Intersection theory and homotopy types with algebraic structure

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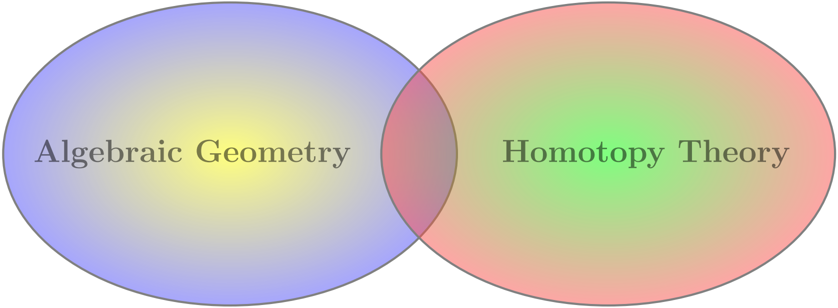
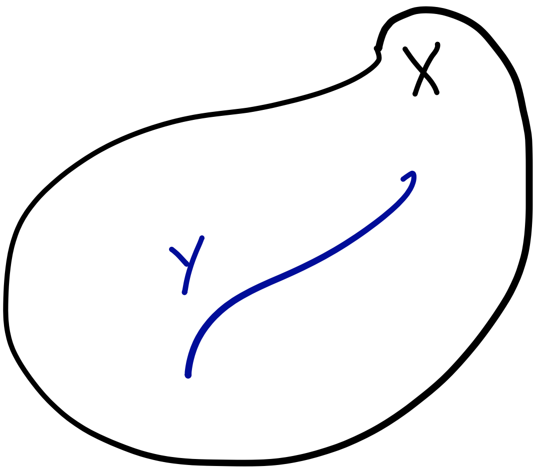

Algebraic Geometry
Homotopy Theory

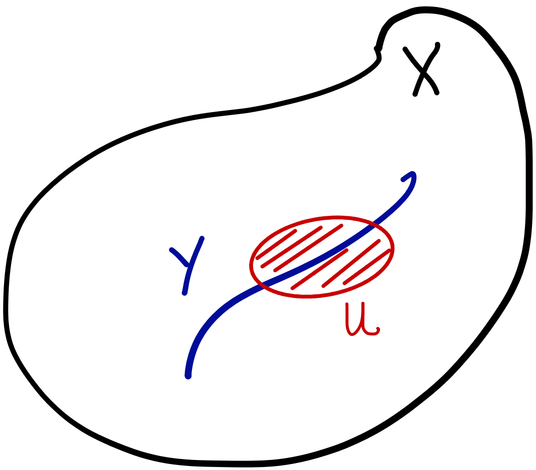
INTERSECTIONS

Classical Objects: Divisors



- X smooth variety over k (field)
- $Y \subseteq X$ has codimension 1

CLASSICAL OBJECTS : DIVISORS



- X smooth variety over k (field)
- $Y \subseteq X$ has codimension 1

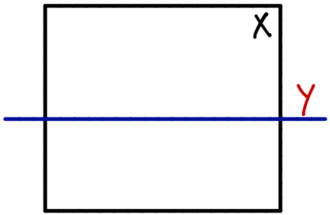
  locally: one equation $f = 0$, $f \in \mathcal{O}_X(U)$
- Y determines an Algebraic/Geometric object: $\mathcal{L}_Y \longrightarrow X$
- This correspondence is reversible:

  $$\mathcal{L} \longrightarrow X \rightsquigarrow Y$$

EXAMPLE: $\mathbb{A}^2_k$ (x,y PLANE)

$X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ Polynomials (x,y)

- $Y = \operatorname{Spec} \dfrac{k[x,y]}{(y)} \cong \operatorname{Spec} k[x] = \mathbb{A}^1_k$



$y = 0$

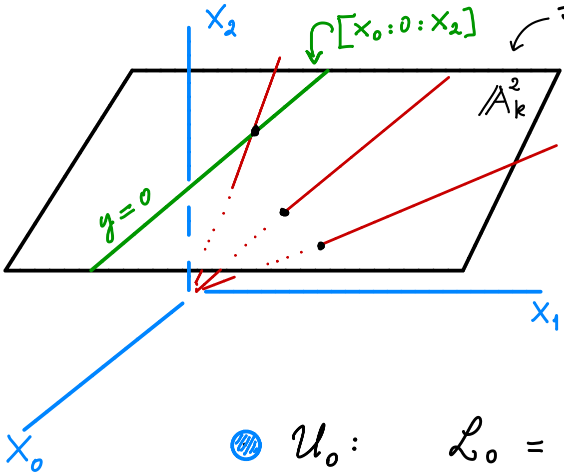
- $\dfrac{1}{y} \in k(x,y) = \left\{ \dfrac{f}{g} \;\middle|\; f, g \in k[x,y] \right\}$

Rational functions (= Meromorphic)

- Define $\mathcal{L} = k[x,y]\langle \frac{1}{y} \rangle = \left\{ f \, \frac{1}{y} \;\middle|\; f \in k[x,y] \right\}$

MODULE GENERATED BY $\dfrac{1}{y}$

EXAMPLE: $\mathbb{P}^2_k$ (PROJECTIVE SPACE)



