DISCRETIZATION OF SPACE-TIME WHITE NOISE IN THE HEAT EQUATION

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Abstract. We consider discretizations of a periodic, one-dimensional spatial white noise. We then propose a particular spectral discretization of space-time white noise and use it to force the heat equation with periodic boundary conditions. We are able to solve explicitly the discretized equations, and then find explicitly the probability distribution of the stationary solution of the heat equation forced by a space-time white noise.

1. Spatial White Noise

We begin by considering a spatial white noise on the interval $[0, 2\pi]$ with periodic boundary conditions. The salient properties of the noise are that it is a centered Gaussian process which is delta-correlated. That is, we wish to have a process $Z(x)$ such that for each $x \in [0, 2\pi]$, $Z(x)$ is a centered normal and $\mathbb{E}Z(x)Z(x') = \delta(x - x')$. This, of course, can only make sense in a weak sense.

Let N be a positive even integer, and define $h = 2\pi/N$. We will consider the points $jh \in [0, 2\pi]$ for $j = 1, \ldots, N$. Suppose that ξ_j^N are independent standard normal random variables (i.e., $\mathbb{E}\xi_j^N = 0$ and $\mathbb{E}(\xi_j^N)^2 = 1$) for $j = 1, \ldots, N$, and $N = 2, 4, 6, \ldots$. We wish to view ξ_j^N, once properly scaled, as a discretized approximation to our desired white noise process $Z(x)$.

Note that $\mathsf{e}^{\mathsf{i}h}$ is a primitive N^{th} root of unity, and that therefore, if k is an integer, then the sum $\sum_{j=1}^{N} \mathsf{e}^{\mathsf{i}jkh}$ is 0, unless k is a multiple of N (usually this will only occur for $k = 0, \pm N$ for us here), in which case the sum is N. We will use this fact repeatedly in what follows.

In order to get reasonable convergence results, and to be able to consider some applications to partial differential equations, we will want to interpolate our discretization

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to define a process on the entire interval $[0, 2\pi]$. We will consider three different interpolations.

1.1. **Spectral interpolation.** We note that if we have real-valued constants $x_1, x_2, \ldots, x_N$, then the function

$$f(x) = \frac{1}{N} \sum_{j=1}^{N} \left(\sum_{k=-N/2}^{N/2}{}' \, \mathrm{e}^{\mathrm{i}k(x-jh)} \right) x_j \tag{1.1}$$

is a real-valued periodic function with period 2π such that $f(jh) = x_j$ for $j = 1, 2, \ldots, N$. The prime on the k-summation denotes that the first and last terms ($k = \pm N/2$) have the understood coefficient $\frac{1}{2}$. This symmetrization of the sum serves to balance the highest modes in the discrete Fourier transfrom so that the interpolation is real-valued, while maintaining the specified values x_j at the points $x = jh$. See, for example, chapter 3 of Trefethen [6] for details of this.

Define

$$Z^N(x) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left(\sum_{k=-N/2}^{N/2}{}' \, \mathrm{e}^{\mathrm{i}k(x-jh)} \right) \xi_j^N. \tag{1.2}$$

Note the scaling $1/\sqrt{N}$ rather than $1/N$. We will see that this will give the desired delta correlation in the limit. Then, $Z^N(jh) = \sqrt{N}\xi_j^N$, and since the ξ_j^N are independent normals, we see that, for each $x \in [0, 2\pi]$, $Z^N(x)$ is a real-valued Gaussian with mean 0.

That is, $Z^N(x)$ is a centered real-valued Gaussian process. We calculate the covariances:

$$\begin{aligned}
\mathbb{E}Z^N(x)Z^N(x') &= \frac{1}{N}\sum_{j=1}^{N}\sum_{j'=1}^{N}\mathbb{E}\xi_j^N\xi_{j'}^N\left(\sideset{}{'}\sum_{k=-N/2}^{N/2}\mathsf{e}^{\mathsf{i}k(x-jh)}\right)\left(\sideset{}{'}\sum_{k'=-N/2}^{N/2}\mathsf{e}^{\mathsf{i}k'(x'-j'h)}\right)\\
&= \frac{1}{N}\sum_{j=1}^{N}\sideset{}{'}\sum_{k=-N/2}^{N/2}\sideset{}{'}\sum_{k'=-N/2}^{N/2}\mathsf{e}^{\mathsf{i}k(x-jh)}\mathsf{e}^{\mathsf{i}k'(x'-jh)}\\
&= \frac{1}{N}\sideset{}{'}\sum_{k=-N/2}^{N/2}\sideset{}{'}\sum_{k'=-N/2}^{N/2}\mathsf{e}^{\mathsf{i}(kx+k'x')}\sum_{j=1}^{N}\mathsf{e}^{-\mathsf{i}(k+k')jh}\\
&= \frac{1}{4}\left(\mathsf{e}^{-\mathsf{i}\frac{N}{2}(x+x')}+\mathsf{e}^{\mathsf{i}\frac{N}{2}(x-x')}+\mathsf{e}^{-\mathsf{i}\frac{N}{2}(x-x')}+\mathsf{e}^{\mathsf{i}\frac{N}{2}(x+x')}\right)+\sum_{k=-N/2+1}^{N/2-1}\mathsf{e}^{\mathsf{i}k(x-x')}\\
&= \cos\frac{N}{2}x\cos\frac{N}{2}x' + \frac{\sin\frac{N-1}{2}(x-x')}{\sin\frac{1}{2}(x-x')}.
\end{aligned}$$

Note that if $\phi(x)$ is a smooth function with period 2π, since $D^N(x) = \frac{\sin\frac{N-1}{2}x}{\sin\frac{1}{2}x}$ is the Dirichlet kernel on $[0, 2\pi]$ (see, for example, chapters 18 and 19 of Körner [2], or section 5.5 of Strauss [5]), we have that

$$\frac{1}{2\pi}\int_0^{2\pi} D^N(x-x')\phi(x')\,\mathrm{d}x' \to \phi(x) \tag{1.3}$$

as $N \to \infty$, for each x. Similarly, the Riemann-Lebesgue lemma (see, for example, theorem 7.5 of Rudin [4]) implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\frac{N}{2}(x-x')\phi(x')\,\mathrm{d}x' \to 0 \tag{1.4}$$

as $N \to \infty$, and therefore, since $\cos\frac{N}{2}x\cos\frac{N}{2}x' = \frac{1}{2}\cos\frac{N}{2}(x-x') + \frac{1}{2}\cos\frac{N}{2}(x+x')$, we see that

$$\frac{1}{2\pi}\int_0^{2\pi} \mathbb{E}Z^N(x)Z^N(x')\phi(x')\,\mathrm{d}x' \to \phi(x) \tag{1.5}$$

as $N \to \infty$. That is, $\mathbb{E}Z^N(x)Z^N(x') \to \delta(x-x')$ in the sense of distributions.

1.2. **Piecewise linear interpolation.** Define the tent functions for $j = 1, \ldots, N$ by

$$\eta_j^N(x) = \begin{cases} \frac{x-(j-1)h}{h}, & (j-1)h \le x \le jh \\ \frac{(j+1)h-x}{h}, & jh \le x \le (j+1)h \\ 0, & \text{otherwise.} \end{cases} \tag{1.6}$$

(Here, x is only defined up to multiples of 2π, so that these functions are periodic with period 2π, and, in particular, η_N^N "wraps around" the interval.)

Now, set $Z^N(x) = \sqrt{N}\sum_{j=1}^N \eta_j^N(x)\xi_j^N$. Again, we see that, for each X, $Z^N(x)$ is a real-valued normal random variable with mean 0, and so again we are interested in the covariances. Now,

$$\begin{aligned}\mathbb{E}Z^N(x)Z^N(x') &= N\sum_{j=1}^N\sum_{j'=1}^N \eta_j^N(x)\eta_{j'}^N(x')\mathbb{E}\xi_j^N\xi_{j'}^N \\ &= N\sum_{j=1}^N \eta_j^N(x)\eta_j^N(x').\end{aligned}$$

Note now that

$$\frac{1}{2\pi}\int_0^{2\pi}\mathbb{E}Z^N(x)Z^N(x')\phi(x')\,\mathrm{d}x' = \frac{N}{2\pi}\sum_{j=1}^N \eta_j^N(x)\int_0^{2\pi}\eta_j^N(x')\phi(x')\,\mathrm{d}x'. \tag{1.7}$$

