A MARTINGALE CONTROL VARIATE METHOD FOR OPTION PRICING WITH CAM MODEL

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Abstract. We propose a variance reduction method for Monte Carlo computation of option prices in the context of the Coupled Additive-Multiplicative Noise model. Four different schemes are applied for the simulation. The methods select control variates which are martingales in order to reduce the variance of unbiased option price estimators. Numerical results for European call options are presented to illustrate the effectiveness and robustness of this martingale control variate method.

Keywords: Monte Carlo method; variance reduction;control variate method;CAM model

1. Introduction

In financial mathematics, stochastic differential equations (SDEs) play an important role as the setting for most of the models used for pricing derivatives. The SDEs describe the evolution of certain financial variables, such as the stock price, volatility of an asset, or interest rate. The classic model which people usually apply for pricing European call options is the Black-Scholes model, where the volatility is assumed to be constant. But this assumption is a limitation of the standard Black-Scholes model, which is proven by the so-called smile effect: that implied volatilities of market prices are not constant with strike price and the time to maturity of the contract. One way to take this into account is to treat volatility as varying in time as well.

People have done plenty of research in a framework for pricing derivatives: for example, Fouque [1], Hull and White [2]. Among these works, mean reversion of the volatility has been used to simplify the basic pricing and estimation problems as well as reflect reality to some extent, as volatility doesn't wander to arbitrarily large or small values.

What is volatility? There are several notions of volatility. Some of them are model dependent, and others are data dependent.

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Like mentioned in Fouque's slides [3], realized volatility, sometimes referred to as the historical volatility, measures one aspect of what actually happened in the past. The measurement of the volatility depends on the particular situation. For example, one could look at the realized volatility for the equity market in November of 2008 by taking the standard deviation of the daily returns within that month. One could also calculate the realized volatility between 11:00AM and 12:00PM of May 16, 2008 by calculating the standard deviation of one minute returns. Here, let $t_0 < t_1 < \cdots < t_N$ be a sequence of times. Then

$$\frac{1}{T-t_0}\int_{t_0}^{t_N} \Sigma_s^2\, ds \sim \frac{1}{N}\sum_{i=1}^{N} \frac{(\log S_{t_i} - \log S_{t_{i-1}})^2}{t_i - t_{i-1}}, \tag{1}$$

where Σ_s is the realized volatility, and S_{t_i} is the stock price at time t_i.

In contrast to realized volatility, implied volatility, as explained by Beckers [4] and Mayhew [5], refers to the market's assessment of future volatility under the assumption that the dynamics can be modelled by a Black-Scholes model. It is an estimation of the volatility of a stock as implied by the price of an option on that stock, as follows. Given an observed European call option price C^{obs} for a contract with strike price K and expiration date T, the implied volatility I is defined to be the value of the volatility parameter that must go into the Black-Scholes formula to match this price:

$$C_{BS}(t, x; K, T; I) = C^{\text{obs}}. \tag{2}$$

Model dependent volatility has two main categories: local volatility and stochastic volatility. People usually set a lognormal model for the asset price X_t:

$$dX_t = \mu X_t\, dt + \sigma X_t\, dW_t. \tag{3}$$

One popular way to modify the lognormal model is to suppose that volatility is a deterministic positive function of time and stock price: $\sigma = \sigma(t, X_t)$. This is called the local volatility. The stochastic differential equation modeling the stock price is then

$$dX_t = \mu X_t\, dt + \sigma(t, X_t) X_t\, dW_t. \tag{4}$$

In stochastic volatility models, the value σ_t, called the volatility process, is allowed to vary stochastically. It does not have to be an Itô process: it can be a jump process, a Markov chain, etc. It should be positive, as it is a volatility. Unlike the local volatility, the stochastic volatility process need not be perfectly correlated with the Brownian motion, W_t, in the asset price model:

$$dX_t = X_t(\mu\, dt + \sigma_t\, dW_t). \tag{5}$$

In other words, the stochastic volatility is a function of some process Y_t, where Y_t contains an additional source of randomness:

$$\sigma_t = f(Y_t). \tag{6}$$

2. Stochastic Volatility Models

2.1. **Standard Models.** As mentioned in the last section, typically, stochastic volatility is taken to be a function of a stochastic process (Y_t) in general (but we consider only the Itô process case here), that is $\sigma_t = f(Y_t)$. The process (Y_t) satisfies a stochastic differential equation driven by a second Brownian motion. The desired models should make the volatility positive.