PLANE $\sim X_2 = 1$: $[X_0 : X_1 : 1] = [x : y : 1]$

$$[X_0 : X_1 : X_2] = \{\lambda(X_0, X_1, X_2) \mid \lambda \in k^*\} \in \mathbb{P}^2_k$$

- $\mathbb{P}^2_k = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2$, $\mathcal{U}_i = \{X_i \neq 0\}$

- $\mathcal{U}_0$: $\mathcal{L}_0 = k[x, y]\langle \frac{1}{y} \rangle$

- $\mathcal{U}_1$: $\mathcal{L}_1 = k[x', z']\langle 1 \rangle$

- $\mathcal{U}_2$: $\mathcal{L}_2 = k[y'', z'']\langle \frac{1}{y''} \rangle$

$\frac{1}{y} = z' \, 1$ $\qquad \mathcal{L}_0 \xrightarrow{\cong} \mathcal{L}_1$

$1 = y'' \, \frac{1}{y''}$ $\qquad \mathcal{L}_1 \xrightarrow{\cong} \mathcal{L}_2$

$\frac{1}{y''} = x \, \frac{1}{y}$ $\qquad \mathcal{L}_2 \xrightarrow{\cong} \mathcal{L}_0$

More Precisely ...

More Precisely...

Cartier

$$H^0(X, \mathcal{K}^*/\mathcal{O}^*) \cong H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$$

Equations

$f \sim f'$ iff $f' = u f$

invertible

Line Bundles

$\mathcal{L} \sim \mathcal{L}'$

Weil

$$CH^1(X)$$

Codimension $1 \subseteq X$

$Y \underset{\text{rat.}}{\simeq} Y'$

More Precisely...

Cartier **Weil**

$$H^0(X, \mathcal{K}^*/\mathcal{O}^*) \cong H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X) \underset{X \text{ "NICE"}}{\cong} CH^1(X)$$

Equations

$f \sim f'$ iff $f' = u f$

invertible

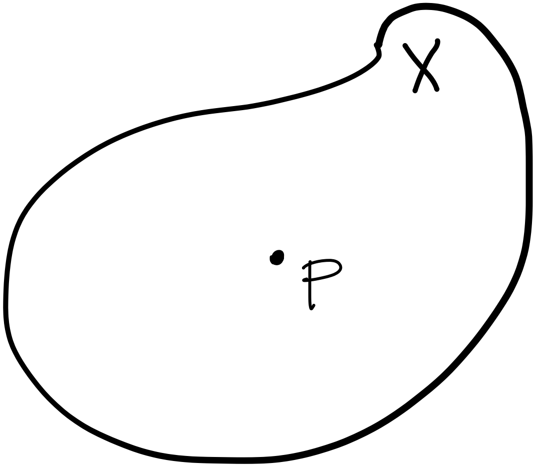
Line Bundles

$\mathcal{L} \sim \mathcal{L}'$

Codimension 1 $\subseteq X$

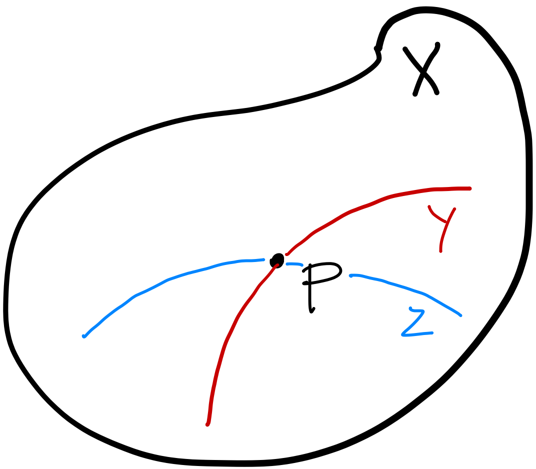
$Y \underset{\text{rat.}}{\simeq} Y'$

CODIMENSION 2 : NON CLASSICAL



- X smooth variety over k (field)
- $P \subseteq X$ has codimension 2
- $P \underset{\text{rat}}{\simeq} P'$: $CH^2(X)$ Codimension 2 Cycles

Codimension 2 : Non Classical



- X smooth variety over k (field)
- $P \subseteq X$ has codimension 2
- $P \underset{\text{rat}}{\simeq} P'$: $CH^2(X)$ Codimension 2 cycles
- Locally, 2 equations NOT NECESSARILY
- Pairing : $CH^1(X) \times CH^1(X) \longrightarrow CH^2(X)$

  $([Y], [Z]) \longmapsto [P]$
- $P \rightsquigarrow$?

