[d] \arrow[dr] & & G_1 \arrow[dl] \arrow[d] \\ H_0 & E \arrow[l] \arrow[r] & G_0 \end{tikzcd}$$

can be completed to

$$\begin{tikzcd} & E_1 \arrow[dl] \arrow[d] \arrow[dr] & \\ H_1 \arrow[d, "\partial"'] \arrow[r] & E \arrow[dl] \arrow[dr] & G_1 \arrow[l] \arrow[d, "\partial"] \\ H_0 & & G_0 \end{tikzcd}$$

where $E_1 = H_1 \times G_1 \longrightarrow E$ is a crossed module. One proves:

$$\mathcal{H} \simeq [H_1 \xrightarrow{\partial} H_0] \simeq [E_1 \longrightarrow E]$$

$E_1 \longrightarrow E \simeq H_1 \longrightarrow H_0$

quasi-isomorphism

Thus there is a fraction of crossed modules:

$$\begin{tikzcd} & E_\bullet \arrow[dl, "\simeq"', "qi"] \arrow[dr] & \\ H_\bullet & & G_\bullet \end{tikzcd}$$

A morphism of butterflies $\alpha : E \xrightarrow{\sim} E'$ yields

$$\begin{tikzcd} & E_\bullet \arrow[dl, "\simeq"', "qi"] \arrow[dr] \arrow[dd, "\alpha"] & \\ H_\bullet & & G_\bullet \\ & E'_\bullet \arrow[ul, "\simeq", "qi"'] \arrow[ur] & \end{tikzcd}$$

Non abelian derived category $\mathcal{D}(\mathrm{XMod})$

(Abelian case: Deligne, SGA 4, XVIII)

APPLICATIONS I

DEFINITION (L. BREEN) A *short exact* sequence of gr-stacks is a sequence of morphisms

$$\mathcal{K} \xrightarrow{i} \mathcal{H} \xrightarrow{p} \mathcal{G}, \qquad p \circ i \simeq 1$$

such that:
- i) $\mathcal{K} \simeq$ Homotopy Kernel (p)
- ii) p essentially surjective
- iii) Suitable notion of exactness at $\mathcal{H}$
- iv) p is a fibration (!)

Reason for iv) The short exact sequence ought to yield a long exact sequence of (non abelian) cohomology objects.

PROPOSITION (E.A.–B.NOOHI) The fibration hypothesis can be dispensed with.

Proof: Given $p : \mathcal{H} \to \mathcal{G}$ essentially surjective, there is a butterfly and, hence, a fraction

$$\begin{tikzcd} & E_\bullet \arrow[dl, "\sim"'] \arrow[dr] & \\ H_\bullet & & G_\bullet \end{tikzcd}$$

which gives an equivalent morphism $p' : \mathcal{E} \to \mathcal{G}$ which is a fibration $\blacksquare$

APPLICATIONS II

The fibration replacement follows from

THEOREM (E.A.—B.N.) Let $F\colon \mathcal{H}\to\mathcal{G}$ be a morphism of gr-stacks, $\mathcal{K}\simeq \mathrm{HKer}(F)$.
There is a long exact sequence of 2-stacks:

$$0\to\pi_1(\mathcal{K})\to\pi_1(\mathcal{H})\to\pi_1(\mathcal{G})\to\mathcal{K}\to\mathcal{H}\xrightarrow{F}\mathcal{G}\xrightarrow{\Delta}\mathcal{F}\to \mathrm{TORS}(\mathcal{H})\xrightarrow{F_*}\mathrm{TORS}(\mathcal{G})$$

* Analog of "turning triangles", but unfortunately it terminates
* $\mathcal{F}$ is the "homotopy fiber" of $F_*\colon \mathrm{TORS}(\mathcal{H})\longrightarrow \mathrm{TORS}(\mathcal{G})$

  2-stack associated to the complex $H_1\longrightarrow E\longrightarrow G_0$, the nonexact diagonal of the butterfly.
* The fraction $H_\bullet \xleftarrow{\sim} E_\bullet \xrightarrow{F'} G_\bullet$ provides an explicit description of F_*:

$$\begin{tikzcd} & \mathcal{X}' := \mathcal{X}\times_{\mathrm{TORS}(H_0)}\mathrm{TORS}(E) \arrow[dr, rightsquigarrow] & \\ \mathcal{X} \arrow[ur, rightsquigarrow] & & F'_*(\mathcal{X}') \end{tikzcd}$$

Sufficiently explicit to provide cocycles.

paves the way to obtain maps in nonabelian cohomology

APPLICATIONS III

Nonabelian cohomology : $H^i(*,\mathcal{G}) = \begin{cases} H^0(*,\pi_1(\mathcal{G})), & i=-1 \\ \pi_0(\mathcal{G}(*)), & i=0 \\ \pi_0(\mathrm{TORS}(\mathcal{G})(*)), & i=1 \end{cases}$

THEOREM (LOTS OF PEOPLE) Let $F:\mathcal{H}\longrightarrow\mathcal{G}$ be a morphism of gr-stacks as above.
There are functorial maps

$$F_*^i : H^i(*,\mathcal{H}) \longrightarrow H^i(*,\mathcal{G})$$

which are induced by

$$\pi_1(F): \pi_1(\mathcal{H}) \longrightarrow \pi_1(\mathcal{G})$$

$$F:\mathcal{H}\longrightarrow\mathcal{G}$$

$$F_* : \mathrm{TORS}(\mathcal{H}) \longrightarrow \mathrm{TORS}(\mathcal{G})$$

found in the long exact sequence.