New Invariants for Algebraic Cycles

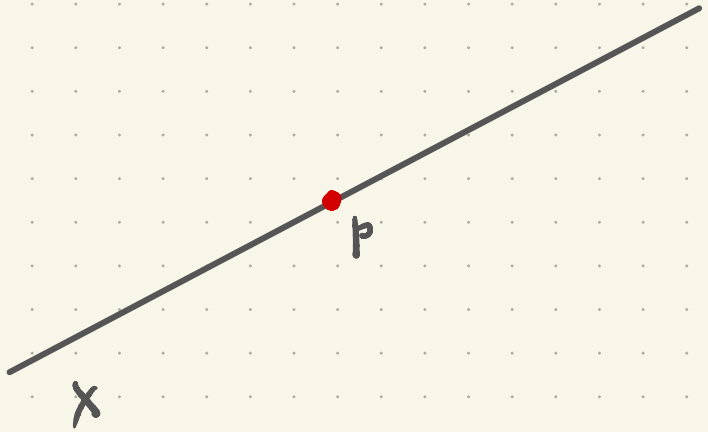
Jan 29, 2021

Colloquium Talk
University of Toledo

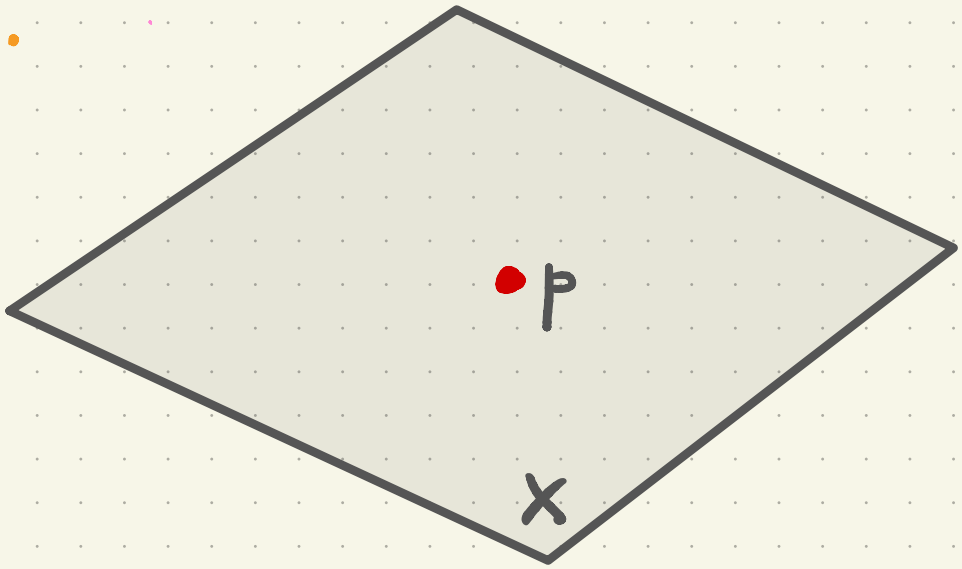
Ettore Aldrovandi

Florida State University

JOINT WORK WITH N. RAMACHANDRAN (UMD)



Vs.



- Codimension: 1
- 1 equation
- Functions on X with a pole at p

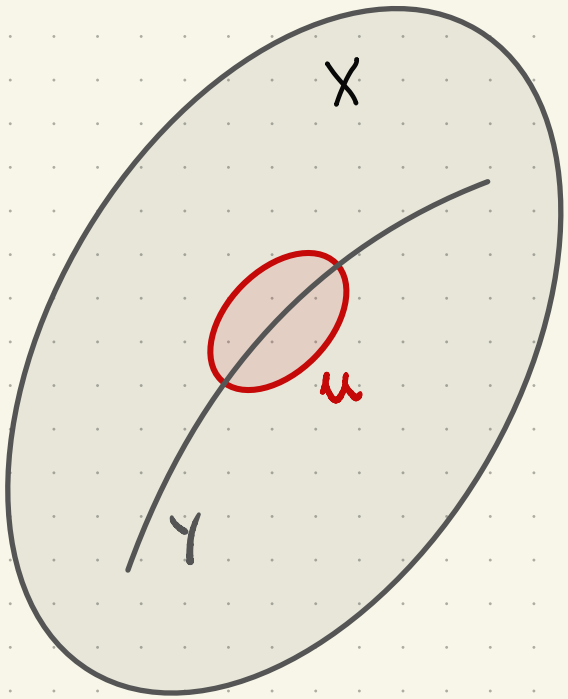
- Codimension: 2
- 2 - equations
- ??