WEIL VS. CARTIER

CARTIER

$$H^0(X, \mathcal{K}^*/\mathcal{O}^*) \cong H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$$

Equations

$f \sim f'$

Line Bundles

$\mathcal{L} \sim \mathcal{L}'$

$\cong$

X "NICE"

WEIL

$$CH^1(X)$$

Codimension 1 $\subseteq X$

$Y \underset{rat.}{\simeq} Y'$

$$CH^2(X)$$

Codimension 2 $\subseteq X$

$P \underset{rat}{\simeq} P'$

$$CH^1(X) \times CH^1(X) \longrightarrow CH^2(X)$$

Weil vs. Cartier

Cartier		Weil
$H^0(X, \mathcal{K}^*/\mathcal{O}^*) \cong H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$	$\cong$ X "NICE"	$CH^1(X)$
Equations $f \sim f'$ Line Bundles $\mathcal{L} \sim \mathcal{L}'$		Codimension 1 $\subseteq X$ $Y \underset{\text{rat.}}{\simeq} Y'$
?		$CH^2(X)$ Codimension 2 $\subseteq X$ $P \underset{\text{rat}}{\simeq} P'$
?		$CH^1(X) \times CH^1(X) \longrightarrow CH^2(X)$

Homotopy Types

What is a Homotopy Type?

Fix a nice category of "Spaces"

Definition Let $n \in \mathbb{N}$, $n \geq 1$.

- X is, or more precisely, represents, an n-Type, if

$$\pi_i(X) = 0, \quad i > n \quad \& \quad i = 0.$$

- $X \sim X'$: $X \simeq_{w.e.} X'$ Same homotopy n-Type

Remark $i = 0$: connected

Remark "Spaces" : Top, sSet, s(Pre)Sheaves, Grothendieck Topo$\left\{\begin{matrix} i \\ ses \end{matrix}\right.$, ...

EXAMPLE: EILENBERG-MAC LANE $K(G,1)$

- G : group (in one of the chosen categories)

- $K(G,1)$ space s.t. $\pi_i(K(G,1)) = \begin{cases} G & i=1 \\ 0 & i\neq 1 \end{cases}$ $\quad$ $K(G,1)$ is a 1-type

- Classifying Property : X space

$$\pi_0 \mathrm{Princ}_G(X) = \left\{ \begin{array}{c} P \\ \downarrow \\ X \end{array} \;\middle|\; \begin{array}{l} \text{principal} \\ \text{homogeneous} \\ \text{fibration} \end{array} \right\} \Big/ \simeq \;\; \cong [X, K(G,1)]$$

$G \times P \longrightarrow P$

$P/G \cong X$

homotopy classes

- Simplicial Model :

$$\mathrm{pt} \leftleftarrows G \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; G \times G \;\overset{\longleftarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longleftarrow}}}\; G \times G \times G \;\; \vdots$$

Example: Eilenberg - MacLane $K(G,1)$

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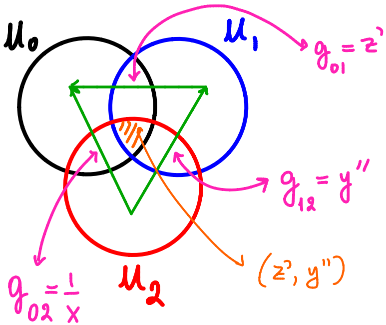
$$(g_1, g_2) \begin{array}{l} \longmapsto g_2 \\ \longmapsto g_1 g_2 \\ \longmapsto g_1 \end{array}$$

EXAMPLE : $Y \subseteq X = \mathbb{P}^2_k$

- Y = line $x_1 = 0 \subseteq \mathbb{P}^2_k$: we have constructed $\mathcal{L}_Y \longrightarrow \mathbb{P}^2_k$, here $G = \mathcal{O}_X^*$

- Classifying map $\mathbb{P}^2_k \longrightarrow K(\mathcal{O}_X^*, 1)$



SAME AS

$$\begin{tikzcd}
\mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \mathcal{U}_2 \arrow[r] & pt \\
\mathcal{U}_{01} \sqcup \mathcal{U}_{02} \sqcup \mathcal{U}_{12} \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & \mathcal{O}_X^* \arrow[u, shift left] \arrow[u, shift right] \\
\mathcal{U}_{012} \arrow[r] \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & \mathcal{O}_X^* \times \mathcal{O}_X^* \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2]
\end{tikzcd}$$

- Simplicial Model :

$$pt \leftleftarrows G \Lleftarrow G \times G \Lleftarrow G \times G \times G \; \cdots$$

$$(g_1, g_2) \longmapsto g_2, \quad (g_1, g_2) \longmapsto g_1 g_2, \quad (g_1, g_2) \longmapsto g_1$$

Example : $Y \subseteq X = \mathbb{P}^2_k$

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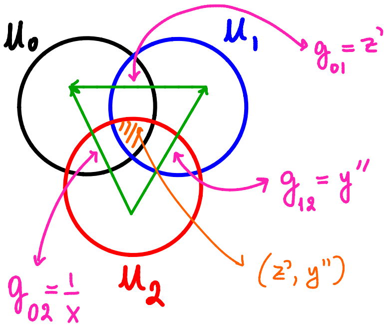

SAME AS

$$\begin{tikzcd}
& \mathbb{P}^2_k \\
& \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \mathcal{U}_2 \arrow[u, "s"] \arrow[r] & pt \\
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\end{tikzcd}$$

- Simplicial Model :

$$pt \leftleftarrows G \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; G \times G \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}\; G \times G \times G \;\substack{\longleftarrow \\ \vdots \\ \longleftarrow}$$

$$(g_1, g_2) \;\begin{matrix} \longmapsto & g_2 \\ \longmapsto & g_1 g_2 \\ \longmapsto & g_1 \end{matrix}$$

SOME RESULTS

ON

2-Types

(E.A.)

