Intersec(K)tions

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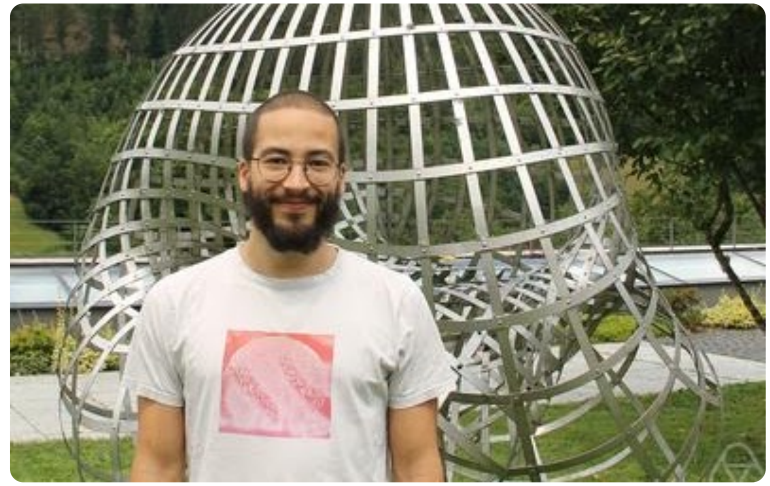
FSU/UF Topology & Geometry Meeting

Jan 23 – 24, 2026



Maxime Ramzi (Münster)

Niranjan Ramachandran (UMD)

- Cup products, the Heisenberg group, and codimension two algebraic cycles. Documenta Math. 21 (2016)
- Fiber Integration of gerbes and Deligne line bundles
  Homology, Homotopy and Applications 25 (2023)
- In progress

Cycles

K-Theory Machine

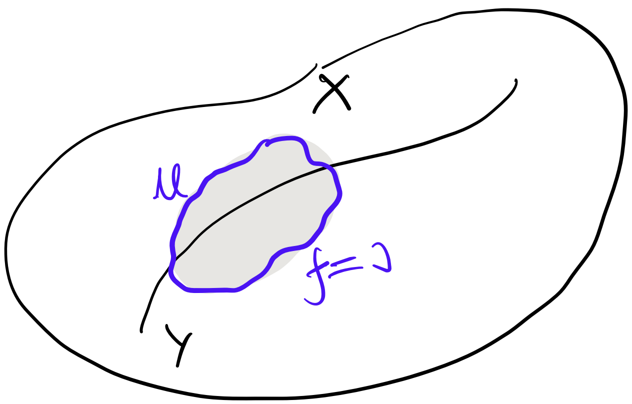
Lower homotopy types of K

Back to Cycles

Epilogue

Divisors

X : complex manifold / Proper Algebraic variety $/_F$



- $f=0$: Codimension 1

  Y locally defined by one equation

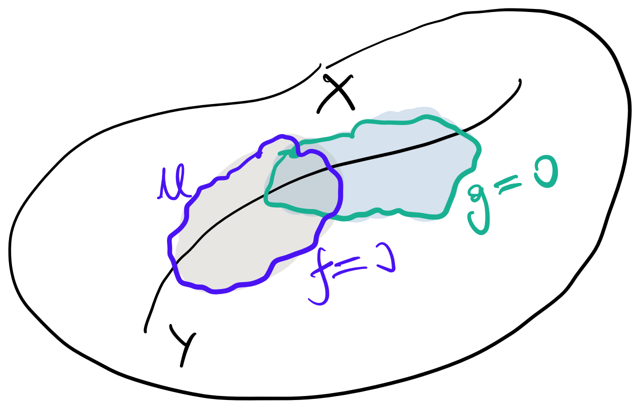
  There are pitfalls

- $\mathcal{L}_{\mathcal{U}} := \mathcal{O}_X(\mathcal{U}) \frac{1}{f}$ free, generated by $\frac{1}{f}$

  $\subseteq \mathcal{M}_X(\mathcal{U})$ = meromorphic / rational functions

Divisors

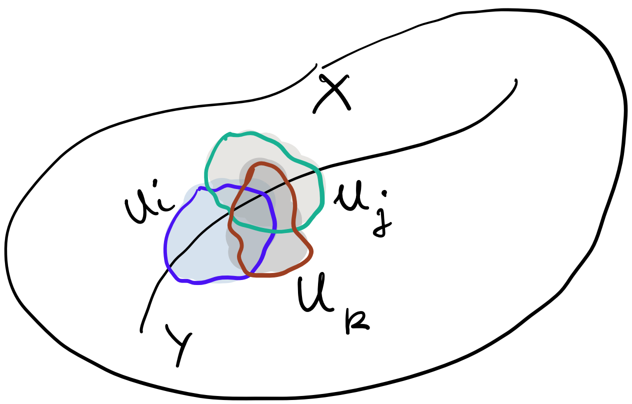
X : complex manifold / Proper Algebraic variety $/F$



- $\frac{g}{f} \in \mathcal{O}_X(U \cap V)^\times$ unit / invertible

Divisors

X : complex manifold / Proper Algebraic variety $/_F$



- $\frac{g}{f} \in \mathcal{O}_X(U \cap V)^\times$ unit / invertible
- $\{U_i\}$ open cover

  $f_i = 0$ defines $Y \cap U_i$

  * $f_i = g_{ij} f_j$, $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^\times$