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PLAN

1. CODIMENSION ONE: DIVISORS & LINE BUNDLES
   An informal introduction (+ 2 definitions)
2. K-THEORY
3. WHIRLWIND TOUR OF (HIGHER) CATEGORIES
4. THE K_2-GERBE OF A POINT

DIVISORS, INFORMALLY

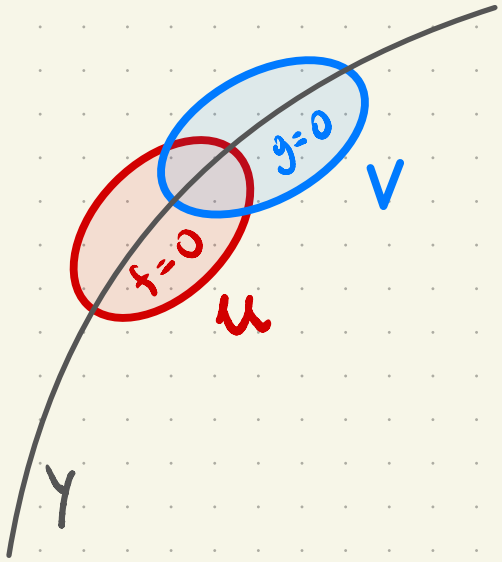


- X smooth variety over k (field)
- $Y \subseteq X$ has **codimension 1**

  locally one equation $f = 0$, $f \in \mathcal{O}_X(U)$: Regular functions
- Y determines an Algebraic/Geometric object $\mathcal{L}_Y \longrightarrow X$

  $U \mapsto \left\{ \frac{1}{f} k \;\middle|\; k \in \mathcal{O}_X(U) \right\} \subseteq K_X(U)$

  Submodule (pole at Y) — rational functions



- Compatibility:

  $\frac{g}{f}$ does not vanish at Y

  $$\left\{ \frac{1}{f} k \;\middle|\; k \in \mathcal{O}_X(U) \right\} = \left\{ \frac{1}{g} k \;\middle|\; k \in \mathcal{O}_X(U) \right\}$$

SOME DEFINITIONS + THINGS THAT GO WRONG

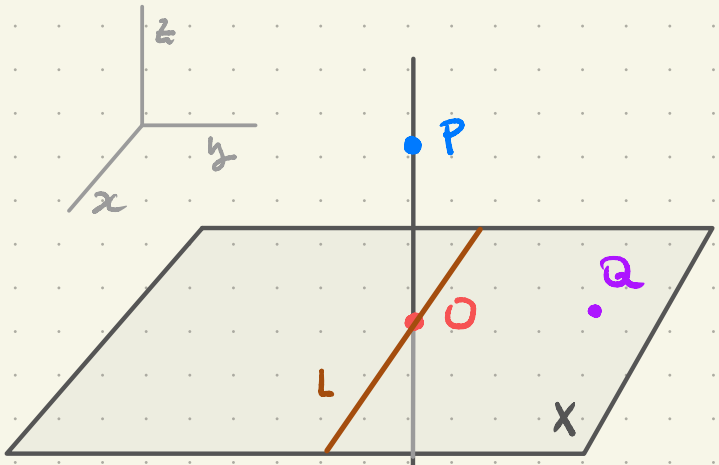
CODIMENSION

INFORMALLY $\operatorname{codim}_X(Y) = \dim X - \#\text{equations}$

FORMALLY $Y \subseteq X$ irreducible closed subset

$$\operatorname{codim}_X(Y) = \sup\left\{ n : Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \right\}$$

Z_i : irreducible closed subs. of X

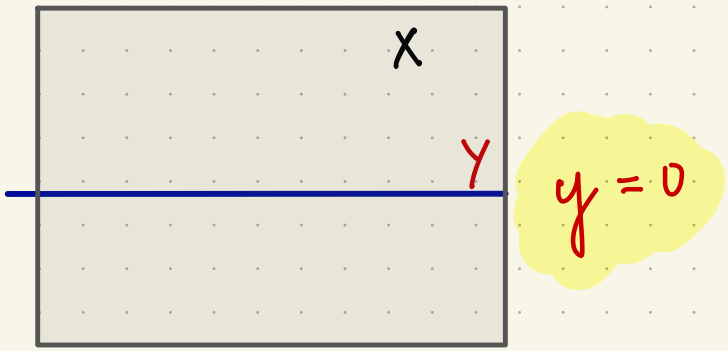


$$X = \operatorname{Spec} \frac{k[x,y,z]}{(xz, yz)}$$

CODIM	EQUATION
1	$z - a = 0$, $a \neq 0$
2	$x = y = z = 0$
2	$x - a = 0$, $y - b = 0$, $(a,b) \neq (0,0)$
1	$y = 0$, $z = 0$