In General...

- $\mathrm{pt} \leftleftarrows G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; G\times G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; G\times G\times G \;\substack{\longleftarrow\\ \vdots\\ \longleftarrow}$ Is a prototype of a simplicial object

In particular

- $G_0 \leftleftarrows G_1 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; G_2 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; G_3 \;\substack{\longleftarrow\\ \vdots\\ \longleftarrow} \;=\; G_\bullet : \Delta^{op} \longrightarrow \mathrm{Grps}$ Simplicial Group

In General...

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In particular

- $G_0 \leftleftarrows G_1 \Lleftarrow G_2 \,\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\, G_3 \,\substack{\leftarrow\\\vdots\\\leftarrow} = G_\bullet : \Delta^{op} \longrightarrow \mathrm{Grps}$ Simplicial Group

- Moore Complex: $N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \cdots \qquad N_j = \bigcap_{i=0}^{j-1} \ker(d_i : G_j \to G_{j-1})$

- Suspension: $G_\bullet \rightsquigarrow BG_\bullet : pt \leftleftarrows G_0\times G_1 \Lleftarrow G_0\times G_1\times G_2 \,\substack{\leftarrow\\\vdots\\\leftarrow}$

In General...

- $pt \leftleftarrows G \overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}} G \times G \; \overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}} \; G \times G \times G \; \vdots$ Is a prototype of a simplicial object

In particular

- $G_0 \leftleftarrows G_1 \Lleftarrow G_2 \;\overset{\longleftarrow}{\underset{\longleftarrow}{\Lleftarrow}}\; G_3 \; \vdots \; = G_\bullet : \Delta^{op} \longrightarrow \mathrm{Grps}$ Simplicial Group
- Moore Complex: $N_0 \leftarrow N_1 \leftarrow N_2 \leftarrow \cdots \qquad N_j = \bigcap_{i=0}^{j-1} \ker(d_i : G_j \to G_{j-1})$
- Suspension: $G_\bullet \rightsquigarrow BG_\bullet : pt \leftleftarrows G_0 \times G_1 \Lleftarrow G_0 \times G_1 \times G_2 \; \vdots$
- DEFINITION $\quad \pi_k(G_\bullet) = H_k(N_\bullet)$

(Kan, Moore, ...)

THEOREM $\quad \pi_k(G_\bullet) = \pi_{k+1}(BG_\bullet)$

COROLLARY $\quad X = BG_\bullet$ is an n-type if $N_\bullet(G)$ is supported on $[0, n-1]$

Morphisms of 2-Types

- A 2-type corresponds to a (Moore) complex $C_1 \xrightarrow{\partial} C_0$ of group objects

- X, Y : 2-types $\rightsquigarrow$ $C_1 \xrightarrow{\partial} C_0$ and $D_1 \xrightarrow{\partial} D_0$ (Moore Complexes)

▷ A butterfly from $D_\bullet$ to $C_\bullet$ is a diagram of group objects

$$\begin{tikzcd}
D_1 \arrow[dd] \arrow[dr] & & C_1 \arrow[dd] \arrow[dl] \\
& E \arrow[dl] \arrow[dr] & \\
D_0 & & C_0
\end{tikzcd}$$

▷ A morphism of butterflies is an isomorphism $E \xrightarrow{\cong} E'$

$$\begin{tikzcd}
D_1 \arrow[r] \arrow[dd] \arrow[dr] & E' \arrow[ddl, bend right] \arrow[ddr, bend left] & C_1 \arrow[l] \arrow[dd] \arrow[dl] \\
& E \arrow[u] \arrow[dl] \arrow[dr] & \\
D_0 & & C_0
\end{tikzcd}$$

(With Behrang Noohi)

Morphisms of 2-Types

Theorem (E. A., Behrang Noohi)

$$X \rightsquigarrow C_\bullet : C_1 \xrightarrow{\partial} C_0 \quad \text{and} \quad Y \rightsquigarrow D_\bullet : D_1 \xrightarrow{\partial} D_0$$

$\mathbb{R}\mathrm{Hom}(D_\bullet, C_\bullet)$ — Category of
$$\begin{tikzcd} D_1 \arrow[d] \arrow[dr] & & C_1 \arrow[dl] \arrow[d] \\ D_0 & E \arrow[l] \arrow[r] & C_0 \end{tikzcd}$$

There is an equivalence:

$$\mathrm{Hom}_{2\text{-Types}}(Y, X) \simeq \mathbb{R}\mathrm{Hom}(D_\bullet, C_\bullet)$$

Remark To work out $X \rightsquigarrow C_1 \xrightarrow{\partial} C_0$ we had to reprove Kan's theory in a topos of sheaves

ALGEBRAIC STRUCTURES (UP TO HOMOTOPY)

- X : 2-type , $C_\bullet : C_1 \xrightarrow{\partial} C_0$

  Reconstruct the simplicial group : $C_0 \leftleftarrows C_0 \times C_1 \Lleftarrow C_0 \times C_1 \times C_1 \cdots$