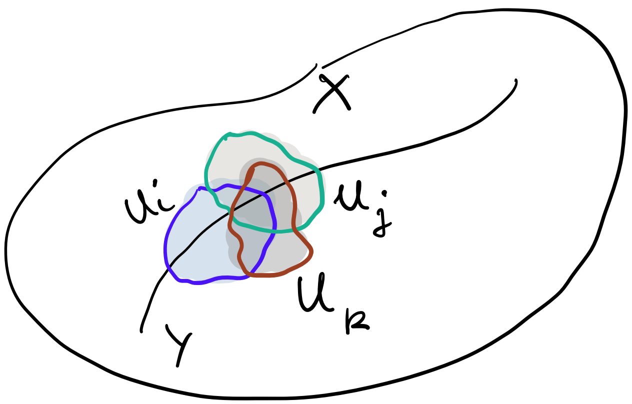
  * $g_{ij}\, g_{jk}\, g_{ki} = 1$

- These data define $\mathcal{L}_Y \longrightarrow X$

  Reversible: $\mathcal{L} \to X \rightsquigarrow Y$

Divisors

X : complex manifold / Proper Algebraic variety $/F$



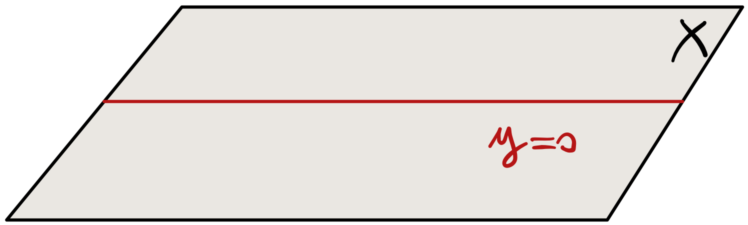
$\mathcal{L}_Y \longrightarrow X$ works nicely :

$$H^1(X, \mathcal{O}_X^\times) \simeq CH^1(X)$$

Line bundles $/\cong$

Codim 1 / rat. equivalence

Ex : Plane



$$X = \mathbb{A}^2 = \operatorname{Spec}\left(F[x,y]\right)$$

$$Y = \operatorname{Spec}\left(\frac{F[x,y]}{(y)}\right) \cong \operatorname{Spec} F[x]$$

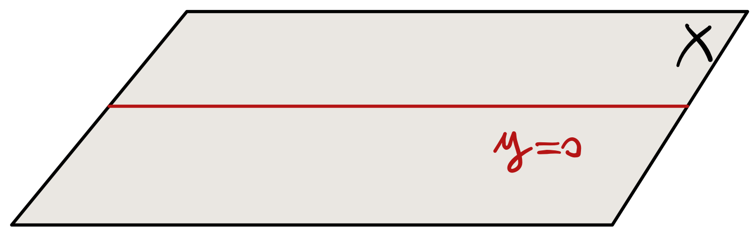
$$\mathcal{L}_Y = \left\{ p(x,y)\frac{1}{y} \;\middle|\; p \in F[x,y] \right\}$$

$$\subseteq F(x,y) = \left\{ \frac{f}{g} \;\middle|\; f,g \in F[x,y] \right\}$$

Rational functions

Fraction field

Ex : Plane



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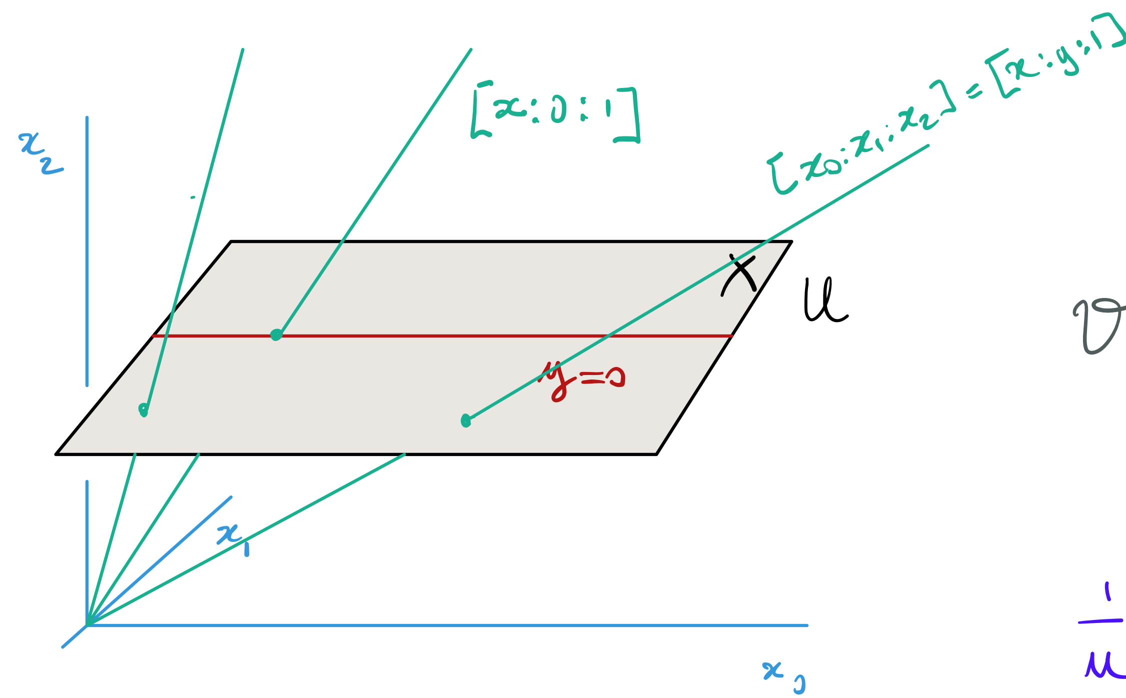
$$\mathcal{L}_Y = \left\{ p(x,y)\frac{1}{y} \;\middle|\; p \in F[x,y] \right\}$$