EXAMPLE LINE IN THE (x,y)-PLANE

$X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ — Polynomials (x,y)

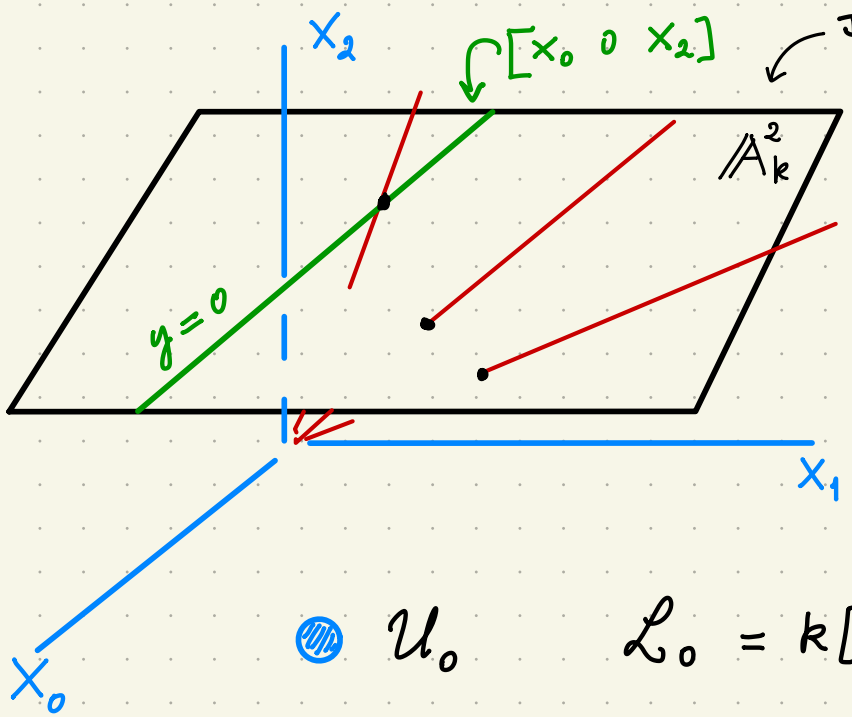


- $Y = \operatorname{Spec} \dfrac{k[x,y]}{(y)} \cong \operatorname{Spec} k[x] = \mathbb{A}^1_k$
- $\dfrac{1}{y} \in k(x,y) = \left\{ \dfrac{f}{g} \mid f, g \in k[x,y] \right\}$

  Rational functions (= Meromorphic)
- Define $\mathcal{L} = k[x,y]\left\langle \dfrac{1}{y} \right\rangle = \left\{ f\,\dfrac{1}{y} \mid f \in k[x,y] \right\}$

  MODULE GENERATED BY $\dfrac{1}{y}$

Extended Example Line in $\mathbb{P}^2_k$ (Projective Space)



$[X_0\ X_1\ 1] = [x\ y\ 1]$

$[X_0\ X_1\ X_2] = \{\lambda(X_0, X_1, X_2) \mid \lambda \in k^*\} \in \mathbb{P}^2_k$

- $\mathbb{P}^2_k = U_0 \cup U_1 \cup U_2$, $U_i = \{X_i \neq 0\}$

- U_0 $\quad \mathcal{L}_0 = k[x,y]\langle \frac{1}{y} \rangle$

- U_1 $\quad \mathcal{L}_1 = k[x', z']\langle 1 \rangle$

- U_2 $\quad \mathcal{L}_2 = k[y'', z'']\langle \frac{1}{y''} \rangle$

$\frac{1}{y} = z'\ 1 \qquad \mathcal{L}_0 \xrightarrow{\cong} \mathcal{L}_1$

$1 = y''\ \frac{1}{y''} \qquad \mathcal{L}_1 \xrightarrow{\cong} \mathcal{L}_2$

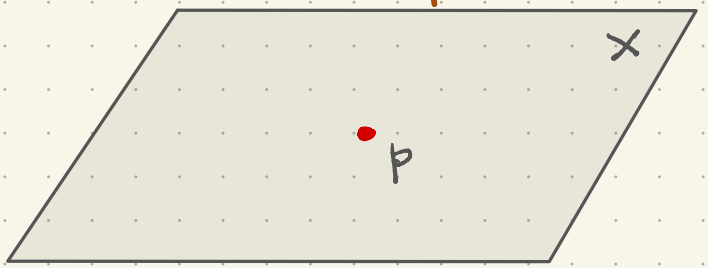
$\frac{1}{y''} = x\ \frac{1}{y} \qquad \mathcal{L}_2 \xrightarrow{\cong} \mathcal{L}_0$

