Groups, Rings and Vector Spaces I

Proposed Problems — Rings and Modules

November 12, 2018

Characteristic.

1. Prove that the characteristic of an integral domain is a nonzero prime number or zero.

2. Prove that if \( \Lambda \) is a subring of a field \( F \), then \( \Lambda \) and \( F \) have the same characteristic.

3. Let \( p > 0 \) be a prime. Prove that if \( F \) is a field, and \( \text{char}(F) = p \), then there is a unique injective homomorphism

\[
\mathbb{F}_p \longrightarrow F,
\]

where \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \).

4. Use Problem 3 to show that if \( F \) is a finite field, say with \( q \) elements, then \( q \) is a power of an appropriate nonzero prime \( p \).

Around the Chinese Remainder Theorem.

5. Let \( R \) be a commutative ring, and \( I, J \) two nontrivial ideals. Prove that if \( I + J = R \), then \( IJ = I \cap J \).

6. Prove that if two nontrivial ideals \( I, J \) in a commutative ring \( R \) satisfy \( I + J = R \), then there is an isomorphism

\[
\frac{R}{IJ} \cong \frac{R}{I} \times \frac{R}{J}.
\]
7. It is not necessary that two ideals $I$ and $J$ be coprime in order that $IJ = I \cap J$. Indeed, prove that, if $k$ is a field, say $\text{char}(k) = 0$ for simplicity, then $(x)(y) = (x) \cap (y)$ in $k[x, y]$.

On the other hand the condition that $I + J = R$ is necessary for the theorem (as proved in 6). Indeed, show that the homomorphism

$$\frac{k[x, y]}{(xy)} \to \frac{k[x, y]}{(x)} \times \frac{k[x, y]}{(y)}$$

induced by projection from $k[x, y]$ is not surjective.

Misc. (ideal) stuff.

8. Let $k$ be a field, assumed of characteristic 0, for simplicity. Find an isomorphism

$$\frac{k[x, y]}{(x^2 + y^2 - 1, y^2 + y - 1)} \cong \frac{k[x]}{(x^4 + x^2 - 1)}.$$

9. (a) Let $F$ be a field such that $\text{char}(F) \neq 2$ and $\sqrt{2}$ exists in $F$. Prove that the ideal $I = (x^2 + y^2 - 1, x + y)$ in $F[x, y]$ is not prime.

(b) Redo the same when $\text{char}(F) = 2$.

10. Prove that the ideal $(x^2 + x + 1)$ in $\mathbb{R}[x]$ is maximal.

Group rings

11. Let $R$ be a commutative ring and let $G$ be a group. Prove that the group ring $R[G]$ has the following universal property. For any $R$-algebra $S$, and any group homomorphism $f : G \to S^\times$, there exists a unique $R$-algebra homomorphism $\phi : R[G] \to S$, such that $f = \phi \circ \iota$, where $\iota : G \to R[G]$ is the inclusion $g \mapsto g1_R$.

12. Use the above to conclude that for any group homomorphism $f : G \to H$ there is a unique ring homomorphism $\phi : R[G] \to R[H]$, compatible with the inclusions $G \to R[G]$ and $H \to R[H]$.

13. Prove that $\mathbb{Z}[C_n] \cong \mathbb{Z}[t]/(t^n - 1)$ and $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[x, y]/(xy - 1)$.

---

$S^\times$ denotes the units of $S$. 