Ex : $\mathbb{P}^2$



Changing coordinates :

$$\mathcal{V} = \{ [x_0 : x_1 : x_2] \mid x_0 \neq 0 \}$$

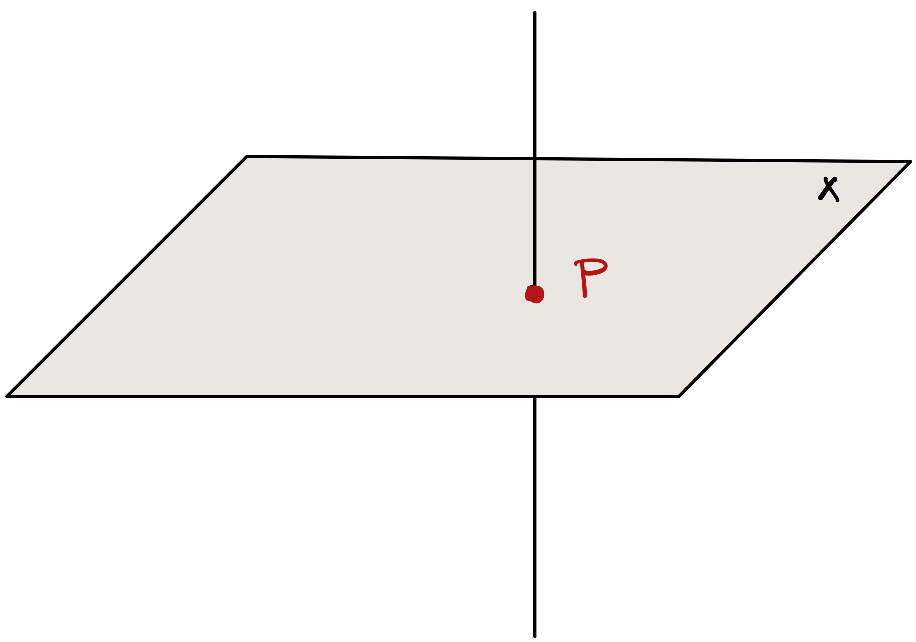
$$= \{ [1 : u : v] \}$$

$$\frac{1}{u} = x \frac{1}{y} \quad \rightsquigarrow \quad x \in \mathcal{O}^*(U \cap \mathcal{V})$$

Pitfalls of "codimension"

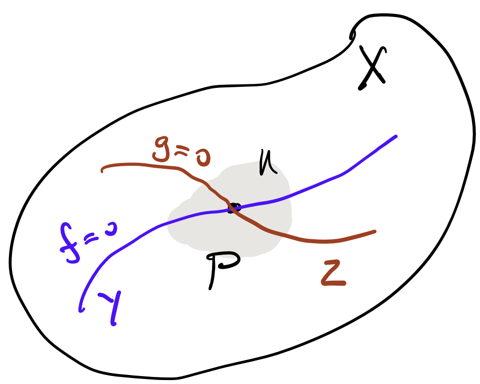
Informally # equations

Formally $\sup\{ n : Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset X \}$



$$X = \operatorname{Spec} \frac{F[x, y, z]}{(xz, yz)}$$

Higher Codimension $P \subseteq X$



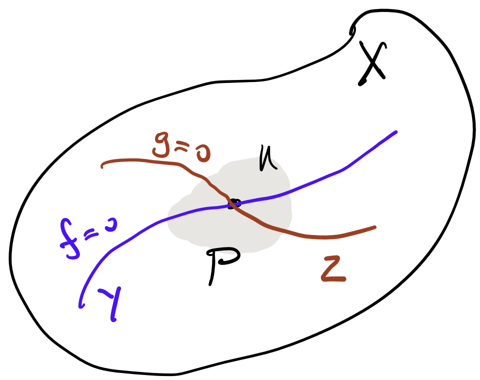
$$P = Y \cap Z$$

Y defines $\mathcal{L} \longrightarrow X$

Z " $\mathcal{M} \longrightarrow Y$

Intrinsic?

Higher Codimension $P \subseteq X$

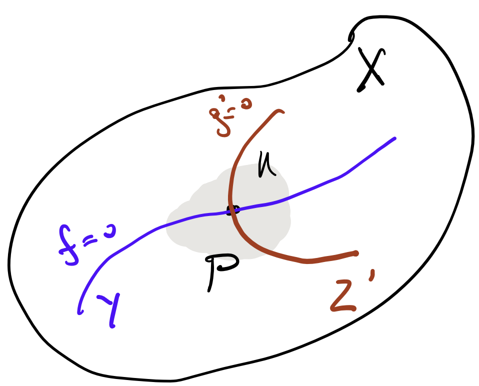


$P = Y \cap Z$

Y defines $\mathcal{L} \longrightarrow X$

Z' " $\mathcal{M}' \longrightarrow Y$

v.



$\mathcal{M} \xrightarrow{\cong} \mathcal{M}'$

- Is there a preferred choice?
- Take all choices into account

An Exercise in Complex Analysis

f, g : meromorphic on $\underset{\text{open}}{\mathcal{U} \subseteq} \mathbb{C}$

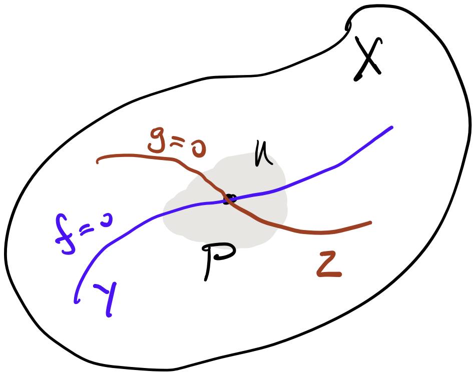
$$\{f,g\}_p = (-1)^{\nu_p(f)\nu_p(g)} \frac{f^{\nu_p(g)}}{g^{\nu_p(f)}}(p)$$ ← Tame symbol

1. $\{f,g\} = \{g,f\}^{-1}$ Antisymmetric
2. $\{f_1 f_2, g\} = \{f_1, g\}\{f_2, g\}$ Bilinear
3. $\{f, f^{-1}\} = 1$
4. $\{f, 1-f\} = 1$ Steinberg Relation

Prove :
$$\{f,g\}_p = \frac{1}{2\pi i} \int_{\gamma_p} \log f \frac{dg}{g}$$

There are algebraic versions

Higher Codimension $P \subseteq X$



$$P = Y \cap Z$$

$$Y : \{f_{ij}\}$$

$$Z : \{g_{ij}\}$$

Use the tame symbols: $\{f_{ij}, g_{jk}\}$ on $U_i \cap U_j \cap U_k$

Good combinatorics:

$$\{f_{jk}, g_{kl}\}\{f_{ik}, g_{kl}\}^{-1}\{f_{ij}, g_{jl}\}\{f_{ij}, g_{jk}\}^{-1} = 1$$