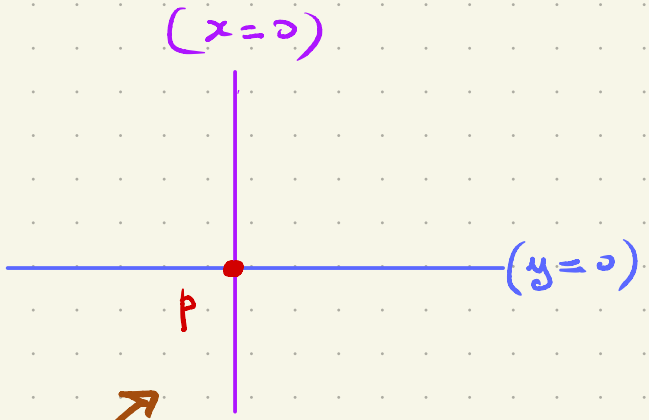
A POINT ON THE PLANE

POINT p, CODIMENSION 2

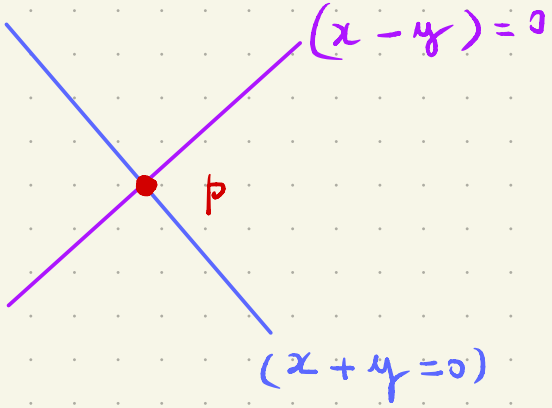
$p \subseteq X$ intrinsic object, but arbitrary extrinsic description by equations

$X = \operatorname{Spec} k[x,y] \ni p = (x,y)$

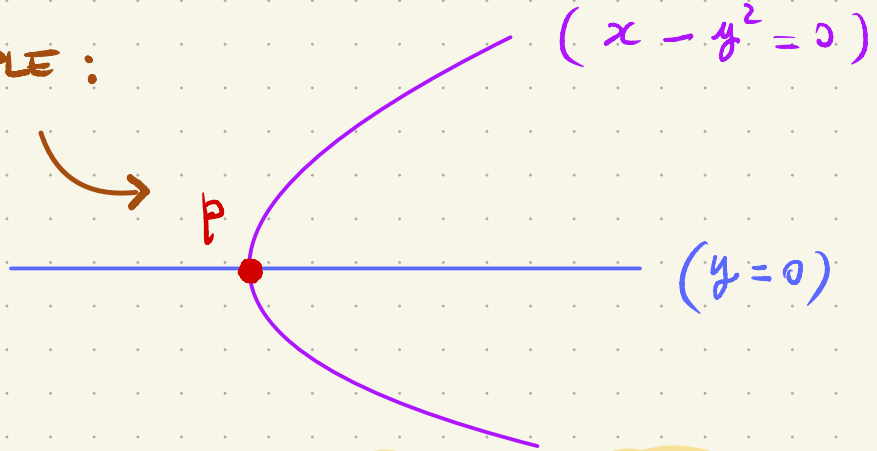




FOR EXAMPLE:

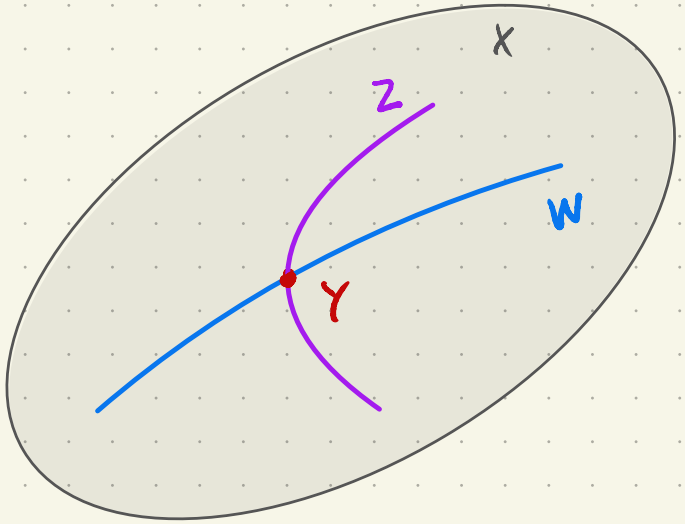


$p = (x-y, x+y)$



$p = (x-y^2, y)$

CODIMENSION 2



$CH^2(X)$ Group of cycles mod $\sim$

- Y a cycle of codimension 2
- If p is represented as an intersection of Z and W, a category of choices
  no "module"
- No intrinsic Y $\rightsquigarrow$?

Some classical Construction

CODIMENSION > 2

$CH^p(X)$

Cod. p - Cycles mod $\sim$

- Same problems as above
- Higher Category of choices for "equations"

K-THEORY

K-THEORY

ALGEBRAIC K-THEORY
INVARIANT OF RINGS

- $\mathcal{C}$ ⇝ [K-Theory] ⇝ $\mathbb{K}(\mathcal{C})$ ⇝ $\pi_i(\mathbb{K}(\mathcal{C}))$

  $\mathcal{C}$: category w/ some structure

  $\mathbb{K}(\mathcal{C})$: Spectrum (∞-loop space)

  $\pi_i(\mathbb{K}(\mathcal{C}))$: K-groups

- R : commutative Ring

  $\mathcal{C} := \mathrm{Proj}^{f.g.}(R)$: fin. gen. Projective R-modules

  Quillen's "$Q=+$" $\quad K_i(R) = \pi_{i+1}(BGL(R)^+)$

- Classical K-Theory (Whitehead, Bass, Milnor, Van der Waerden, ...)

  $K_0(R) = \mathbb{Z} \qquad K_1(R) = R^\times \qquad K_2(R)$

K-THEORY

$K_2(R)$

- $0 \to K_2(R) \to St(R) \xrightarrow{\phi} E(R) \to 1$

  Universal Central Extension

  $E(R) = [GL(R), GL(R)]$ Elementary Matrices

- Symbol : $K_1(R) \otimes_{\mathbb{Z}} K_1(R) = R^\times \otimes_{\mathbb{Z}} R^\times \xrightarrow{\{,\}} K_2(R)$

  intersection product

  (1) Bilinear
  (2) Antisymmetric
  (3) $\{1-r, r\} = 1$

- MATSUMOTO'S THM $R = F$ field $K_2(F) = \langle \{x,y\} ; x,y \in F^\times \mid \text{same relations} \rangle$

- If R is a local ring $R^\times \otimes_{\mathbb{Z}} R^\times \twoheadrightarrow K_2(R)$

  $K_2(R)$ is generated by symbols

K-THEORY OF SCHEMES

X regular scheme

(We usually assume more, X smooth over F)

- $U \subseteq X$ open

$U \mapsto K_i(U) := K_i(\mathcal{O}_X(U))$ — $\mathcal{O}_X(U)$: Ring of regular functions

"Sheafify" $\rightsquigarrow$ $\mathcal{K}_{i,X}$ — i^{th} K-Theory Sheaf on X

- BLOCH-QUILLEN FORMULA

$$CH^i(X) \cong H^i(X, \mathcal{K}_{i,X})$$

$CH^i(X)$: codim i cycles / $\sim$ rat equivalence

K-THEORY OF SCHEMES

- GERSTEN CONJECTURE

rat. functions

$$0 \to K_{i,X} \longrightarrow K_i(F(X)) \to \bigoplus_{x \in X^{(1)}} K_{i-1}(k(x)) \to \cdots$$

$K_i(F(X))$: at generic point

$$\cdots \to \bigoplus_{x \in X^{(i-1)}} K_1(k(x)) \to \bigoplus_{x \in X^{(i)}} K_0(k(x)) \to 0$$