An important feature often applied in the stochastic volatility models is mean reversion. The definition for the term "mean reversion" is a linear pull-back term in the drift of the volatility process itself, or in the drift of some (underlying) process of which volatility is a function. It also refers to the characteristic time it takes for a process to get back to the mean level of its invariant distribution. The stochastic differential equation for (Y_t) introduces a new Brownian motion Z'_t

$$dY_t = \alpha(m - Y_t)\, dt + \beta(t, Y_t)\, dZ'_t. \tag{7}$$

Here the parameters are α and m. α is the rate of the mean reversion and m is the long-run mean level of Y_t. Y_t will approach m with speed α, on average.

The simplest mean-reverting model is an Ornstein-Uhlenbeck (OU) process, which is defined as a solution of equation (7) where $\beta(t, Y_t) = \beta$ is constant. Also notice that the second Brownian motion (Z'_t) is typically correlated with the Brownian motion (W_t) from the asset price equation (5). $\rho \in [-1, 1]$ is the instantaneous correlation coefficient defined by equation (8). ρ is often found to be negative because of the leverage effect between stock price and volatility shocks. It's often convenient to write it like equation (9), where (Z_t) is a standard Brownian motion independent from (W_t):

$$dW_t\, dZ'_t = \rho\, dt \tag{8}$$

with

$$Z'_t = \rho W_t + \sqrt{1 - \rho^2} Z_t. \tag{9}$$

Besides the Ornstein-Uhlenbeck process, there are some other common mean-reverting processes. The Feller or Cox-Ingersoll-Ross (CIR) process is another common one:

$$dY_t = \kappa(m' - Y_t)\, dt + \nu\sqrt{Y_t}\, dZ'_t. \tag{10}$$

The popular Heston model [6] is based on the CIR process with $f(Y_t) = \sqrt{Y_t}$.

To revise the lognormality assumption (3) of Black-Scholes [7], the Constant Elasticity of Variance (CEV) model [8] is also focused on by researchers. The CEV model is in the form:

$$dX_t = \mu X_t\, dt + \sigma X_t^{\frac{\theta}{2}}\, dW_t, \tag{11}$$

that is,

$$\frac{dX_t}{X_t} = \mu\, dt + \frac{\sigma}{X_t^{1-\frac{\theta}{2}}}\, dW_t. \tag{12}$$

The return variance with respect to price X_t, $\nu(X_t, t) = \sigma^2 X_t^{\theta-2}$, has the relationship

$$\frac{d\nu(X_t,t)/dX_t}{\nu(X_t,t)/X_t} = \theta - 2, \tag{13}$$

which implies that

$$d\nu(X_t,t)/\nu(X_t,t) = (\theta-2)dX_t/X_t. \tag{14}$$

The quantity $\theta - 2$ is called elasticity of return.

In particular, if $\theta = 2$, then the elasticity is zero and the stock price is lognormally distributed as in the Black-Scholes model. If $\theta = 1$, then the elasticity is -1. This is the model proposed by Cox and Ross.

The CEV model has been exploited a lot recently, for example, Anderson [9] and Lord [10]. As mentioned in [10], the asset price process (X_t) and the variance process (Y_t) evolve according to the following SDEs:

$$dX_t = \mu X_t\, dt + \lambda \sqrt{Y_t} X_t^{\beta}\, dW_t, \tag{15}$$

$$dY_t = \kappa(m' - Y_t)\, dt + \omega Y_t^{\alpha}\, dZ_t'. \tag{16}$$

Here the process is specified under the risk-neutral probability measure. The parameter μ is the risk neutral drift of the asset price, $\kappa \geq 0$ is the speed of mean-reversion of the variance, $m' > 0$ is the invariant average variance, $\omega \geq 0$ is the so-called volatility of variance, and λ is a scaling constant. Finally, as explained above, W_t and Z_t' are correlated Brownian motions, with instantaneous correlation coefficient ρ. β is restricted to lie in $(0, 1]$ and α to be positive. The popular Heston model is the special case when $\alpha = \frac{1}{2}$ and $\beta = 1$.

2.2. **CAM Model.** The coupled additive-multiplicative noise (CAM) model was introduced by Sardeshmukh and Sura in their papers [11] and [12]. In [11], they found a link between the skewness and kurtosis of daily sea surface temperature (SST) variations. If the standard deviation of SST anomalies T_0' at a particular point on the ocean's surface is denoted by σ, the skewness (skew) and kurtosis (kurt) become

$$\text{skew} \equiv \frac{\langle T_0'^3 \rangle}{\sigma^3} \text{ and } \text{kurt} \equiv \frac{\langle T_0'^4 \rangle}{\sigma^4} - 3. \tag{17}$$

Skewness is a measure of asymmetry of a probability density function. If the left tail is heavier than the right tail, the probability density function has negative skewness. If the reverse is true, it has positive skewness. If the probability density function is symmetric, like the Gaussian, it has zero skewness. Kurtosis (or more accurately, "excess kurtosis", since the kurtosis of 3 for a Gaussian distribution is subtracted) measures the excess probability (fatness) in the tails, where excess is defined in relation to a Gaussian distribution.