Let $\varepsilon > 0$. There exists an N_0 such that if $N \geq N_0$ and $|x - x'| < h = \frac{2\pi}{N}$, then $|\phi(x) - \phi(x')| < \varepsilon$. Therefore,

$$\begin{aligned}\left|\frac{N}{2\pi}\int_0^{2\pi}\eta_j^N(x')\phi(x')\,\mathrm{d}x' - \phi(jh)\right| &\leq \frac{N}{2\pi}\int_0^{2\pi}\eta_j^N(x')|\phi(x') - \phi(jh)|\,\mathrm{d}x' \\ &= \frac{N}{2\pi}\int_{(j-1)h}^{(j+1)h}\eta_j^N(x')|\phi(x') - \phi(jh)|\,\mathrm{d}x' \\ &\leq \frac{N}{2\pi}\int_{(j-1)h}^{(j+1)h}\eta_j^N(x')\varepsilon\,\mathrm{d}x' \\ &= \varepsilon.\end{aligned}$$

and so, if $Jh \leq x < (J+1)h$, our error is

$$\begin{aligned}\text{Error} &= \left|\frac{1}{2\pi}\int_0^{2\pi}\mathbb{E}Z^N(x)Z^N(x')\phi(x')\,\mathrm{d}x' - \phi(x)\right| \\ &= \left|\sum_{j=1}^N \eta_j^N(x)\frac{N}{2\pi}\int_0^{2\pi}\eta_j^N(x')\phi(x')\,\mathrm{d}x' - \phi(x)\right| \\ &\leq \left|\sum_{j=J}^{J+1} \eta_j^N(x)\frac{N}{2\pi}\int_0^{2\pi}\eta_j^N(x')\phi(x')\,\mathrm{d}x' - \phi(x)\right| \\ &\leq \sum_{j=J}^{J+1}\eta_j^N(x)\left|\frac{N}{2\pi}\int_0^{2\pi}\eta_j^N(x')\phi(x')\,\mathrm{d}x' - \phi(jh)\right| + \sum_{j=J}^{J+1}\eta_j^N(x)|\phi(jh) - \phi(x)| \\ &\leq 2\varepsilon.\end{aligned}$$

So we see that in this case, too, we have that $\mathbb{E}Z^N(x)Z^N(x') \to \delta(x - x')$, as $N \to \infty$.

1.3. **Step interpolation.** Finally, let

$$\theta_j^N(x) = \begin{cases} 1, & (j-\frac{1}{2})h \le x \le (j+\frac{1}{2})h \\ 0, & \text{otherwise.} \end{cases} \tag{1.8}$$

Again, we wrap around the interval as necessary.

It is very similar to the piecewise linear interpolation to show that $Z^N(x) = \sqrt{N}\sum_{j=1}^N \theta_j^N(x)\xi_j^N$ also approximates a white noise.

2. Space-Time White Noise

We also need to consider the temporal noise. We aim to apply our discretizations and interpolations to partial stochastic differential equations written in Itô form, so for this it suffices to replace the normal random variables ξ_j^N by Brownian motions in time $W_j^N(t)$. For example, in the case of the spectral interpolation (and we will confine ourselves to considering this case in the remainder of this paper), we define

$$Z^N(x,t) = \frac{1}{\sqrt{N}}\sum_{j=1}^N \left({\sum_{k=-N/2}^{N/2}}' \mathrm{e}^{\mathrm{i}k(x-jh)} \right) W_j^N(t). \tag{2.1}$$

Then our space-time white noise is approximated by the Itô differential

$$\mathrm{d}Z^N(x,t) = \frac{1}{\sqrt{N}}\sum_{j=1}^N \left({\sum_{k=-N/2}^{N/2}}' \mathrm{e}^{\mathrm{i}k(x-jh)} \right) \mathrm{d}W_j^N(t). \tag{2.2}$$

Now, note that

$$ {\sum_{k=-N/2}^{N/2}}' \mathrm{e}^{\mathrm{i}kx} = \frac{\sin\frac{N}{2}x}{\tan\frac{1}{2}x}. \tag{2.3}$$

Therefore, since the W_j^N are independent for distinct j, we have that

$$(\mathrm{d}Z^N(x,t))^2 = \frac{1}{N}\sum_{j=1}^N \left({\sum_{k=-N/2}^{N/2}}' \mathrm{e}^{\mathrm{i}k(x-jh)} \right)^2 \mathrm{d}t = \frac{1}{N}\sum_{j=1}^N \frac{\sin^2\frac{N}{2}k(x-jh)}{\tan^2\frac{1}{2}(x-jh)}\,\mathrm{d}t. \tag{2.4}$$