$K_0(k(x)) \cong \mathbb{Z}$

$X^{(i)}$: primes/pts of codim i

$\bigoplus_{x \in X^{(i)}}$: cod i-cycles : $Z^i(X)$

$K_0(k(x))$: Global Sections / X

- BLOCH-QUILLEN

$$CH^i(X) = H^i(X, \mathcal{K}_{i,X}) = \mathbb{H}^0(X, \mathcal{G}_i^\bullet)$$

K-THEORY OF SCHEMES

$i = 1$ $\quad 0 \to K_{1,X} = \mathcal{O}_X^* \to K_1(F(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} \mathbb{Z} \to 0$

$$Z^1(X) \cong H^0(X, K_1/\mathcal{O}_X^*) \twoheadrightarrow H^1(X, \mathcal{O}_X^*) \cong CH^1(X)$$

Cod 1-cycles — Divisors (Cartier) — Line Bundles — Cod 1-cycles mod $\sim$

$i = 2$ $\quad 0 \to K_{2,X} \to K_2(F(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} \kappa(x)^* \longrightarrow \bigoplus_{x \in X^{(2)}} \mathbb{Z} \longrightarrow 0$

rat. functions on x

$$Z^2(X) \twoheadrightarrow H^1(X, K_2(F(X))/K_{2,X}) \xrightarrow{\cong} H^2(X, K_{2,X}) \cong CH^2(X)$$

Cycles — "Cartier"

HIGHER CATEGORIES

HIGHER CATEGORIES

SIMPLICIAL HOMOTOPY DEFINITION

- $\mathcal{C}$ a category is identified with a Simplicial Set

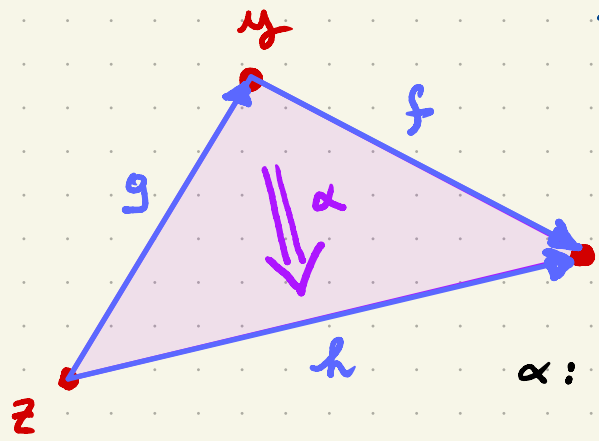
Think: Simplicial Complex

Objects = 0-Cells : $x, y, z, \dots$

1-Morphisms = 1-Cells : $f, g, h, \dots$

2-Morphisms = 2-Cells : $\alpha, \dots$

$\dots$



$\alpha : f \circ g \Rightarrow h : z \to x$

- An ordinary category $\mathcal{C}$ is a "category" as above via its Nerve

(no i-cells, $i \geq 2$)

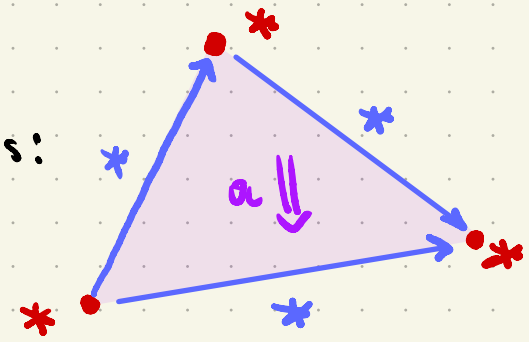
HIGHER CATEGORIES

EILENBERG-MACLANE, DOLD-PUPPE-KAN, ...

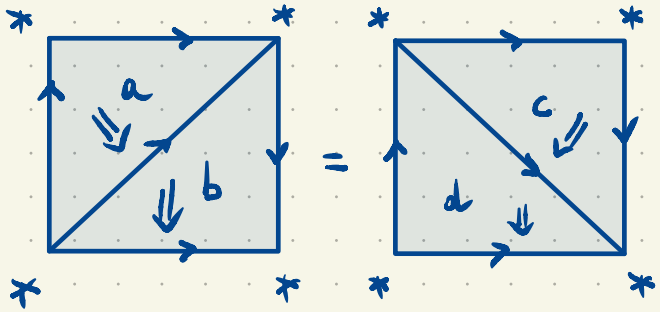
- Representation thm: $H^i(X,A) \cong [X, K(A,i)]$

- $K(A,2)$ as a category:

Only 2-cells:



match faces →



$a + b = c + d$

- Write X as $X = \bigcup_i U_i$

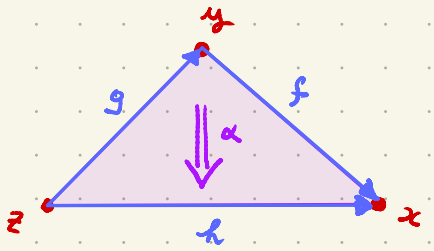
  - $U_i \to *$
  - $U_i \cap U_j \to *$
  - $U_i \cap U_j \cap U_k \to a_{ijk}$