  $\cong$

  Nerve ($\mathcal{C}$) $\quad$ $\mathcal{C}$: Category (Groupoid)

  Objects $= C_0$

  Morphisms $= C_0 \times C_1$

- Better : $\mathcal{C}$ is the stack quotient of $\underbrace{C_0 \times C_1 \longrightarrow C_0}_{\text{action via } \partial}$

- GROUP LAW ON $\mathcal{C}$ $\quad$ $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

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  LOCALLY

ALGEBRAIC STRUCTURES (UP TO HOMOTOPY)

- X : 2-type , $C_\bullet : C_1 \xrightarrow{\partial} C_0$

  Reconstruct the simplicial group : $C_0 \leftleftarrows C_0 \times C_1 \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; C_0 \times C_1 \times C_1 \;\substack{\leftarrow\\ \vdots\\ \leftarrow}$

  $\cong$

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  Objects $= C_0$

  Morphisms $= C_0 \times C_1$

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- GROUP LAW ON $\mathcal{C}$ $\mathcal{C} \times \mathcal{C} \xrightarrow{m} \mathcal{C}$

  $\boxed{\text{up to homotopy}}$

  E.g.

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] \arrow[dr, phantom, "\overset{\simeq}{\mu}"] & \mathcal{C}\times\mathcal{C} \arrow[d, "m"] \\
\mathcal{C}\times\mathcal{C} \arrow[r, "m"'] & \mathcal{C}
\end{tikzcd}$$

Algebraic Structures (up to homotopy)

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] \arrow[dr, phantom, "\overset{\simeq}{\mu}"] & \mathcal{C}\times\mathcal{C} \arrow[d, "m"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "m"'] & \mathcal{C} \end{tikzcd}$$

x, y, z : Objects of $\mathcal{C}$

$$m(m(x,y),z) \xrightarrow[\sim]{\mu} m(x, m(y,z))$$

Algebraic Structures (up to homotopy)

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] \arrow[dr, phantom, "\overset{\simeq}{\mu}"] & \mathcal{C}\times\mathcal{C} \arrow[d, "m"] \\
\mathcal{C}\times\mathcal{C} \arrow[r, "m"'] & \mathcal{C}
\end{tikzcd}$$

$x \to x'$, $y \to y'$, $z \to z'$: morphisms

$$\begin{tikzcd}
m(m(x,y),z) \arrow[r, "\mu", "\sim"'] \arrow[d] & m(x,m(y,z)) \arrow[d] \\
m(m(x',y'),z') \arrow[r, "\mu", "\sim"'] & m(x',m(y',z'))
\end{tikzcd}$$

Include μ into the axioms E.g.

$$\begin{tikzcd}
 & (x(yz))w \arrow[r] & x((yz)w) \arrow[dr] & \\
((xy)z)w \arrow[ur] \arrow[drr] & & & x(y(zw)) \\
 & & (xy)(zw) \arrow[ur] &
\end{tikzcd}$$

NON ABELIAN COHOMOLOGY

$$H^i(T, \mathscr{C}) =_{\text{def}} \begin{cases} \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Spaces})}(T, BG_\bullet) & i = 1 \\ \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Spaces})}(T, G_\bullet) & i = 0 \\ \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Spaces})}(T, \Omega^{-i} G_\bullet) & i < 0 \end{cases}$$

THEOREM (E. A., BEHRANG NOOHI)

- $H^1(T, \mathscr{C}) \cong \pi_0(\mathrm{Prim}_{\mathscr{C}}(T)) \cong \left\{ \begin{array}{c} \mathscr{X} \\ \downarrow \\ T \end{array} \middle| \ \mathscr{C} \times \mathscr{X} \to \mathscr{X} \right\} \Big/ \simeq$

- $\begin{tikzcd} D_1 \arrow[d] \arrow[dr] & & C_1 \arrow[d] \arrow[dl] \\ D_0 & E \arrow[l] \arrow[r] & C_0 \end{tikzcd} \rightsquigarrow \mathscr{D} \to \mathscr{C} \rightsquigarrow H^1(-, \mathscr{D}) \longrightarrow H^1(-, \mathscr{C})$

  EXPLICIT COMPUTATION

APPLICATION (LATER)

$$C_\bullet : C_1 \xrightarrow{\partial} C_0 = A \longrightarrow 0 \ , \ A \text{ abelian}$$

- $H^1(-,\mathcal{C}) = \mathrm{Hom}_{\mathrm{Ho(Spaces)}}(-, BK(A,1)) = \mathrm{Hom}_{\mathrm{Ho(Spaces)}}(-, K(A,2)) = H^2(-,A)$

- $\mathrm{Princ}_{\mathcal{C}}(T) \cong \mathrm{GERBES}(T, A)$

Detour:

Connective Spectra

&

Categorical Rings

DETOUR :

RING

CONNECTIVE SPECTRA

&

CATEGORICAL RINGS

WHAT ARE THEY?