Finally, we note that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{N}\sum_{j=1}^N \frac{\sin^2\frac{N}{2}k(x-jh)}{\tan^2\frac{1}{2}(x-jh)}\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin^2\frac{N}{2}x}{\tan^2\frac{1}{2}x}\,\mathrm{d}x \to 1, \tag{2.5}$$

an increasing limit as $N\to\infty$. That is, formally,

$$\frac{1}{2\pi}\int_0^{2\pi} (\mathrm{d}Z(x,t))^2\,\mathrm{d}x = \mathrm{d}t. \tag{2.6}$$

3. The Stochastic Heat Equation

We consider the equation, for $x \in [0, 2\pi]$, $t \geq 0$,

$$\mathrm{d}u(x,t) = \nu \frac{\partial^2 u}{\partial x^2}\,\mathrm{d}t + \alpha\,\mathrm{d}Z(x,t), \tag{3.1}$$

where ν and α are positive constants, and $\frac{\mathrm{d}Z}{\mathrm{d}t}$ is a space-time white noise, as above. We will assume the initial condition $u(x,0) = 0$, and periodic boundary conditions. As the equation is linear, nonzero initial conditions just superpose the solution of the deterministic heat equation (i.e., $\alpha = 0$) onto the solution of this equation, so we needn't concern ourselves with this case here.

We will, as above, discretize in space with N equally spaced points, and interpolate spectrally. That is, we have the equation

$$\mathrm{d}u^N(x,t) = \nu \frac{\partial^2 u^N}{\partial x^2}\,\mathrm{d}t + \alpha\,\mathrm{d}Z^N(x,t), \tag{3.2}$$

with $u^N(x,0) = 0$. Here,

$$Z^N(x,t) = {\sum_{k=-N/2}^{N/2}}' \mathsf{e}^{\mathsf{i}kx} \sum_{j=1}^{n} \mathsf{e}^{-\mathsf{i}jkh} \frac{1}{\sqrt{N}} W_j^N(t). \tag{3.3}$$

If we take the Fourier expansion

$$u^N(x,t) = {\sum_{k=-N/2}^{N/2}}' \mathsf{e}^{\mathsf{i}kx} u_k^N(t), \tag{3.4}$$

we get the system of stochastic differential equations

$$\mathrm{d}u_k^N(t) = -\nu k^2 u_k^N(t)\,\mathrm{d}t + \frac{\alpha}{\sqrt{N}} \sum_{j=1}^{N} \mathsf{e}^{-\mathsf{i}jkh}\,\mathrm{d}W_j^N(t) \tag{3.5}$$

for the coefficients, with initial conditions $u_k^N(0) = 0$.

These SDEs can be solved explicitly. If we define $v_k^N(t) = \mathsf{e}^{\nu k^2 t} u_k^N(t)$, Itô's formula tells us that

$$\mathrm{d}v_k^N(t) = \frac{\alpha}{\sqrt{N}} \sum_{j=1}^{N} \mathsf{e}^{-\mathsf{i}jkh} \mathsf{e}^{\nu k^2 t}\,\mathrm{d}W_j^N(t), \tag{3.6}$$

so that we can immediately integrate to get that

$$v_k^N(t) = \frac{\alpha}{\sqrt{N}} \sum_{j=1}^{N} \mathsf{e}^{-\mathsf{i}jkh} \int_0^t \mathsf{e}^{\nu k^2 s}\,\mathrm{d}W_j^N(s), \tag{3.7}$$

and so

$$u_k^N(t) = \frac{\alpha}{\sqrt{N}} \sum_{j=1}^{N} \mathsf{e}^{-\mathsf{i}jkh} \int_0^t \mathsf{e}^{-\nu k^2(t-s)}\, \mathrm{d}W_j^N(s). \tag{3.8}$$

Define

$$X_{jk}^N(t) = \int_0^t \mathsf{e}^{-\nu k^2(t-s)}\, \mathrm{d}W_j^N(s). \tag{3.9}$$

For $j = 1, \ldots, N$, $k = -N/2, \ldots, N/2$ and at any finite list of distinct times t, these random variables are jointly normal with mean 0, and so their distribution is completely determined by their covariance structure.