The kurt-skew relationship was gained from the scatterplot of emprically calculated kurtosis vs skewness of the time series of all high-resolution observational data points at most locations around the globe. The scatterplot evinced a lower parabolic bound on kurtosis in their dataset: kurt $\geq (3/2)$ skew^2. All of the data points lay above this parabola, and this is evidently a very strong constraint on the non-Gaussian character of the SST variability. From these observations, a detailed dynamical explanation was provided. They introduced a univariate linear model with multiplicative noise to capture the observed non-Gaussianity of SST anomalies over almost all the globe:

$$\frac{\partial T_0'}{\partial t} = -\lambda T_0' - \phi F' T_0' + F' + R' + \phi \overline{F' T_0'}. \tag{18}$$

Here T_0' is the SST anomalies, $-\lambda$ and $-\phi$ are locally constant parameters, F' and R' are rapidly varying forcing terms. They assumed that the rapidly varying terms F' and R' can be approximated as independent, zero mean Gaussian white noise processes, under which (18) becomes an SDE for SST anomalies T_0'. They also derived an analytical equation from (18) to explain the kurtosis-skewness relationship shown in the scatterplot figure, and they finally concluded that the CAM model is applicable for anomalous SST variability.

Empirical plots of skewness vs kurtosis for log volatility of commodity or stock prices also exhibits the parabolic lower bound. So the log volatility of a commodity also has a non-Gaussian distribution. In order to capture this non-Gaussian behavior, we propose to model stochastic volatility by a CAM model. So we apply this new model for pricing the European call option in this paper. If we make the volatility of option price $\sigma(Y_t) = \exp(Y_t)$, the log volatility is just the diffusion process Y_t. Based on the asset price model (5) and (6), we consider the CAM model for the diffusion process Y_t in this way:

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t^{(1)} + \gamma Y_t d\hat{Z}_t^{(2)}. \tag{19}$$

This is the so-called CAM process. This SDE is developed from the simplest Ornstein-Uhlenbeck (OU) process, and it has a mean-reverting drift term. A second source of randomness $\hat{Z}_t^{(2)}$ is added to the equation besides $\hat{Z}_t^{(1)}$. And the three white noises W_t, $\hat{Z}_t^{(1)}$, $\hat{Z}_t^{(2)}$ are correlated. We can use the coefficients of correlation ρ_1, ρ_2, ρ_3 and three independent white noises W_t, $Z_t^{(1)}$, $Z_t^{(2)}$ to represent the correlations:

$$W_t = W_t, \tag{20}$$

$$\hat{Z}_t^{(1)} = \rho_1 W_t + \sqrt{1-\rho_1^2} Z_t^{(1)} \tag{21}$$

and

$$\hat{Z}_t^{(2)} = \rho_2 W_t + \frac{\rho_3 - \rho_1\rho_2}{\sqrt{1-\rho_1^2}} Z_t^{(1)} + \sqrt{1-\rho_2^2 - \frac{(\rho_3-\rho_1\rho_2)^2}{1-\rho_1^2}} Z_t^{(2)}. \tag{22}$$

Besides its mean-reverting property, we apply the CAM model for pricing the European call option because of its several advantages. First, it's analytically tractable in some ways. We can ask this question: when does the moment $\mathbb{E}Y_t^n$ stay bounded as $t \to \infty$? We can solve the ordinary differential equation for $\mathbb{E}Y_t^n$ to find a relationship between the parameters which can ensure that a particular moment stays bounded for all the time. This relationship turns out to be

$$\alpha \geq \frac{(n-1)}{2}\gamma^2. \tag{23}$$

For example, in order for fifth moments of the stationary distribution to exist, we would need that

$$\alpha \geq 2\gamma^2. \tag{24}$$

The proof of (23) and (24) will be shown in the appendix A.

3. Option Pricing using CAM Model

3.1. Different Numerical Monte Carlo Schemes.

3.1.1. *Black-Scholes Formula.* The price of the European call option for a non-dividend paying underlying stock in terms of the Black-Scholes parameters is

$$C(t,S) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

$$d_1 = \frac{\log(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\log(\frac{S}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Here $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. The time to maturity is $T-t$, the stock price is S, the strike price is K, the risk free rate is r, and the volatility of returns of the underlying asset is σ.