- $E_\bullet : E_0 \leftleftarrows E_1 \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} E_2 \cdots \longleftrightarrow N_\bullet(E) : N_0 \longleftarrow N_1$

  Simplicial group Object — Moore Complex

  $\mathcal{E}$ Stack quotient N_0/N_1

- $\mathcal{E}$ is a Ring up to homotopy: $+ : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$, $\times : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$

  - $(x+y)+z \xrightarrow{\sim} x+(y+z)$, $(xy)z \xrightarrow{\sim} x(yz)$ — More...
  - $x(y+z) \xrightarrow{\sim} xy + xz$, $(x+y)z \xrightarrow{\sim} xz + yz$
  - Stability: $\begin{tikzcd} x+y \arrow[r, "c_{x,y}"] \arrow[rr, bend right, "id"'] & y+x \arrow[r, "c_{y,x}"] & x+y \end{tikzcd}$ $\Longrightarrow$ Spectrum Condition

    Braiding

WHAT ARE THEY?

- $E_\bullet : E_0 \leftleftarrows E_1 \leftleftarrows E_2 \cdots$ $\longleftrightarrow$ $N_\bullet(E) : N_0 \longleftarrow N_1$

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  $\mathcal{E}$ Stack quotient N_0/N_1

- $\mathcal{E}$ is a Ring up to homotopy: $+ : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$, $\times : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$
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  - Stability: $x+y \xrightarrow{c_{x,y}} y+x \xrightarrow{c_{y,x}} x+y$, $\mathrm{id}: x+y \to x+y$ $\Longrightarrow$ Spectrum Condition

- K-Theory, TMF (topological Modular Forms), ...

WHAT ARE THEY?

- $E_\bullet : \; E_0 \leftleftarrows E_1 \leftleftarrows E_2 \; \cdots$ $\longleftrightarrow$ $N_\bullet(E) : \; N_0 \xleftarrow{\partial} N_1$

  Simplicial group Object — Moore Complex

  $\mathcal{E}$ Stack quotient N_0/N_1

- $\mathcal{E}$ is a Ring up to homotopy: $+ : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$, $\times : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$

  - $(x+y)+z \xrightarrow{\sim} x+(y+z)$, $(xy)z \xrightarrow{\sim} x(yz)$ — More...
  - $x(y+z) \xrightarrow{\sim} xy + xz$, $(x+y)z \xrightarrow{\sim} xz + yz$
  - Stability:

$$\begin{tikzcd} x+y \arrow[r, "c_{x,y}"] \arrow[rr, bend right, "id"'] & y+x \arrow[r, "c_{y,x}"] & x+y \end{tikzcd}$$

- $$\boxed{\begin{array}{ll} \pi_0(\mathcal{E}) \equiv \pi_0(E_\bullet) \equiv H_0(N_\bullet(E)) \equiv \operatorname{Coker} \partial & \text{RING} \\ \pi_1(\mathcal{E}) \equiv \pi_1(E_\bullet) \equiv H_1(N_\bullet(E)) \equiv \ker \partial & \pi_0\text{-BIMODULE} \end{array}}$$

Some Results about Cat-Rings

Theory Appl. Cat. 30 (2015)

E.A. (ArXiv.org: 1501.07592)

- **Strict Picard Condition** $\quad c_{x,x} : x + x \xrightarrow{\cong} x + x \; = \; \text{identity}$

SOME RESULTS ABOUT CAT-RINGS

Theory Appl. Cat. 30 (2015)

E.A.

(ArXiv.org: 1501.07592)

- STRICT PICARD CONDITION $\quad c_{x,x} : x+x \xrightarrow{\cong} x+x \;=\; \text{identity}$

- **THEOREM** $\quad \mathcal{E}$ cat ring with Moore complex $N_1 \xrightarrow{\partial} N_0$

  Then we have a crossed extension of algebras

$$0 \longrightarrow \pi_1(\mathcal{E}) \longrightarrow N_1 \xrightarrow{\partial} N_0 \longrightarrow \pi_0(\mathcal{E}) \longrightarrow 0$$

- $N_0 \longrightarrow \pi_0(\mathcal{E})$ Ring homomorphism

  N_1 : N_0 - bimodule , $\partial : N_1 \longrightarrow N_0$ bimodule homomorphism

  $\pi_1(\mathcal{E})$: $\pi_0(\mathcal{E})$ - bimodule

Some Results about Cat-Rings

Theory Appl. Cat. 30 (2015)

(ArXiv.org: 1501.07592)

E. A.

- Strict Picard Condition $\quad c_{x,x} : x+x \xrightarrow{\cong} x+x \; = \; \text{identity}$

- Computing Morphisms $\mathcal{E} \longrightarrow \mathcal{F}$:

Theorem Let $\mathcal{E}, \mathcal{F}$ be as above. There exist equivalences

$$\mathrm{Hom}_{\mathrm{Cat\,Rings}}(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Hom}_{Sp}(BE_\bullet, BF_\bullet) \simeq \mathbb{R}\mathrm{Hom}(N_\bullet(E), N_\bullet(F))$$

$$\begin{tikzcd}
N_1(E) \arrow[d] \arrow[dr] & & N_1(F) \arrow[dl] \arrow[d] \\
 & S \arrow[dl] \arrow[dr] & \\
N_0(E) & & N_0(F)
\end{tikzcd}$$

Algebra Extension

Some Results about Cat-Rings

Theory Appl. Cat. 30 (2015)

E.A.