We calculate, if $0 \le t \le t'$,

$$\mathbb{E}X_{j0}^N(t)X_{j'0}^N(t') = \mathbb{E}W_j^N(t)W_{j'}^N(t') = \begin{cases} t, & \text{if } j = j' \\ 0, & \text{if } j \neq j'. \end{cases} \tag{3.10}$$

If $(k, k') \neq (0, 0)$, we have

$$\begin{aligned}
\mathbb{E}X_{jk}^N(t)X_{j'k'}^N(t') &= \mathbb{E}\int_0^t \mathsf{e}^{-\nu k^2(t-s)}\, \mathrm{d}W_j^N(s) \int_0^{t'} \mathsf{e}^{-\nu k'^2(t'-s')}\, \mathrm{d}W_{j'}^N(s') \\
&= \delta_{jj'}\mathbb{E}\left[\int_0^t \mathsf{e}^{-\nu k^2(t-s)}\, \mathrm{d}W_j^N(s) \left(\int_0^t + \int_t^{t'}\right) \mathsf{e}^{-\nu k'^2(t'-s')}\, \mathrm{d}W_j^N(s')\right] \\
&= \delta_{jj'} \int_0^t \mathsf{e}^{-\nu k^2(t-s)} \mathsf{e}^{-\nu k'^2(t'-s)}\, \mathrm{d}s \\
&= \delta_{jj'} \frac{(1 - \mathsf{e}^{-\nu(k^2+k'^2)t})\mathsf{e}^{-\nu k'^2(t'-t)}}{\nu(k^2 + k'^2)}.
\end{aligned}$$

Notice, in particular, that, when $j = j'$, this does not depend on j.

Then, for a finite list of values of x and t, the

$$u^N(x, t) = \frac{\alpha}{\sqrt{N}} \sum_{k=-N/2}^{N/2}{}' \mathsf{e}^{\mathsf{i}kx} \sum_{j=1}^{N} \mathsf{e}^{-\mathsf{i}jkh} X_{jk}^N(t) \tag{3.11}$$

are also real, jointly normal random variables with mean 0, and so again, we need only find their covariance structure. We recall again that $\sum_{j=1}^{N} \mathsf{e}^{-\mathsf{i}jkh}$ is zero unless k is a multiple of N, in which case it is N. Thus, if $0 \leq t \leq t'$,

$$\begin{aligned}
\mathbb{E}u^N(x,t)u^N(x',t') &= \frac{\alpha^2}{N} \sideset{}{'}\sum_{k=-N/2}^{N/2} \sideset{}{'}\sum_{k'=-N/2}^{N/2} \mathsf{e}^{\mathsf{i}(kx+k'x')} \sum_{j=1}^{N} \sum_{j'=1}^{N} \mathsf{e}^{-\mathsf{i}(jk+j'k')h} \mathbb{E}X_{jk}^N(t) X_{j'k'}^N(t') \\
&= \frac{\alpha^2}{N} \sideset{}{'}\sum_{k=-N/2}^{N/2} \sideset{}{'}\sum_{k'=-N/2}^{N/2} \mathsf{e}^{\mathsf{i}(kx+k'x')} \sum_{j=1}^{N} \mathsf{e}^{-\mathsf{i}j(k+k')h} \frac{(1-\mathsf{e}^{-\nu(k^2+k'^2)t})\mathsf{e}^{-\nu k'^2(t'-t)}}{\nu(k^2+k'^2)} \\
&= \alpha^2 \left[\frac{1}{2} \left(\cos\frac{N}{2}(x-x') + \cos\frac{N}{2}(x+x') \right) \frac{(1-\mathsf{e}^{-\frac{1}{2}\nu N^2 t})\mathsf{e}^{-\frac{1}{4}\nu N^2(t'-t)}}{\frac{1}{2}\nu N^2} \right. \\
&\quad \left. + \sum_{k=1}^{N/2-1} \left(2\cos k(x-x') \frac{(1-\mathsf{e}^{-2\nu k^2 t})\mathsf{e}^{-\nu k^2(t'-t)}}{2\nu k^2} \right) + t \right] \\
&= \frac{4\alpha^2}{\nu N^2} \cos\frac{N}{2}x \cos\frac{N}{2}x' (1-\mathsf{e}^{-\frac{1}{2}\nu N^2 t})\mathsf{e}^{-\frac{1}{4}\nu N^2(t'-t)} \\
&\quad + \frac{\alpha^2}{\nu} \sum_{k=1}^{N/2-1} \frac{\cos k(x-x')}{k^2} (1-\mathsf{e}^{-2\nu k^2 t})\mathsf{e}^{-\nu k^2(t'-t)} + \alpha^2 t.
\end{aligned}$$

In the last two expressions, the first term comes from $k, k' = \pm N/2$; the second term (the summation) comes when $k' = -k$ are nonzero, and not $\pm N/2$; and the final term comes from $k = k' = 0$.