Spencer Bloch, '74 (Ann. Math. 99 (1974))

- $$H^1(X, \mathcal{O}_X^*) \times H^1(X, \mathcal{O}_X^*) \xrightarrow{\text{cup product}} H^2(X, \mathcal{O}_X^* \underset{\mathbb{Z}}{\otimes} \mathcal{O}_X^*) \xrightarrow{\{\, ,\,\}} H^2(X, K_2(\mathcal{O}_X^*))$$
$$((f_{ij}), (g_{ij})) \longmapsto (\{f_{ij}, g_{jk}\})$$

- $$H^2(X, K_2(\mathcal{O}_X^*)) \cong \underset{\substack{\uparrow \\ \text{Codim 2 cycles}/\text{equivalence}}}{CH^2(X)}$$ The Bloch Quillen formula

Cycles

K-Theory Machine

Lower homotopy types of K

Back to Cycles

Epilogue

K-Theory

Category $\mathcal{C}$ with interesting structure $\longrightarrow \boxed{K} \longrightarrow$ Space/Spectrum $K(\mathcal{C})$

Quillen, Waldhausen, Thomason

K-groups of $\mathcal{C}$

$$K_i(\mathcal{C}) := \pi_i K(\mathcal{C})$$
$$= [S^i, K(\mathcal{C})]$$

homotopy classes

K-Theory

Category $\mathcal{C}$ with interesting structure $\longrightarrow \boxed{K} \longrightarrow$ Space/Spectrum $K(\mathcal{C})$

Quillen, Waldhausen, Thomason

K-groups of $\mathcal{C}$

$$\mathcal{C} = \mathrm{Proj}^{fg}_R = Ch^b(\mathrm{Mod}_R) = \mathrm{Perf}_R$$

$$\mathcal{C} = \mathrm{Fin} = CW^f_* = CW_*$$

$$K_i(\mathcal{C}) := \pi_i K(\mathcal{C}) = \underbrace{[S^i, K(\mathcal{C})]}_{\text{homotopy classes}}$$

Classical K-Theory groups (Whitehead, Bass, Milnor, Steinberg ...)

R : Ring

$$K_0(R) = \frac{\mathbb{Z}\{[P] \mid P \text{ f g projective module}\}}{\begin{array}{c}\text{subgr. gen by } [P'] - [P] + [P''] \\ \text{whenever } 0 \to P' \to P \to P'' \to 0\end{array}}$$

$$K_1(R) = GL(R)^{ab} = GL(R)/E(R)$$

$GL(R) = \lim_{n \to \infty} GL_n(R)$

elementary matrices

$$K_2(R) = \ker\left(St(R) \longrightarrow E(R)\right)$$

Only these

$[e_{ij}(\lambda), e_{kl}(\mu)] = 1$ $\quad j \neq k,\ i \neq l$

$[e_{ij}(\lambda), e_{jl}(\mu)] = e_{il}(\lambda\mu)$ $\quad i \neq l$

$[e_{ij}(\lambda), e_{ki}(\mu)] = e_{kj}(-\mu\lambda)$ $\quad j \neq k$

+ many others (possibly not all known)

Classical K-Theory groups (Whitehead, Bass, Milnor, Steinberg ...)

R : Ring

$$K_0(R) = \frac{\mathbb{Z}\{[P] \mid P \text{ f g projective module}\}}{\text{Subgr. gen by } [P'] - [P] + [P''] \text{ whenever } 0 \to P' \to P \to P'' \to 0}$$

$\mathbb{Z} \simeq$

$$K_1(R) = GL(R)^{ab} = GL(R)/E(R)$$

$R^* \simeq$

elementary matrices

$$GL(R) = \lim_{n\to\infty} GL_n(R)$$

$$K_2(R) = \ker\left(St(R) \longrightarrow E(R)\right)$$

generated by symbols $\{r, s\}$

(Equal if R field)

Only these

$[e_{ij}(\lambda), e_{kl}(\mu)] = 1$ $\quad j \neq k$, $i \neq l$

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$[e_{ij}(\lambda), e_{ki}(\mu)] = e_{kj}(-\mu\lambda)$ $\quad j \neq k$

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Classical K-Theory groups (Whitehead, Bass, Milnor, Steinberg ...)

R: Ring

$$K_0(R) = \frac{\mathbb{Z}\{[P] \mid P \text{ f g projective module}\}}{\text{Subgr. gen by } [P'] - [P] + [P''] \text{ whenever } 0 \to P' \to P \to P'' \to 0}$$

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elementary matrices

$$K_2(R) = \ker\left(St(R) \longrightarrow E(R)\right)$$

Exact sequence for $I \subset R$ ideal:

$$K_2(I) \to K_2(R) \to K_2(R/I) \to \cdots \to K_0(I) \to K_0(R) \to K_0(R/I)$$

Milnor

Quillen, Waldhausen

Simplicial Object

S. construction

$$\mathcal{C} \longrightarrow \boxed{\mathcal{K}} \longrightarrow K(\mathcal{C}) = \Omega \left| N_{\bullet}\, w S_{\bullet} \mathcal{C} \right|$$

Geometric Realization

Loop

cofibrations $\rightarrowtail$

weak equivalences $\xrightarrow{\sim}$

Waldhausen Categories

Quillen, Waldhausen

Simplicial Object

S. construction

$$\mathcal{C} \longrightarrow \boxed{\;K\;} \longrightarrow K(\mathcal{C}) = \Omega \left| N_\bullet\, w S_\bullet \mathcal{C} \right|$$