(ArXiv.org: 1501.07592)

- Strict Picard Condition $\quad c_{x,x} : x+x \xrightarrow{\cong} x+x \;=\; \text{identity}$

- Postnikov Invariant$(\mathcal{E})$ = Char Class$(0 \to \pi_1 \to N_1 \to N_0 \to \pi_0)$

  Theorem $\quad k(\mathcal{E}) \in \mathbb{H}^3(\pi_0(\mathcal{E}), \pi_1(\mathcal{E}))$ André-Quillen cohomology of rings

- Remark $\quad$ My calculation holds in a general Topos

Some Results about Cat-Rings

E.A. ArXiv.org: 1501.04664

- MetaTheorem :

You Can Drop

- Strict Picard Condition $c_{x,x} : x + x \xrightarrow{\cong} x + x = \text{identity}$

Some Results about Cat-Rings

E.A. ArXiv.org: 1501.04664

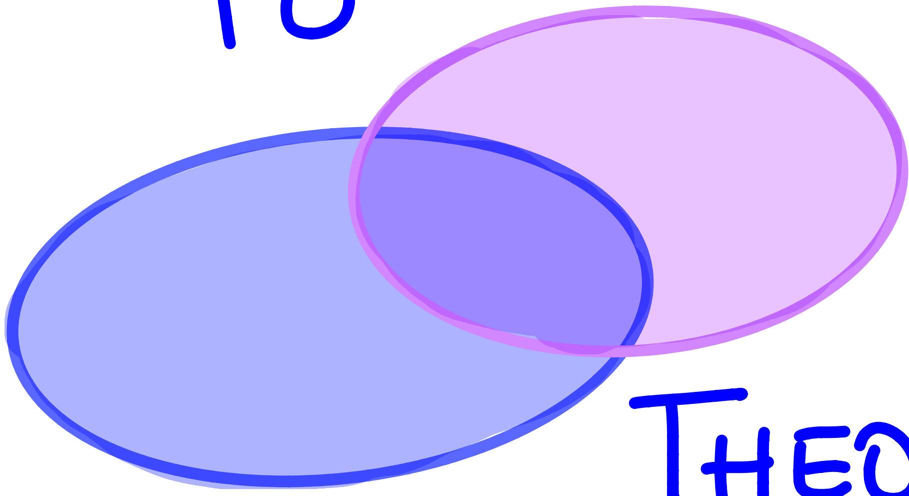
- MetaTheorem :

You Can Drop

- Strict Picard Condition $c_{x,x} : x+x \xrightarrow{\cong} x+x$ = identity

But it's hard!

Back To Theory



ArXiv.org: 1510.01825

WITH

NIRANJAN RAMACHANDRAN

(UMD)

RECALL :

CARTIER	WEIL
$H^0(X, \mathcal{K}^*/\mathcal{O}^*) \cong H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$ $\underset{X \text{ "NICE"}}{\cong}$	$CH^1(X)$
Equations $f \sim f'$ Line Bundles $\mathcal{L} \sim \mathcal{L}'$	Codimension 1 $\subseteq X$ $Y \underset{rat.}{\simeq} Y'$
?	$CH^2(X)$ Codimension 2 $\subseteq X$ $P \underset{\text{rat}}{\simeq} P'$
?	$CH^1(X) \times CH^1(X) \longrightarrow CH^2(X)$

RECALL :

CARTIER	WEIL
$H^0(X, \mathcal{K}^*/\mathcal{O}^*) \cong H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$ $\underset{X \text{ "NICE"}}{\cong}$	$CH^1(X)$
Equations: $f \sim f'$ Line Bundles: $\mathcal{L} \sim \mathcal{L}'$	Codimension 1 $\subseteq X$: $Y \underset{\text{rat.}}{\simeq} Y'$
✓	$CH^2(X)$ Codimension 2 $\subseteq X$: $P \underset{\text{rat}}{\simeq} P'$
?	$CH^1(X) \times CH^1(X) \longrightarrow CH^2(X)$

Bloch-Quillen Formula $CH^2(X) \cong H^2(X, \mathcal{K}_{2,X})$

Gersten Resolution

$$0 \to \mathcal{K}_{2,X} \to K_2(k(X)) \to \bigoplus_{C \subseteq X} \underset{\substack{\cong \\ k(C)^*}}{K_1(k(C))} \to \bigoplus_{p \in X} \underset{\cong\, \mathbb{Z}}{K_0(k(p))} \to 0$$

Bloch-Quillen Formula $CH^2(X) \cong H^2(X, \mathcal{K}_{2,X})$

Gersten Resolution

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_{2,X} \arrow[r] & K_2(k(X)) \arrow[r] & \bigoplus_{C \subseteq X} K_1(k(C)) \arrow[r] & \bigoplus_{p \in X} K_0(k(p)) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{K}_{2,X} \arrow[r] \arrow[u, equal] & K_2(k(X)) \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u, "p"] & 0
\end{tikzcd}$$