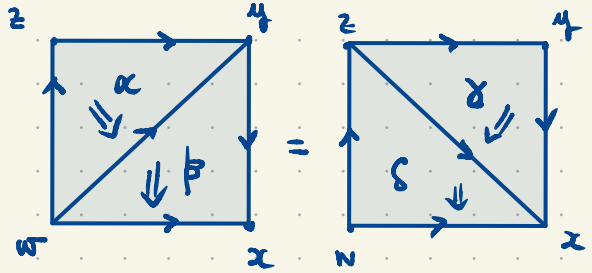
Then:

$$a_{jkl} - a_{ikl} + a_{ijl} - a_{ijk} = 0$$

HIGHER CATEGORIES

- We can do this with a **complex**: $L_2 \xrightarrow{\partial} L_1 \xrightarrow{\partial} L_0$ (α, f, x)





- Complex $\to L_3 \xrightarrow{\partial} L_2 \xrightarrow{\partial} L_1 \xrightarrow{\partial} L_0 \rightsquigarrow$ Category $\Gamma(L_\bullet)$

- Write X as $X = \bigcup_i U_i$
  - $U_i \to x_i$
  - $U_i \cap U_j \to f_{ij}$
  - $U_i \cap U_j \cap U_k \to \alpha_{ijk}$

$\rightsquigarrow$

$$x_j - x_i = \partial f_{ij}$$

$$f_{jk} - f_{ik} + f_{ij} = \partial \alpha_{ijk}$$

$$\alpha_{jkl} - \alpha_{ikl} + \alpha_{ijl} - \alpha_{ijk} = 0$$

K_2 - GERBES AND CODIMENSION 2 CYCLES

BACK TO K-THEORY

RESOLUTION OF $\mathcal{K}_{i,X}$ BY K-THEORY OF FIELDS

$$\begin{array}{ccccccc} \mathcal{K}_{i,X} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathcal{G}_i^{(i)} & \longrightarrow & \mathcal{G}_{i-1}^{(i)} & \longrightarrow & \cdots & \longrightarrow & \mathcal{G}_0^{(i)} \end{array}$$

$$\mathcal{G}_j^{(i)} = \bigoplus_{x \in X^{(i-j)}} K_j(\kappa(x))$$

Res. field at x

- $CH^i(X) \cong H^i(X, \mathcal{K}_{i,X}) \cong [X, \Gamma(\mathcal{G}_\bullet^{(i)})]$ ← Hom classes of maps

in particular a codim i cycle $\alpha \in Z^i(X) = \mathcal{G}_0^{(i)}(X)$ is a map

$$\alpha : * \longrightarrow \mathcal{G}_0^{(i)} = \bigoplus_{x \in X^{(i)}} K_0(\kappa(x)) \cong \bigoplus_{x \in X^{(i)}} \mathbb{Z}$$

- You get maps $(i \geq 2)$

$$Z^i(X) \cong H^0\left(X, \bigoplus_{x \in X^{(i)}} \mathbb{Z}\right) \longrightarrow H^{i-1}\left(X, \frac{K_i(F_X)}{\mathcal{K}_{i,X}}\right) \cong CH^i(X)$$

EXAMPLE

CLASSICAL DIVISOR SEQUENCE

$i = 1$

$$0 \to K_{1,X} = \mathcal{O}_X^* \to K_1(F(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} \mathbb{Z} \to 0$$

$$Z^1(X) \cong H^0(X, K_1/\mathcal{O}_X^*) \twoheadrightarrow H^1(X, \mathcal{O}_X^*) \cong CH^1(X)$$

Cod 1-cycles — Divisors (Cartier) — Line Bundles — Cod 1-cycles mod $\sim$

$i = 2$

$$0 \to K_{2,X} \to K_2(F(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} \kappa(x)^\times \longrightarrow \bigoplus_{x \in X^{(2)}} \mathbb{Z} \longrightarrow 0$$

$$Z^2(X) \twoheadrightarrow H^1(X, K_2(F(X))/K_{2,X}) \xrightarrow{\cong} H^2(X, K_{2,X}) \cong CH^2(X)$$

Cycles — "Cartier"