Loop

Geometric Realization

cofibrations $\rightarrowtail$

weak equivalences $\xrightarrow{\sim}$

Waldhausen Categories

$$\mathcal{C} = \mathrm{Proj}^{fg}_R \qquad\qquad \mathcal{C} = \mathrm{Fin}$$

$$= Ch^b(\mathrm{Mod}_R) \qquad\qquad = CW^f_* \qquad \rightsquigarrow \qquad K(\mathcal{C}) =: K(R)$$

$$= \mathrm{Perf}_R \qquad\qquad = CW_*$$

Simplicial Sets (Objects)

$$X_0 \;\substack{\longleftarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; X_2 \;\substack{\longleftarrow \\ \vdots \\ \longleftarrow}\; \cdots \; X_{n-1} \;\underset{n+1}{\substack{\longleftarrow \\ \vdots \\ \longleftarrow}}\; X_n \;\underset{n+2}{\substack{\longleftarrow \\ \vdots \\ \longleftarrow}}$$

face maps

$$d_i : X_n \longrightarrow X_{n-1}$$

Simplicial Sets (Objects)

$$\begin{tikzcd}
X_0 \arrow[r, bend left] & X_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[r, bend left] \arrow[r, bend left=50] & X_2 \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] & \arrow[l, shift left] \arrow[l, phantom, "\vdots"] \arrow[l, shift right] \cdots & X_{n-1} \arrow[r, bend left] \arrow[r, bend left=60, "\vdots"] & X_n \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2, "n"'] \arrow[r, bend left] \arrow[r, bend left=60, "\vdots"] & \arrow[l, shift left=2] \arrow[l, phantom, "\vdots"] \arrow[l, shift right=2, "n+1"']
\end{tikzcd}$$

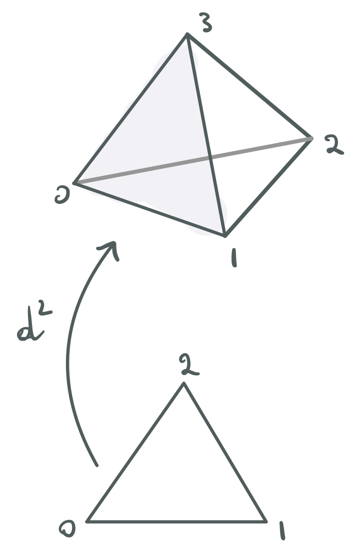
$s_i : X_{n-1} \longrightarrow X_n$

degeneracies

Simplicial Sets (Objects)

$$X_0 \leftleftarrows X_1 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; X_2 \;\substack{\longleftarrow\\ \vdots\\ \longleftarrow}\; \cdots \; X_{n-1} \;\substack{\longleftarrow\\ \vdots\\ \longleftarrow}\; X_n \;\substack{\longleftarrow\\ \vdots\\ \longleftarrow}$$

$$\begin{array}{ccc} & X_n & \\ \downarrow & \cdots & \downarrow \\ & X_{n-1} & \end{array} \qquad\qquad \begin{array}{ccc} & \Delta^n & \\ \uparrow & \cdots & \uparrow \\ & \Delta^{n-1} & \end{array}$$



Standard simplex:

$$\Delta^n_{top} = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid t_0 + \dots + t_n = 1\}$$

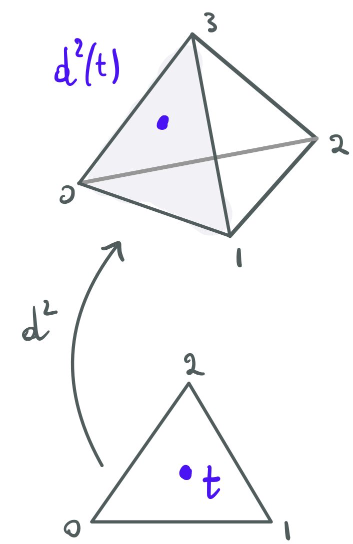
Simplicial Sets (Objects)

$$X_0 \leftleftarrows X_1 \leftleftarrows\!\!\leftarrow X_2 \;\substack{\leftarrow \\ \vdots \\ \leftarrow}\; \cdots\; X_{n-1} \;\substack{\leftarrow \\ \vdots \\ \leftarrow}\; X_n \;\substack{\leftarrow \\ \vdots \\ \leftarrow}\; \cdots$$

$$x \in X_n \qquad \Delta^n_{top}$$

$$\downarrow \;\cdots\; \downarrow \qquad \uparrow \;\cdots\; \uparrow$$

$$X_{n-1} \qquad \Delta^{n-1}_{top} \ni t$$



$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n_{top} \Big/ \sim$$

Geometric Realization

$$(d_i x, t) \sim (x, d^i(t))$$

Ex $\quad G$: group

$$BG_\bullet : \quad \underset{0}{*} \leftleftarrows \underset{1}{G} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \underset{2}{G^2} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \underset{3}{G^3} \;\substack{\longleftarrow\\ \cdots\\ \longleftarrow}\; \cdots$$

$$\begin{matrix} h \longleftarrow\!\shortmid \\ gh \longleftarrow\!\shortmid \\ g \longleftarrow\!\shortmid \end{matrix} \; (g, h)$$

$$\begin{matrix} (b, c) \longleftarrow\!\shortmid \\ (ab, c) \longleftarrow\!\shortmid \\ (a, bc) \longleftarrow\!\shortmid \\ (a, b) \longleftarrow\!\shortmid \end{matrix} \; (a, b, c)$$

$|BG|$ Classifying space of G

Ex Nerve of a Category $\qquad \mathcal{C}$: small category

$$N\mathcal{C}_\bullet \qquad \mathrm{Ob}\,\mathcal{C} \leftleftarrows \mathrm{Mor}\,\mathcal{C} \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; \cdots \;\overset{\longleftarrow}{\underset{\longleftarrow}{\vdots}}\; N\mathcal{C}_n \;\overset{\longleftarrow}{\underset{\longleftarrow}{\vdots}}$$

$$\left(x_0 \xleftarrow{f_1} \cdots \, x_{i-1} \xleftarrow{f_i f_{i+1}} x_{i+1} \longleftarrow \cdots\right) \overset{d_i}{\longmapsfrom} \left(x_0 \xleftarrow{f_1} x_1 \longleftarrow \cdots \xleftarrow{f_n} x_n\right)$$

compose f_i with f_{i+1}

n-tuple of composable morphisms of $\mathcal{C}$

$$B\mathcal{C} \overset{\mathrm{def}}{=} |N\mathcal{C}| \qquad \text{Classifying Space of } \mathcal{C}$$