3.1.2. *Euler Scheme.* Here our model is

$$dX_t = \mu X_t\,dt + \sigma_t X_t\,dW_t \tag{25}$$

with the CAM model diffusion process

$$dY_t = \alpha(m-Y_t)\,dt + \beta\,d\hat{Z}_t^{(1)} + \gamma Y_t\,d\hat{Z}_t^{(2)}, \text{ and } \mathrm{\sigma_t = exp(Y_t)}. \tag{26}$$

And apply (20), (21) and (22), we can rewrite the diffusion process Y_t in the form of (27)

$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \mu X_t \\ \alpha(m-Y_t) \end{pmatrix} dt$$

$$+\begin{pmatrix} \exp(Y_t)X_t & 0 & 0 \\ \beta\rho_1+\gamma Y_t\rho_2 & \beta\sqrt{1-\rho_1^2}+\gamma Y_t\frac{\rho_3-\rho_1\rho_2}{\sqrt{1-\rho_1^2}} & \gamma Y_t\sqrt{1-\rho_2^2-\frac{(\rho_3-\rho_1\rho_2)^2}{1-\rho_1^2}} \end{pmatrix} \times \begin{pmatrix} dW_t \\ dZ_t^{(1)} \\ dZ_t^{(2)} \end{pmatrix},$$

where W_t, $Z_t^{(1)}$, and $Z_t^{(2)}$ are three independent standard Brownian Motions. We use some simple notations here: $a_0 = \mu X_t$, $b_0 = \exp(Y_t)X_t$, $a = \alpha(m-Y_t)$, $b_1 = \beta\rho_1+\gamma Y_t\rho_2$, $b_2 = \beta\sqrt{1-\rho_1^2}+\gamma Y_t\frac{\rho_3-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}$, and $b_3 = \gamma Y_t\sqrt{1-\rho_2^2-\frac{(\rho_3-\rho_1\rho_2)^2}{1-\rho_1^2}}$. As in [1], using the risk-neutral theory, there is an equivalent martingale measure $\mathbb{P}^*$ under which the discounted stock price $\tilde{X}_t = e^{-rt}X_t$ is a martingale. And we can compute the European call option price with time-T payoff H using the formula

$$C_t = \mathbb{E}^*\{e^{-r(T-t)}H|\mathcal{F}_t\} \tag{28}$$

for all $t \le T$, when there is no arbitrage opportunity. Thus C_t is a possible price for the European call option. We can try to construct the equivalent martingale measures now. Like what Fouque's group did, we absorb the drift term of $\tilde{X}_t$ in its martingale term by setting

$$W_t^* = W_t + \int_0^t \frac{(\mu-r)}{\exp(Y_s)}ds. \tag{29}$$

Any shift of the second and the third independent Brownian motions of the form

$$Z_t^{(j)*} = Z_t^{(j)} + \int_0^t \theta_s^{(j)}ds \;\; (j=1,2) \tag{30}$$

will not change the drift of $\tilde{X}_t$. By the multiple dimensional Girsanov's theorem from [13], (W^*) and $(Z^{(j)*})$ are independent standard Brownian motions under a measure $\mathbb{P}^{*(\theta^{(1)}\theta^{(2)})}$ defined by

$$\frac{d\mathbb{P}^{*(\theta^{(1)}\theta^{(2)})}}{d\mathbb{P}} =$$
$$\exp\left(-\tfrac{1}{2}\int_0^T ((\theta_s)^2 + (\theta_s^{(1)})^2 + (\theta_s^{(2)})^2)ds - \int_0^T \theta_s dW_s - \int_0^T \theta_s^{(1)} dZ_s^{(1)} - \int_0^T \theta_s^{(2)} dZ_s^{(2)}\right),$$

$$\theta_t = \frac{(\mu - r)}{\exp(Y_t)}.$$

Here $(\theta_t^{(j)})$ are any adapted (and suitably regular) processes. We assume that the newly defined measure $\mathbb{P}^{*(\theta^{(1)}\theta^{(2)})}$ is well-defined, so that f is bounded away from zero and $(\theta_t^{(j)})$ are bounded. Then, under this new risk-neutral measure, the SDEs (25) and (26) become

(31)
$$d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} rX_t \\ \alpha(m-Y)t - \Phi(t,x,y) \end{pmatrix} dt$$
$$+\begin{pmatrix} \exp(Y_t)X_t & 0 & 0 \\ \beta\rho_1 + \gamma Y_t\rho_2 & \beta\sqrt{1-\rho_1^2} + \gamma Y_t \frac{\rho_3 - \rho_1\rho_2}{\sqrt{1-\rho_1^2}} & \gamma Y_t \sqrt{1-\rho_2^2 - \frac{(\rho_3-\rho_1\rho_2)^2}{1-\rho_1^2}} \end{pmatrix} \times \begin{pmatrix} dW_t^* \\ dZ_t^{(1)*} \\ dZ_t^{(2)*} \end{pmatrix},$$

where

$$\begin{aligned} \Phi(t,x,y) = (\beta\rho_1 + \gamma Y_t\rho_2)\tfrac{(\mu-r)}{e^{Y_t}} + \left(\beta\sqrt{1-\rho_1^2} + \gamma Y_t \tfrac{\rho_3-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\right)\theta_t^{(1)} \\ + \gamma Y_t \sqrt{1-\rho_2^2 - \tfrac{(\rho_3-\rho_1\rho_2)^2}{1-\rho_1^2}}\theta_t^{(2)}. \end{aligned} \tag{32}$$

