Groups, Rings and Vector Spaces I

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Proposed Problems — Cats

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1. Let $\mathcal{C} = \mathcal{S}et$. Identify the categories $\mathcal{C}_{/\{*\}}$ and ${}^{\varnothing}/\mathcal{C}$.

2. Still let $\mathcal{C} = \mathcal{S}et$. Identify the category $\mathcal{C}_{/\varnothing}$

3. Let $\mathcal{C}$ be any category. Fix an object S of $\mathcal{C}$ and consider the slice (=comma) category $\mathcal{C}_{/S}$. Now, let $f:\ T \to S$ be an object of $\mathcal{C}_{/S}$. Identify $\left(\mathcal{C}_{/S}\right)_{/f}$.

4. Let $\mathcal{C}$ be a category in which products and coproducts exist, and assume that I is an initial object and T is a terminal one. Prove that, for any object X of $\mathcal{C}$, there are isomorphisms

$$T \times X \cong X \cong X \times T, \quad I \sqcup X \cong X \cong X \sqcup I.$$

5. It is not going to be possible to formulate a general result such as that of the preceding problem if we switch the rôles of I and T. *However,* do the following: if $\mathcal{C} = \mathcal{S}et$, and S is any set, what are the results of

$$\varnothing \times S, \quad \{*\} \sqcup S\,?$$

6. Consider the category $\mathcal{S}et_*$ of pointed sets, and use the short-hand notation (T,t) to denote a pointed set $t:\ \{*\} \to T$. If S is any set, and (T,t) any pointed one, prove that there is an isomorphism

$$\mathrm{Hom}_{\mathcal{S}et_*}(\{*\} \sqcup S, (T,t)) \cong \mathrm{Hom}_{\mathcal{S}et}(S,T),$$

where in the right-hand side T simply denotes the underlying set of (T,t) (i.e. you forget the distinguished point).

7. Let $\mathcal{G}$ be a groupoid, assumed to be small to avoid set-theoretic complications. Define a relation on the objects of $\mathcal{G}$ as follows: *X is equivalent to Y, written $X \sim Y$, if there exists an arrow $X \to Y$.* Prove $\sim$ is an equivalence relation.
(Ob$(\mathcal{G})/\sim$ is typically denoted by $\pi_0(\mathcal{G})$.)

8. Let $f : T \to S$ be a surjective function. Define the following *category $\mathcal{G}$:* the objects are the points of T, i.e. Ob$(\mathcal{G}) = T$; the morphisms are the pairs $(t_1, t_2) \in T \times T$ such that $f(t_1) = f(t_2)$.

   (a) Verify that $\mathcal{G}$ is a category and prove that it is in fact a groupoid;

   (b) Prove that $\pi_0(\mathcal{G}) \cong S$;

   (c) Prove that any equivalence relation arises in this way.

   (d) More generally, drop that the assumption that f be surjective; prove that in this case $\pi_0(\mathcal{G})$ is isomorphic to the image of f.

9. Using the notions expounded in Problems 7 and 8, can you reconstruct a groupoid $\mathcal{G}$ from the knowledge of $\pi_0(\mathcal{G})$? Why or why not?

10. *The skeleton of a category.* Let $\mathcal{C}$ be a category, and define a new category Sk $\mathcal{C}$ in the following way. Select one object for each *isomorphism class* of objects of $\mathcal{C}$, where, again, X and Y are in the same class if there exists an isomorphism $X \to Y$. If U, V are two such objects, define
$$\mathrm{Hom}_{\mathrm{Sk}\,\mathcal{C}}(U,V) = \mathrm{Hom}_{\mathcal{C}}(U,V).$$
Note that it is possible that $V = U$. What happens then?

   (a) Verify that Sk $\mathcal{C}$ is well defined.

   (b) Let k be your favorite field, and let $\mathcal{C} = \mathcal{V}ect_k$ be the category of vector spaces over k. What is Sk$(\mathcal{V}ect_k)$? (Hint: This category appears in one of the problems in the textbook...)

   (c) Let $\mathcal{F}in$ the category of *finite* sets, and $\mathcal{F}in_*$ that of finite, *pointed* sets. Give descriptions of Sk$(\mathcal{F}in)$ and Sk$(\mathcal{F}in_*)$.