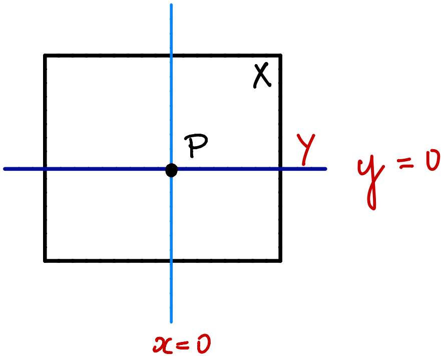
$\leftarrow$ Select a Codim 2 cycle

$\underbrace{K_2(k(X)) \longrightarrow M}_{\text{Moore complex of a } \mathcal{K}_{2,X}\text{-gerbe}}$

Objects (C,f) : $C \ni p$, $f \in k(C)$, $v_p(f) = 1$

Morphisms $(C,f) \longrightarrow (D,g)$: $u \in K_2(k(X))$ $T_C(u) = f$, $T_D(u) = g^{-1}$

Back to $\mathbb{A}^2_k$



Two objects:

$$(C, f) = ((y), x)$$

$$(D, g) = ((x), y)$$

$$k(C) = k\left(\frac{k[x,y]}{(y)}\right) \cong k(x) \ni x = f$$

$$k(D) = k\left(\frac{k[x,y]}{(x)}\right) \cong k(y) \ni y = g$$

Morphism $((y), x) \longrightarrow ((x), y)$

$$\{x, y\} \in K_2(k(x,y))$$

Tame Symbol

DICTIONARY

CARTIER		WEIL
$H^0(X, \mathcal{K}^*/\mathcal{O}^*) \cong H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$ Equations $f \sim f'$ Line Bundles $\mathcal{L} \sim \mathcal{L}'$	$\cong$ X "NICE"	$CH^1(X)$ Codimension 1 $\subseteq X$ $Y \underset{\text{rat.}}{\simeq} Y'$
$H^1(X, K_1(k(X))/\mathcal{K}_{2,X}) \cong H^2(X, \mathcal{K}_{2,X})$ Codim. 2 Cartier $\mathcal{K}_{2,X}$-GERBES		$CH^2(X)$ Codimension 2 $\subseteq X$ $P \underset{\text{rat}}{\simeq} P'$
?		$CH^1(X) \times CH^1(X) \longrightarrow CH^2(X)$

DICTIONARY

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$H^0(X, \mathcal{K}^*/\mathcal{O}^*) \cong H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$ Equations $\quad$ Line Bundles $f \sim f'$ $\quad$ $\mathcal{L} \sim \mathcal{L}'$	$\cong$ X "NICE"	$CH^1(X)$ Codimension 1 $\subseteq X$ $Y \underset{rat.}{\simeq} Y'$
$H^1(X, K_1(k(X))/\mathcal{K}_{2,X}) \cong H^2(X, \mathcal{K}_{2,X})$ Codim. 2 Cartier $\quad$ $\mathcal{K}_{2,X}$-GERBES		$CH^2(X)$ Codimension 2 $\subseteq X$ $P \underset{rat}{\simeq} P'$
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DICTIONARY

CARTIER		WEIL
$H^0(X, \mathcal{K}^*/\mathcal{O}^*) \cong H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$ Equations $f \sim f'$ Line Bundles $\mathcal{L} \sim \mathcal{L}'$	$\cong$ X "NICE"	$CH^1(X)$ Codimension 1 $\subseteq X$ $Y \underset{\text{rat.}}{\simeq} Y'$
$H^1(X, K_1(k(X))/\mathcal{K}_{2,X}) \cong H^2(X, \mathcal{K}_{2,X})$ Codim. 2 Cartier $\mathcal{K}_{2,X}$-GERBES		$CH^2(X)$ Codimension 2 $\subseteq X$ $P \underset{\text{rat}}{\simeq} P'$
$\mathrm{TORS}(\mathcal{O}_X^*) \times \mathrm{TORS}(\mathcal{O}_X^*) \longrightarrow \mathrm{GERBES}(X, \mathcal{K}_{2,X})$		$CH^1(X) \times CH^1(X) \longrightarrow CH^2(X)$

Animating a Cup Product

$$\begin{array}{ccc} \mathrm{Tors}(\mathcal{O}_X^*) \times \mathrm{Tors}(\mathcal{O}_X^*) & \longrightarrow & \mathrm{Gerbes}(X, \mathcal{K}_{2,X}) \\ \downarrow & & \downarrow \\ CH^1(X) \times CH^1(X) & \longrightarrow & CH^2(X) \end{array}$$

is based on the following

Theorem (E.A., N. Ramachandran, '15)

In any topos there exists a canonical central extension of groups

$$0 \longrightarrow A \underset{\mathbb{Z}}{\otimes} B \longrightarrow \mathcal{H}_{A,B} \longrightarrow A \times B \longrightarrow 0$$

inducing the morphism of types

$$K(A \times B, 1) \simeq K(A,1) \times K(B,1) \longrightarrow K(A \underset{\mathbb{Z}}{\otimes} B, 2)$$

Then apply to $A = B = \mathcal{O}_X^* = \mathcal{K}_{1,X}$, follow by $\mathcal{K}_{1,X} \underset{\mathbb{Z}}{\otimes} \mathcal{K}_{1,X} \longrightarrow \mathcal{K}_{2,X}$

THANK YOU!