And the three Brownian motions W_t^*, $Z_t^{(1)*}$ and $Z_t^{(2)*}$ under the new measure are independent. The function $\Phi(t,x,y)$, as explained in [1], is related to the risk premium factor from the second and the third sources of the randomness that drive the volatility. For the Monte Carlo computation of the derivative prices, it is used to treat $\Phi(t,x,y) = 0$ for simplification as in [16], and this won't affect the computation results.

We will have the time intervals equal to each other, so $T = N\Delta$ where T is the time to maturity of the option with $t_0 = 0$ and N is the number of time steps. As stated in [14], there are general strong and weak Itô-Taylor approximations. For strong approximations, the stochastic process X_t satisfies the convergence condition $[\mathbb{E}[(X_t - X_t^\delta)^2]]^{\frac{1}{2}} \leq O(\Delta t^{\alpha_1})$. For weak schemes, any function, f, of X_t should satisfy the convergence condition $\mathbb{E}|f(X_t) - f(X_t^\delta)| \leq O(\Delta t^{\beta_1})$ provided f and enough of its partial derivatives have polynomial growth. Here X_t^δ is the numerical discretization of X_t, and α_1, β_1 are orders of the schemes. Since the payoff function of the European call option is just a simple function of the stock price at maturity X_T, the weak scheme here is sufficient for pricing the option.

The Euler scheme corresponds to the truncated Itô-Taylor expansion which contains only the ordinary time integral and the simple Itô integral. We shall see from a general convergence result for weak Taylor approximations, as stated in Theorem 14.5.1 of chapter 14 of [14], that the Euler approximation has order of weak convergence 1.0, if amongst other assumptions, aa, bb, a, b_1, b_2 and b_3 are four times continuously differentiable. This means that the Euler scheme is the order 1.0 weak Taylor approximation.

So for the SDEs

$$dX_t = a_0(t, X_t)\, dt + b_0(t, X_t)\, dW_t^* \tag{33}$$

and

$$dY_t = a(t, Y_t)\, dt + b_1(t, Y_t)\, dW_t^* + b_2(t, Y_t)\, dZ_t^{(1)*} + b_3(t, Y_t)\, dZ_t^{(2)*}, \tag{34}$$

the *Euler scheme* has the form

$$X_{t+\Delta} = X_t + a_0(X_t)\Delta + b_0(X_t)\Delta W_t^* \tag{35}$$

and

$$Y_{t+\Delta} = Y_t + a(Y_t)\Delta + b_1(Y_t)\Delta W_t^* + b_2(Y_t)\Delta Z_t^{(1)*} + b_3(Y_t)\Delta Z_t^{(2)*}, \tag{36}$$

with initial value $X_0 = x$ and $Y_0 = y$, where

$$\Delta = t_{n+1} - t_n \;\;,\;\; \Delta W^* = W^*_{t_{n+1}} - W^*_{t_n} \;\text{ and }\; \Delta \mathrm{Z}^{(\mathrm{j})*} = \mathrm{Z}^{(\mathrm{j})*}_{\mathrm{t_{n+1}}} - \mathrm{Z}^{(\mathrm{j})*}_{\mathrm{t_n}}$$

with $j = 1, 2$.

3.1.3. *Simplified Weak Euler Scheme.* From [14], for weak convergence we only need to approximate the measure induced by the Itô process Y_t, so we can replace the Gaussian increments ΔW_t^* and $\Delta Z_t^{(j)*}$ in (36) by other random variables $\Delta\tilde{Z}_t^{(i)}$ ($i = 1, 2, 3$) with similar moment properties. We can thus obtain a simpler scheme by choosing more easily generated noise increments. This leads to the *simplified weak Euler scheme*

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b_1(Y_n)\Delta\tilde{Z}_t^{(1)} + b_2(Y_n)\Delta\tilde{Z}_t^{(2)} + b_3(Y_n)\Delta\tilde{Z}_t^{(3)}, \tag{37}$$

where the $\Delta\tilde{Z}_t^{(i)}$ for $i = 1, 2, \ldots, m$ (here $m = 3$) must be independent measurable random variables with moments satisfying the convergence condition:

$$|\mathbb{E}(\Delta\tilde{Z}_t^{(i)})| + |\mathbb{E}((\Delta\tilde{Z}_t^{(i)})^3)| + |\mathbb{E}((\Delta\tilde{Z}_t^{(i)})^2) - \Delta| \leq C\Delta^2 \tag{38}$$