E. A., N. RAMACHANDRAN ('16, '20)

DEFINITION CODIMENSION TWO CARTIER CYCLES

$(i=2)$

$K_2(F(X))/K_{2,X}$ – PRINCIPAL BUNDLES / X

THEOREM TO α WE ASSIGN THE HIGHER CATEGORY $\mathcal{C}_\alpha$ BY

$(i=2)$

PULLING BACK $\Gamma\left(\left[\mathcal{G}_2^{(2)} \to \mathcal{G}_1^{(2)}\right]\right)$ VIA

$$* \xrightarrow{\alpha} \bigoplus_{x \in X^{(2)}} K_0(k(x))$$ DEFINED BY α

Homotopy Fiber

$$\begin{array}{ccc} \mathcal{C}_\alpha & \longrightarrow & \Gamma \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{\alpha} & \bigoplus_{x \in X^2} K_0(k(x)) \end{array}$$

Computing $\mathcal{C}_\alpha$

(E.A., N. Ramachandran)

$$\alpha \in Z^2(X) \longleftrightarrow * \longrightarrow \bigoplus_{x \in X^{(2)}} K_0(k(x)) = \bigoplus_{x \in X^{(2)}} \mathbb{Z}$$

$\mathcal{C}_\alpha$ has the following description:

Objects: (D, f) $\quad D \in \bigoplus_{x \in X^{(1)}} \mathbb{Z}$ $\quad \alpha \subseteq |D|$

$f \in k(D)$ $\quad \operatorname{div}(f) = \alpha$

Morphisms: $(D, f) \xrightarrow{u} (E, g)$ $\quad u \in K_2(F(X))$

$$T_D(u) = f \,, \quad T_E(u) = -g$$

$$u = \{\alpha_1, \beta_1\} \cdots \{\alpha_k, \beta_k\} \,, \quad \alpha_1, \ldots, \beta_k \in F(X)$$

THE TAME SYMBOL

$x \in X^{(1)}$ irreducible (prime), codimension 1

The TAME SYMBOL of two rational functions $f, g \in F(X)^*$

$$T_x(f,g) = (-1)^{v_x(f)v_x(f)} \frac{f^{v_x(g)}}{g^{v_x(f)}}(x)$$

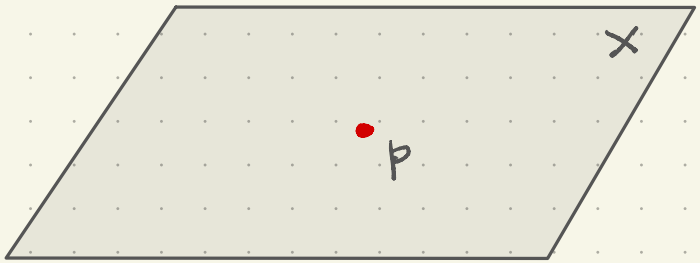
Tame Symbol at x

1. Bilinear
2. Antisymmetric
3. $\{1-r, r\} = 1$

Example X: Riemann Surface, $x \in X$, f, g meromorphic

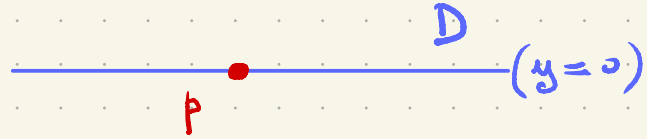
WEIL RECIPROCITY: $\prod_{p \in X} T_p(f,g) = 1$

A POINT IN THE PLANE

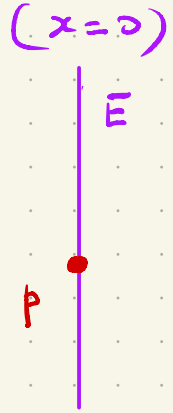


$X = \operatorname{Spec} k[x,y]$

TWO OBJECTS OF $\mathcal{C}_\alpha$:

① 

$f = x \in k(D) \cong k(x)$

② 

$g = y \in k(E) \cong k(y)$

(ISO)MORPHISM

③ $(D, f) \xrightarrow{\cong} (E, g)$

Consider $x, y \in k(x,y)$

$\nu_D(x) = 0$ $\quad$ $\nu_E(x) = 1$

$\nu_D(y) = 1$ $\quad$ $\nu_E(y) = 0$

So:

$T_D(\{x,y\}) = \frac{x}{1}\Big|_C = x \in k(x)$

$T_E(\{x,y\}) = \frac{1}{y}\Big|_C = y^{-1} \in k(y)$

THANK YOU