Now, we consider what happens when $N \to \infty$, i.e., the discretization becomes arbitrarily fine. We see easily that

$$\mathbb{E}u^N(x,t)u^N(x',t') \to \frac{\alpha^2}{\nu} \sum_{k=1}^{\infty} \frac{\cos k(x-x')}{k^2} (1-\mathsf{e}^{-2\nu k^2 t})\mathsf{e}^{-\nu k^2(t'-t)} + \alpha^2 t. \tag{3.12}$$

If we define

$$\tilde{u}^N(x,t) = \frac{\alpha}{\sqrt{N}} \sideset{}{'}\sum_{k\neq 0} \mathsf{e}^{\mathsf{i}kx} \sum_{j=1}^{N} \mathsf{e}^{-\mathsf{i}jkh} X_{jk}^N(t), \tag{3.13}$$

that is, $\tilde{u}^N$ is merely u^N without the constant ($k = 0$) term in the expansion, then we see that

$$\mathbb{E}\tilde{u}^N(x,t)\tilde{u}^N(x',t') \to \frac{\alpha^2}{\nu} \sum_{k=1}^{\infty} \frac{\cos k(x-x')}{k^2} (1-\mathsf{e}^{-2\nu k^2 t})\mathsf{e}^{-\nu k^2(t'-t)}. \tag{3.14}$$

Now, since the series is dominated by the summable series $\sum \frac{1}{k^2}$, we have that for fixed t, as $t' \to \infty$,

$$\frac{\alpha^2}{\nu} \sum_{k=1}^{\infty} \frac{\cos k(x-x')}{k^2} (1-\mathsf{e}^{-2\nu k^2 t})\mathsf{e}^{-\nu k^2(t'-t)} \searrow 0. \tag{3.15}$$

That is, the large-time behavior decorrelates from the initial small-time behavior. To see what this large time behavior is, we let $t = t' \to \infty$ and we see that

(3.16)
$$\frac{\alpha^2}{\nu}\sum_{k=1}^{\infty}\frac{\cos k(x-x')}{k^2}(1-\mathsf{e}^{-2\nu k^2 t}) \nearrow \frac{\alpha^2}{\nu}\sum_{k=1}^{\infty}\frac{\cos k(x-x')}{k^2} = \frac{\alpha^2}{\nu}\left(\frac{1}{4}(x-x')^2 - \frac{\pi}{2}(x-x') + \frac{\pi^2}{6}\right),$$
for $0 \le x - x' \le 2\pi$.

We note that for each fixed time $0 \le t \le \infty$, the covariance structure of $\tilde{u}(x,t)$ satisfies Fernique's continuity condition (see Fernique [1] or Marcus [3]), and therefore a process with continuous realizations (in x) exists for each t, and in the large-time limit.

Finally, we note that that the constant term $u_0(t) = u_0(x,t)$ doesn't depend on x and is Gaussian with mean zero and covariance given by $\mathbb{E}u_0(t)u_0(t') = \alpha^2 t$. Therefore, $B(t) = \frac{1}{\alpha}u_0(t)$ is a standard Brownian motion, and it is easy to check that it is uncorrelated with, and therefore independent from, $\tilde{u}(x,t)$. Therefore we can write the solution to the stochastic heat equation (3.1) with initial value $u(x,0) = 0$ in the form

$$u(x,t) = \alpha B(t) + \tilde{u}(x,t), \tag{3.17}$$

where $B(t)$ is a standard Brownian motion, and $\tilde{u}$ is a centered Gaussian process independent of B with covariance structure given by

$$\mathbb{E}\tilde{u}(x,t)\tilde{u}(x',t') = \frac{\alpha^2}{\nu}\sum_{k=1}^{\infty}\frac{\cos k(x-x')}{k^2}(1-\mathsf{e}^{-2\nu k^2 t})\mathsf{e}^{-\nu k^2(t'-t)}. \tag{3.18}$$

We note further that the solution of the stochastic heat equation (3.1), with arbitrary initial condition, once centered (i.e., we look at $\tilde{u}(x,t) = u(x,t) - \frac{1}{2\pi}\int_0^{2\pi} u(x,t)\,\mathrm{d}x$), converges to a continuous-in-space stationary distribution with covariance

$$\mathbb{E}\tilde{u}(x,t)\tilde{u}(x',t') = \frac{\alpha^2}{\nu}\left(\frac{1}{4}(x-x')^2 - \frac{\pi}{2}(x-x') + \frac{\pi^2}{6}\right), \tag{3.19}$$

for $0 \le x - x' \le 2\pi$, $0 \le t \le t'$.

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