Groups, Rings and Vector Spaces I

Proposed Problems — Rings and Modules

November 12, 2018

Characteristic.

1. Prove that the characteristic of an integral domain is a nonzero prime number or zero.

2. Prove that if Λ is a subring of a field F, then Λ and F have the same characteristic.

3. Let $p > 0$ be a prime. Prove that if F is a field, and $\mathrm{char}(F) = p$, then there is a unique injective homomorphism
$$\mathbb{F}_p \longrightarrow F,$$
where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

4. Use Problem 3 to show that if F is a finite field, say with q elements, then q is a power of an appropriate nonzero prime p.

Around the Chinese Remainder Theorem.

5. Let R be a commutative ring, and I, J two nontrivial ideals. Prove that if $I + J = R$, then $IJ = I \cap J$.

6. Prove that if two nontrivial ideals I, J in a commutative ring R satisfy $I + J = R$, then there is an isomorphism
$$\frac{R}{IJ} \cong \frac{R}{I} \times \frac{R}{J}.$$

7. It is *not* necessary that two ideals I and J be coprime in order that $IJ = I \cap J$. Indeed, prove that, if k is a field, say $\mathrm{char}(k) = 0$ for simplicity, then $(x)(y) = (x) \cap (y)$ in $k[x, y]$.

   On the other hand the condition that $I + J = R$ is necessary for the theorem (as proved in 6). Indeed, show that the homomorphism

$$\frac{k[x,y]}{(xy)} \longrightarrow \frac{k[x,y]}{(x)} \times \frac{k[x,y]}{(y)}$$

   induced by projection from $k[x, y]$ is *not* surjective.

MISC. (IDEAL) STUFF.

8. Let k be a field, assumed of characteristic 0, for simplicity. Find an isomorphism

$$\frac{k[x,y]}{(x^2+y^2-1, y^2+y-1)} \cong \frac{k[x]}{(x^4+x^2-1)}.$$

9. (a) Let F be a field such that $\mathrm{char}(F) \neq 2$ and $\sqrt{2}$ exists in F. Prove that the ideal $I = (x^2 + y^2 - 1, x + y)$ in $F[x, y]$ is not prime.

   (b) Redo the same when $\mathrm{char}(F) = 2$.

10. Prove that the ideal $(x^2 + x + 1)$ in $\mathbb{R}[x]$ is maximal.

GROUP RINGS

11. Let R be a commutative ring and let G be a group. Prove that the group ring $R[G]$ has the following universal property. For any R-algebra S, and any group homomorphism $f : G \to S^\times$,[1] there exists a unique R-algebra homomorphism $\phi : R[G] \to S$, such that $f = \phi \circ \iota$, where $\iota : G \to R[G]$ is the inclusion $g \mapsto g1_R$.

12. Use the above to conclude that for any group homomorphism $f : G \to H$ there is a unique ring homomorphism $\phi : R[G] \to R[H]$, compatible with the inclusions $G \to R[G]$ and $H \to R[H]$.

13. Prove that $\mathbb{Z}[C_n] \cong \mathbb{Z}[t]/(t^n - 1)$ and $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[x, y]/(xy - 1)$.

[1] $S^\times$ denotes the units of S.