Rmk $\quad \mathcal{C}$: $\mathrm{Ob}\,\mathcal{C} = \{*\}$ $\qquad$ Then $B\mathcal{C} = BG$

$\qquad \mathrm{Hom}_{\mathcal{C}}(*,*) = G$

Back to: $\mathcal{C} \longrightarrow \boxed{K} \longrightarrow K(\mathcal{C}) = \Omega \left| N_\bullet w S_\bullet \mathcal{C} \right|$

Waldhausen Categories

$\mathcal{C} = \mathrm{Fin}$ $\qquad \rightarrowtail \;=\;$ inclusions

$\qquad\qquad \xrightarrow{\sim} \;=\;$ bijections

$$\begin{tikzcd}[row sep=small, column sep=small]
& A' \arrow[rr, tail] \arrow[dd] & & B' \arrow[dd] \\
A \arrow[ur, "\sim"] \arrow[rr, tail, crossing over] \arrow[dd] & & B \arrow[ur, "\sim"] & \\
& 0 \arrow[rr, tail] & & B'/A' \\
0 \arrow[ur, equal] \arrow[rr, tail] & & B/A \arrow[ur, "\sim"] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

$\mathcal{C} = \mathrm{Proj}^{fg}_R$ $\qquad \rightarrowtail \;=\;$ monomorphisms

$\qquad\qquad \xrightarrow{\sim} \;=\;$ isomorphisms

$\mathcal{C} = Ch(\mathrm{Mod}_R)$ $\qquad \rightarrowtail \;=\;$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r] \arrow[d, hook] & C_i \arrow[r] \arrow[d, hook] & C_{i-1} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & D_{i+1} \arrow[r] & D_i \arrow[r] & D_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

$= \mathrm{Perf}(R)$

$= \ldots$

(split) monomorphisms

$\xrightarrow{\sim} \;=\; C_\bullet \longrightarrow D_\bullet$ inducing $H_i(C_\bullet) \xrightarrow{\cong} H_i(D_\bullet)$

(quasi isomorphisms)

Back to: $\mathcal{C} \longrightarrow \boxed{K} \longrightarrow K(\mathcal{C}) = \Omega \left| N_\cdot w S_\cdot \mathcal{C} \right|$

$S_\cdot$ – Construction

simplicial Category

At level n: Category $S_n \mathcal{C}$ with

$$
\begin{tikzcd}
0 \arrow[r, tail] & A_{0,1} \arrow[r, tail] \arrow[d] & A_{0,2} \arrow[r, tail] \arrow[d, two heads] & A_{0,3} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{0,n-1} \arrow[r, tail] \arrow[d, two heads] & A_{0,n} \arrow[d, two heads] \\
& 0 \arrow[r, tail] & A_{1,2} \arrow[r, tail] \arrow[d] & A_{1,3} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & A_{1,n-1} \arrow[r, tail] \arrow[d, two heads] & A_{1,n} \arrow[d, two heads] \\
& & 0 \arrow[r, tail] & A_{2,3} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & A_{2,n-1} \arrow[r, tail] \arrow[d, two heads] & A_{2,n} \arrow[d, two heads] \\
& & & \cdots & \cdots & \cdots \arrow[d] & \cdots \arrow[d] \\
& & & & 0 \arrow[r, tail] & A_{n-1,n-1} \arrow[r, tail] \arrow[d] & A_{n-1,n} \arrow[d, two heads] \\
& & & & & 0 \arrow[r, tail] & A_{n,n} \arrow[d, two heads] \\
& & & & & & {}
\end{tikzcd}
$$

- $S_0 \mathcal{C} = *$
- $S_1 \mathcal{C} = \mathcal{C}$

Back to: $\mathcal{C} \longrightarrow \boxed{\mathcal{K}} \longrightarrow K(\mathcal{C}) = \Omega \left| N_\bullet w S_\bullet \mathcal{C} \right|$

3-Cells in $N_\bullet wS_\bullet\mathcal{C}$

$$
\begin{tikzcd}
0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d] & B \arrow[r, tail] \arrow[d, two heads] & C \arrow[d] \\
 & 0 \arrow[r, tail] & B/A \arrow[r, tail] \arrow[d] & C/A \arrow[d, two heads] \\
 & & 0 \arrow[r, tail] & C/B \arrow[d] \\
 & & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "\wr"] & B \arrow[r, two heads] \arrow[d, "\wr"] & B/A \arrow[d, "\wr"] \\
A' \arrow[r, tail] & B' \arrow[r, two heads] & B'/A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "\wr"] \\
A' \arrow[d, "\wr"] \\
A''
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[d, equal] \\
0 \arrow[d, equal] \\
0 \arrow[d, equal] \\
0
\end{tikzcd}
$$

Faces

$$\Big\downarrow \quad i = 0, \dots, 3$$

$$
\begin{tikzcd}
A \arrow[r, tail] & B \arrow[d, two heads] \\
 & B/A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail] & C \arrow[d, two heads] \\
 & C/A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, tail] & C \arrow[d, two heads] \\
 & C/B
\end{tikzcd}
\qquad
\begin{tikzcd}
B/A \arrow[r, tail] & C/A \arrow[d, two heads] \\
 & C/B
\end{tikzcd}
$$

Cycles

K-Theory Machine

Lower homotopy types of K

Back to Cycles

Epilogue

Lower homotopy types of $K(e)$

Recall $K_i(e) = \pi_i K(e) = [S^i, K(e)]$

Find $P_n K(e)$ such that

$$\pi_i P_n K(e) = \begin{cases} \pi_i K(e) = K_i(e) & i \leq n \\ 0 & i > n \end{cases}$$

Eg

$$\begin{tikzcd}
 & \vdots \arrow[d, dashed] \\
 & P_n K(e) \arrow[d] \\
K(e) \arrow[uur] \arrow[ur] \arrow[r] \arrow[ddr] & P_{n-1} K(e) \arrow[d] \\
 & \vdots \arrow[d, dashed] \\
 & P_1 K(e) \arrow[d] \\
 & *
\end{tikzcd}$$

$\rightsquigarrow$ Explicit models ?