for some constant C, and this is from [14]. A very simple example of such $\Delta\tilde{Z}_t^{(i)}$ in (37) are two-point distributed random variables with

$$P\left(\Delta\tilde{Z}_t^{(i)} = \pm\sqrt{\Delta}\right) = \frac{1}{2}.$$

3.1.4. *Order 2.0 Weak Taylor Scheme.* First, we need to introduce the stochastic Taylor expansions. From [14] we denote the following notations. The *multi-index* $\alpha = (j_1, j_2, \dots, j_l)$ is a row vector with $j_i \in \{0, 1, \dots, m\}$ for $i \in \{1, 2, \dots, l\}$ and $m = 1, 2, 3, \dots$. The length of α is $l = l(\alpha) \in \{1, 2, \dots\}$. The vector ν denotes the multi-index of length zero, which means $l(\nu) = 0$. In addition, the number $n(\alpha)$ denotes the number of components of a multi-index α which are equal to 0. We denote the set of all multi-indices by $\mathcal{M} = \{(j_1, j_2, \dots, j_l) : j_i \in \{0, 1, \dots, m\}, i \in \{1, \dots, l\}\} \cup \{\nu\}$, for $l = 1, 2, 3, \dots$.

For adapted right continuous stochastic processes $f(t)$, we can define certain function spaces $\mathcal{H}_\alpha$. The first such is the totality of all such processes, which is $\mathcal{H}_\nu$. It contains all the f with $|f(t)|$ being almost surely finite, for each $t \geq 0$. The second space, $\mathcal{H}_{(0)}$, is the subspace of $\mathcal{H}_\nu$ consisting of those f with

$$\int_0^t |f(s)|\, ds < \infty \tag{39}$$

almost surely, for every $t \geq 0$. And the third space, $\mathcal{H}_{(j)}$ with $j \neq 0$, is the subspace of $\mathcal{H}_\nu$ consisting of those f with

$$\int_0^t |f(s)|^2\, ds < \infty \tag{40}$$

almost surely, for every $t \geq 0$.

Let ρ and τ be two stopping times with $0 \leq \rho(\omega) \leq \tau(\omega) \leq T$, w.p. 1. Then the *multiple Itô integral* $I_\alpha[f(\cdot)]_{\rho,\tau}$ is defined by

$$I_\alpha[f(\cdot)]_{\rho,\tau} := \begin{cases} f(\tau) & : l = 0 \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s} ds & : l \geq 1 \text{ and } j_l = 0 \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s} dW_s^{j_l} & : l \geq 1 \text{ and } j_l \geq 1, \end{cases} \tag{41}$$

and $\alpha-$ denotes α with its last component j_l removed. Now $\mathcal{H}_\alpha$ with $\alpha \in \mathcal{M}$ and length $l(\alpha) > 1$, is considered recursively by

$$I_{\alpha-}[f(\cdot)]_{0,t} \in \mathcal{H}_{(j_l)} \tag{42}$$

almost surely, for every $t \geq 0$.

To define the Itô Taylor expansion we also need to learn the Itô coefficient functions. There are two types of differential operators related to a SDE. These are, for the general SDE

$$du_t = a(t, u_t)\,dt + b(t, u_t)\,dW_t, \tag{43}$$

we have

$$L^0 = \frac{\partial}{\partial t} + \sum_k a^k \frac{\partial}{\partial u^k} + \frac{1}{2}\sum_{j,k,l} b^{kj} b^{lj} \frac{\partial^2}{\partial u^k \partial u^l} \tag{44}$$

and

$$L^j = \sum_k b^{kj} \frac{\partial}{\partial u^k}. \tag{45}$$

For each $\alpha = (j_1, \dots, j_l)$, we define $L^\alpha = L^{j_1} \cdots L^{j_l}$, and $f_\alpha = L^\alpha = L^\alpha f$, $f_\nu = f$.

The Itô-Taylor expansion is considered for the Itô process

$$X_t = X_{t_0} + \int_{t_0}^t a(s, X_s)\,ds + \sum_{j=1}^m \int_{t_0}^t b^j(s, X_s)\,dW_s^j, \tag{46}$$

with $t_0 \le t \le T$, the equivalent, integral form of SDE above. Using the previous contents in this section, let's define a hierarchical set $\mathcal{A} \subset \mathcal{M}$ as a nonempty set of multiindices such that $\sup_{\alpha \in \mathcal{A}} l(\alpha) < \infty$, and $-\alpha \in \mathcal{A}$ whenever $\alpha \neq \nu$ is in $\mathcal{A}$. The remainder set $\mathcal{B}(\mathcal{A})$ consists of all those α not in $\mathcal{A}$ such that $-\alpha$ is in $\mathcal{A}$. Finally, we get the Itô Taylor expansion of the function f applied to a solution X of (46):

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}} I_\alpha[f_\alpha(\rho, X_\rho)]_{\rho,\tau} + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(\cdot, X_\cdot)]_{\rho,\tau}, \tag{47}$$

and for $\gamma = 1, 2, \dots$, we denote by $\mathcal{A}_\gamma$ the hierarchical set consisting of all as of length at most γ, and we call the Itô Taylor expansion with $\mathcal{A} = \mathcal{A}_\gamma$ the (weak) Itô Taylor expansion to order γ. A weak Taylor scheme is simply the strong Taylor-expansion with high order terms truncated. To see the derivation, refer to [14].

