Groups, Rings and Vector Spaces I

Proposed Problems — Groups

October 17, 2018

1. Construct a homomorphism $C_2 * C_n \to D_{2n}$. Optionally, if you dare, prove it is surjective.

2. Let G be a group. For any $g \in G$, prove that the map $\iota_g \colon G \to G$ defined by $\iota_g(h) = ghg^{-1}, \forall h \in G$, is an isomorphism.

3. For any group G, prove that the map

$$\begin{aligned} \iota \colon G &\longrightarrow \mathrm{Aut}(G) \\ g &\longmapsto \iota_g \end{aligned}$$

   is a homomorphism

4. For a group G, prove that G is abelian if and only if the binary operation in G, as a function

$$m \colon G \times G \longrightarrow G,$$

   is a group homomorphism.

5. For a group G, prove that the map $\sigma \colon G \to G$, $\sigma(g) = g^{-1}$, is a group homomorphism if and only if G is abelian.

6. Let C_k be the cyclic group of order k. Recall that if $k|n$ there is a canonical homomorphism $\pi^n_k \colon C_n \to C_k$. Prove that $(\pi^{mn}_m, \pi^{mn}_n) \colon C_{mn} \to C_m \times C_n$ is an isomorphism iff $\gcd(m, n) = 1$.

7. Let S and T be two sets with the same cardinality. Prove that the corresponding permutation groups $\mathrm{Aut}(S)$ and $\mathrm{Aut}(T)$ are isomorphic.

8. Prove that in the category $\mathcal{G}rp$ an epimorphism is necessarily surjective. Provide an example to show that in the category $\mathcal{G}rp$ an epimorphism does not necessarily have a right inverse.

9. Prove that D_{24} is not isomorphic with S_4.

10. There are two interesting, nonabelian, groups of order 8: D_8, and Q_8, the *Quaternion Group*. $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, plus cyclic permutation of these, and ± 1 behave in the expected way. Prove that D_8 is not isomorphic to Q_8.

11. Let G be a group and H, K two subgroups. Prove that if $G = H \cup K$, then either $H = G$, or $K = G$.

12. Let G be a group such that it only has two subgroups. Prove that G is of prime order.

13. Let F be a field and let $\mathrm{GL}_n(F)$ be the groups of $n \times n$ invertible matrices with entries in F. Let $\mathrm{SL}_n(F)$ be the subgroup of $\mathrm{GL}_n(F)$ of matrices whose determinant is equal to one. Denote by I the identity matrix and define:

$$\mathrm{PGL}_n(F) = \mathrm{GL}_n(F)/\{\lambda I | \lambda \in F^*\}.$$

   - Verify that $\mathrm{PGL}_n(F)$ is a group, in other words, verify that $\{\lambda I | \lambda \in F^*\}$ is normal;
   - Prove that there is an isomorphism $\mathrm{PGL}_2(\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C})$, but that $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PGL}_2(\mathbb{R})$ are *not* isomorphic to each other.
   - Can you explain this phenomenon in terms of F? (*Hint:* It helps thinking about these isomorphisms in terms of the first isomorphism theorem.)

14. Let $\mathbb{F}_2$ be the field with two elements (as an abelian group we call it $\mathbb{Z}/2\mathbb{Z}$) and consider the group $\mathrm{SL}_2(\mathbb{F}_2)$. Note that this is equal to the group $\mathrm{GL}_2(\mathbb{F}_2)$ (why?) Identify it (i.e. find an isomorphism) with a previously know group.

15. Let $\mathrm{SU}(1,1) = \left\{g \in M_2(\mathbb{C}) \mid \bar{g}^t \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\}$ (the group of 2×2 matrices with complex entries satisfying the stated condition). Define $\mathrm{PSU}(1,1)$ in the way analogous to the previous problem.

    Identify $\mathrm{PSU}(1,1)$ with a group you know from another course.

16. Let G be a group acting on a set S. Define a groupoid $\mathcal{G}$ by letting $\mathrm{Ob}(\mathcal{G}) = S$, and declaring that there is an arrow from s to t if and only if there exists $g \in G$ such that $t = gs$.

(a) Verify that $\mathcal{G}$ is indeed a groupoid

(b) What is $\pi_0(\mathcal{G})$?

(c) If $s \in S$, what is $\mathrm{Aut}_{\mathcal{G}}(s)$?

17. Let $G = \mathrm{GL}_2(\mathbb{F}_2)$, and consider $\widehat{\mathbb{F}_2} = \mathbb{F}_2 \cup \{\infty\}$. Let G act on $\widehat{\mathbb{F}_2}$ by fractional linear transformations (check that it does it make sense). Is this action faithful, transitive, free? Analyze the orbits. Identify the action (i.e. find an isomorphism) with that of a known group (that you should have identified from a previous problem)