Lower homotopy types of $K(\mathcal{C})$

Muro - Tonks ('07), Muro-Tonks-Witte ('15)

$D_\bullet\mathcal{C} : D_1\mathcal{C} \xrightarrow{\partial} D_0\mathcal{C}$ algebraic model for $P_1 K(\mathcal{C})$

* $H_1(\partial) = K_1(\mathcal{C})$, $H_0(\partial) = K_0(\mathcal{C})$
* $D_\bullet\mathcal{C}$ generated by : $[A]$ for any object of $\mathcal{C}$

  nil-2 groups $\qquad [A \xrightarrow{\sim} A']$ for any weak equiv.

  $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofibration
* Relations (some)

$$\partial[A \xrightarrow{\sim} A'] = -[A'] + [A]$$

$$\partial[A \rightarrowtail B \twoheadrightarrow B/A] = -[B] + [B/A] + [A]$$

* Functorial & Universal

Lower homotopy types of $K(\mathcal{C})$

E. A, M. Gunnjel ('25)

quadratic module (H-J. Baues)

$Q.\mathcal{C}$: $Q_2(\mathcal{C}) \xrightarrow{\partial} Q_1(\mathcal{C}) \xrightarrow{\partial} Q_0(\mathcal{C})$ algebraic model for $P_2 K(\mathcal{C})$

nil-2 groups

- $H_i(\partial) = K_i(\mathcal{C}) \quad i = 0, 1, 2$
- Generators $\quad |A|, \ A \in \mathrm{Ob}\,\mathcal{C}$

$$\left|\begin{tikzcd} X \arrow[r, tail] & Y \\ A \arrow[u, "\sim"'] \end{tikzcd}\right| , \quad \left|\begin{tikzcd} U \arrow[r, tail] & V \arrow[r, tail] & W \\ X \arrow[u, "\sim"'] \arrow[r, tail] & Y \arrow[u, "\sim"'] & \\ A \arrow[u, "\sim"'] & & \end{tikzcd}\right|$$

- Relations

- Functorial in $\mathcal{C}$ and universal

Cycles

K-Theory Machine

Lower homotopy types of K

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Epilogue

K-Theory Sheaves

X: complex manifold or proper alg. variety $/_F$

$\underset{\text{open}}{\mathcal{U} \subset X} \rightsquigarrow \mathcal{O}_X(\mathcal{U})$ nicely working with $V \subset \mathcal{U} \subset X$

$\rightsquigarrow$ sheaf of rings / regular functions

- $\mathcal{U} \subset X \longmapsto K(\mathcal{O}_X(\mathcal{U}))$

restriction $V \subset \mathcal{U} \longmapsto K(\mathcal{O}_X(\mathcal{U})) \longrightarrow K(\mathcal{O}_X(\mathcal{U}))$

presheaf of spaces (spectra)

- $\mathcal{U} \subset X \longmapsto K_i(\mathcal{O}_X(\mathcal{U}))$ K_i-group

$K_{i,X}$ K_i-theory sheaf

Sheaf of abelian groups

Codimension q cycles

$$Z^q(X) := \mathbb{Z}\{\text{codimension } q \text{ subvarieties}\}$$

Rmk : If A is an abelian group, $B(\cdot)$ can be iterated :

$$A, \; BA, \; B^2A, \ldots$$

Therefore we can consider (varying across $\underset{\text{open}}{U \subset X}$)

$$B^q K_{q,X}$$

Key Lemma (E.A., Ramachandran, Ramzi ; 2016 $\longrightarrow$ 26)

$$\alpha \in Z^q(X) \rightsquigarrow \quad X \xrightarrow{f_\alpha} B^q K_{q,X}$$

Classifying map $\rightsquigarrow$ With a caveat

Construction/Definition (E.A., N.R. '16 ; w/ MR '25-26)

$B^{q-1}K_{q,X}$ -torsor

$$\begin{tikzcd} e_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d, "\mathrm{can}"] \\ X \arrow[r, "f_\alpha"'] & B^q K_{q,X} \end{tikzcd}$$

$$\begin{tikzcd} B^{q-1}K_{q,X} \simeq \Omega B^q K_{q,X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & e_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d, "\mathrm{can}"] \\ U \arrow[r, hook] & X \arrow[r, "f_\alpha"'] & B^q K_{q,X} \end{tikzcd}$$

composition is $* \simeq \alpha|_U \sim 0$

The Gersten conjecture (Thm by Quillen, 72)

The sequence:

skyscrapers

$$0 \to \mathcal{K}_{q,X} \to K_q(F(X)) \to \cdots \to \bigoplus_{x \in X^{(i)}} K_{(q-i)}(k(x)) \to \cdots \to \underbrace{\bigoplus_{x \in X^{(q)}} K_0(k(x))}_{Z^q(X)} \to 0$$

$\bigoplus_{x \in X^{(q)}} K_0(k(x)) = \mathbb{Z}$

function field: $k(\eta_X)$

"points" of codimension i = irr. subvarieties

$k(x)$ = residue field $= \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$