Now we can consider the *order 2.0 weak Taylor scheme*, which is obtained by adding all of the double stochastic integrals from the Itô-Taylor expansion (47) to the Euler scheme.

Applying the Itô Taylor expansion (47) in the case $d = 2$, $m = 3$ for $f \equiv y$ (or x), we obtain the following for the CAM model:

$$\begin{aligned} Y_{t+\Delta} = Y_t &+ a\Delta + b_1 \Delta W^* + b_2 \Delta Z^{(1)*} + b_3 \Delta Z^{(2)*} + L^1 b_1 I_{(1,1)} \\ &+ L^2 b_1 I_{(2,1)} + L^3 b_1 I_{(3,1)} + L^1 b_2 I_{(1,2)} + L^2 b_2 I_{(2,2)} + L^3 b_2 I_{(3,2)} \\ &+ L^1 b_3 I_{(1,3)} + L^2 b_3 I_{(2,3)} + L^3 b_3 I_{(3,3)} + L^0 b_1 I_{(0,1)} + L^0 b_2 I_{(0,2)} \\ &+ L^0 b_3 I_{(0,3)} + L^1 a I_{(1,0)} + L^2 a I_{(2,0)} + L^3 a I_{(3,0)} \\ &+ \tfrac{1}{2} L^0 a \Delta^2 + R_2^\Delta(t), \end{aligned} \tag{48}$$

$$X_{t+\Delta} = X_t + a_0\Delta + b_0\Delta W^* + L^1 b_0 I_{(1,1)} + L^2 b_0 I_{(2,1)} + L^3 b_0 I_{(3,1)} \\ + L^0 b_0 I_{(0,1)} + L^1 a_0 I_{(1,0)} + L^2 a_0 I_{(2,0)} + L^3 a_0 I_{(3,0)} + \tfrac{1}{2} L^0 a_0 \Delta^2 + \tilde{R}_2^{\Delta}(t), \tag{49}$$

where $R_2^{\Delta}(t)$ and $\tilde{R}_2^{\Delta}(t)$ are remainders. The differential operators here are

$$L^0 = \tfrac{\partial}{\partial t} + rX_t \tfrac{\partial}{\partial X_t} + \alpha(m - Y_t)\tfrac{\partial}{\partial Y_t} + \tfrac{1}{2} e^{2Y_t} X_t^2 \tfrac{\partial^2}{\partial X_t^2} + \tfrac{1}{2}(\beta\rho_1 + \gamma Y_t \rho_2)^2 \tfrac{\partial^2}{\partial X_t^2} \\ + e^{Y_t} X_t (\beta\rho_1 + \gamma Y_t\rho_2) \tfrac{\partial^2}{\partial X_t \partial Y_t} + \tfrac{1}{2}(\beta\sqrt{1-\rho_1^2} + \gamma Y_t \tfrac{\rho_3 - \rho_1\rho_2}{\sqrt{1-\rho_1^2}})^2 \tfrac{\partial}{\partial Y_t^2} \\ + \tfrac{1}{2}\gamma^2 Y_t^2 (1 - \rho_2^2 - \tfrac{(\rho_3-\rho_1\rho_2)^2}{1-\rho_1^2}) \tfrac{\partial^2}{\partial Y_t^2}, \tag{50}$$

$$L^1 = e^{Y_t} X_t \frac{\partial}{\partial X_t} + (\beta\rho_1 + \gamma Y_t \rho_2)\frac{\partial}{\partial Y_t}, \tag{51}$$

$$L^2 = \left(\beta\sqrt{1-\rho_1^2} + \gamma Y_t \frac{\rho_3 - \rho_1\rho_2}{\sqrt{1-\rho_1^2}}\right)\frac{\partial}{\partial Y_t}, \tag{52}$$

and

$$L^3 = \gamma Y_t \sqrt{1 - \rho_2^2 - \frac{(\rho_3 - \rho_1\rho_2)^2}{1-\rho_1^2}} \frac{\partial}{\partial Y_t}. \tag{53}$$