Codimension q cycles in X

is exact

Bloch - Quillen Formula

$$H^q(X, \mathcal{K}_{q,X}) \cong H^0(X, B^q \mathcal{K}_{q,X}) \cong CH^q(X)$$

Rmk: The complex $\mathcal{G}^{\bullet}_{q,X} : K_q(F(X)) \to \cdots \to \bigoplus_{x \in X^{(q)}} K_0(k(x))$

is a model for $B^q K_{q,X}$:

$$\begin{tikzcd}
K_{q,X}[q]: \arrow[d, "q.i."] & K_{q,X} \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\
\mathcal{G}^{\bullet}_{q,X}: & K_q(F(X)) \arrow[r] & \cdots \arrow[r] & \bigoplus_{x \in X^{(i)}} K_{(q-i)}(k(x)) \arrow[r] & \cdots \arrow[r] & \bigoplus_{x \in X^{(q)}} K_0(k(x)) \arrow[r] & 0
\end{tikzcd}$$

This is an equivalence

$$B^q K_{q,X} \xrightarrow{\sim} \mathrm{Ani}(\mathcal{G}^{\bullet}_{q,X})$$

in $\mathrm{Sh}(X, \mathrm{Ani}) = \mathcal{D}^{\leq 0}(\mathbb{Z})$

Rmk: The complex $\mathcal{G}^{\bullet}_{q,X} : K_q(F(X)) \to \cdots \to \bigoplus_{x \in X^{(q)}} K_0(k(x))$

is a model for $B^q \mathcal{K}_{q,X}$:

$$\begin{tikzcd}
\mathcal{K}_{q,X}[q]: \arrow[d, "q.i."'] & \mathcal{K}_{q,X} \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \\
\mathcal{G}^{\bullet}_{q,X}: & K_q(F(X)) \arrow[r] & \cdots \arrow[r] & \bigoplus_{x \in X^{(i)}} K_{(q-i)}(k(x)) \arrow[r] & \cdots \arrow[r] & \bigoplus_{x \in X^{(q)}} \underbrace{K_0(k(x))}_{\mathbb{Z}} \arrow[r] & 0 \\
& & & & & X \arrow[u, "f_\alpha"] &
\end{tikzcd}$$

We can explicitly write the classifying map

$\alpha \in \sum_{x \in X^{(q)}} \alpha_x x$

Construction/Definition (E.A., N.R. '16; w/ MR '25-26)

$B^{q-1}K_{q,X}$-torsor

$$\begin{tikzcd} e_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "\mathrm{can}"] \\ X \arrow[r, "f_\alpha"'] & B^q K_{q,X} \end{tikzcd}$$

Rmk: Since $\mathcal{G}^\bullet_{q,X}$ is a model for $B^q K_{q,X}$ the truncation

$$\tau_{\geq 1}\mathcal{G}^\bullet_{q,X} : K_q(F(X)) \longrightarrow \cdots \longrightarrow \bigoplus_{x \in X^{(q-1)}} K_1(k(x))$$

is a model for e_α

$q = 1$:

$$\begin{tikzcd} K_1(F(X)) = F(X)^* \arrow[r, "\mathrm{Div}"] & \bigoplus_{x \in X^{(1)}} \mathbb{Z} \\ e_\alpha \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner", very near start] & X \arrow[u, "\alpha"'] \end{tikzcd}$$

$q = 2$: (E.A., NR '16)

e_α is a $K_{2,X}$-gerbe

Questions

- By Grayson '78, $H^i(X,\mathcal{K}_{i,X}) \times H^j(X,\mathcal{K}_{j,X}) \xrightarrow[\text{cup}]{\text{smash} + } H^{i+j}(X,\mathcal{K}_{i+j,X})$

  is compatible with

  $$CH^i(X) \times CH^j(X) \xrightarrow[\text{intersection}]{\cdot} CH^{i+j}(X)$$

We have

$$X \xrightarrow{\alpha\wedge\beta} B^i\mathcal{K}_{i,X} \times B^j\mathcal{K}_{j,X} \longrightarrow B^{i+j}(\mathcal{K}_{i,X}\otimes\mathcal{K}_{j,X}) \longrightarrow B^{i+j}\mathcal{K}_{i+j,X}$$

Call $\mathcal{E}_{\alpha\wedge\beta}$ the resulting torsor

- Can we describe $\mathcal{E}_{\alpha\wedge\beta}$ in terms of $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$?
- Is $\mathcal{E}_{\alpha\wedge\beta}$ equivalent to $\mathcal{E}_{\alpha\cdot\beta}$?

- If $\alpha \underset{\text{rat}}{\sim} \alpha'$ is $\mathcal{E}_\alpha \xrightarrow{\sim} \mathcal{E}_{\alpha'}$?

All YES if $p = 1, 2$

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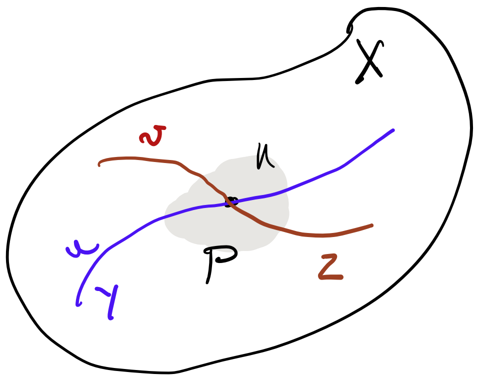
Description of $\mathcal{C}_P$



$(Y,u)\,,\,(Z,v)$: objects (0-cells) of $\mathcal{C}_P$

$u\in F(Y)\,,\ v\in F(Z)$ rational functions w/ prescribed order at P

1-cell: $(Y,u)\xrightarrow{\{f,g\}}(Z,v)$

$\{f,g\}\in K_2(F(X))$ such that

$$T_Y\{f,g\} = u \qquad T_Z\{f,g\} = v^{-1}$$

Tame Symbols

E.g.

$$T_Z(\{f,g\}) = (-1)^{\nu_Z(f)\nu_Z(g)} \left. \frac{f^{\nu_Z(g)}}{g^{\nu_Z(f)}} \right|_Z$$

Thank you