We have the multiple Itô integrals involving different components of the Wiener process, which is not easy to generate in reality. Under weak convergence, we can still use some $\Delta\tilde{Z}_t^{(i)}$ to replace ΔW_t^* and $\Delta Z_t^{(j)*}$, use $\frac{1}{2}\Delta\tilde{Z}_t^{(i)}\Delta$ to replace $I_{(0,i)}$ and $I_{(i,0)}$. The last type of multiple integrals $I_{(i_1,i_2)}$ can be replaced by $\frac{1}{2}(\Delta\tilde{Z}_t^{(i_1)}\Delta\tilde{Z}_t^{(i_2)} + V_{i_1,i_2})$. Here the $\Delta\tilde{Z}_t^{(i)}$ for $i = 1, 2, 3$ are independent random variables satisfying the moment conditions explained in [14] and three-point distributed with

$$P(\Delta\tilde{Z}_t^{(i)} = \pm\sqrt{3\Delta}) = \frac{1}{6}, \;\; P(\Delta\tilde{Z}_t^{(i)} = 0) = \frac{2}{3}. \tag{54}$$

And the independent variables V_{i_1,i_2} are in a two-point distribution with

$$P(V_{i_1,i_2}) = \frac{1}{2} \tag{55}$$

for $i_2 = 1, \ldots, i_1 - 1$,

$$V_{i_1,i_2} = -\Delta \tag{56}$$

and

$$V_{i_2,i_1} = -V_{i_1,i_2} \tag{57}$$

for $i_2 = i_1 + 1, \ldots, m$ and $i_1 + 1, \ldots, m$. Finally, the convergence of this weak order 2.0 scheme was proved by [14].

3.1.5. *A Stochastic Adams-Bashforth Scheme.* The Stochastic Adams-Bashforth (SAB) scheme can be represented in several versions. The simplest one is for the ordinary differential equation $\phi' = F(\phi)$:

$$\phi_{n+1} = \phi_n + \frac{\Delta t}{2}[3F(\phi_n) - F(\phi_{n-1})], \tag{58}$$

and is of order Δt^2. The paper [15] listed a stochastic analog of the previous one with strong convergence. The version we applied here is a weak convergent form. This is derived from the order 2.0 Itô-Taylor expansion which is

$$\begin{aligned} U_{t+\Delta} &= U_t + \textstyle\sum_j b^j \Delta W^j + a\Delta + \sum_{j,k} L^j b^k I_{(j,k)} \\ &+ \textstyle\sum_k L^0 b^k I_{(0,k)} + \sum_j L^j a I_{(j,0)} + \frac{1}{2} L^0 a \Delta^2 + R_2^{\Delta}(t) \\ &= U_t + a\Delta + \tfrac{1}{2} L^0 a \Delta^2 + M^{\Delta}(t), \end{aligned} \tag{59}$$

where each coefficient is evaluated at the point (t, U_t), and each stochastic integral is from t to $t + \Delta$, $\Delta = \Delta t$.

We can also apply the Itô-Taylor expansion for the coefficient a in orders 1 and 0:

$$a(t + \Delta, U_{t+\Delta}) = a + L^0 a\Delta + N^{\Delta}(t), \tag{60}$$

where$N^{\Delta}(t) = \sum_j L^j a \Delta W^j + R_1^{\Delta}(t)$, and

$$L^0 a(t + \Delta, U_{t+\Delta}) = L^0 a + P^{\Delta}(t), \tag{61}$$

where $P^{\Delta}(t) = R_0\Delta(t)$.

We combine these results to yield, for any η and θ,

$$\begin{aligned} U_{t+\Delta} &= U_t + [\eta a(t + \Delta, U_{t+\Delta}) + (1 - \eta)a]\Delta \\ &+ (\tfrac{1}{2} - \eta)[\theta L^0 a(t + \Delta, U_{t+\Delta}) + (1 - \theta)L^0 a]\Delta^2 \\ &- \eta\Delta N^{\Delta}(t) - (\tfrac{1}{2} - \eta)\theta\Delta^2 P^{\Delta}(t) + M^{\Delta}(t). \end{aligned} \tag{62}$$

So, if $t = t_n$, $\Delta = 2\Delta t$, $\eta = \theta = 0$, and writing U_n for U_{t_n},

$$U_{n+2} = U_n + 2a(t_n, U_n)\Delta t + 2L^0 a(t_n, U_n)\Delta t^2 + M^{2\Delta t}(t_n), \tag{63}$$

and if $t = t_n$, $\Delta = \Delta t$, $\eta = -\frac{3}{2}$, and $\theta = 0$,

$$\begin{aligned} U_{n+1} &= U_n - \tfrac{3}{2} a(t_{n+1}, U_{n+1})\Delta t + \tfrac{5}{2} a(t_n, U_n)\Delta t \\ &+ 2L^0 a(t_n, U_n)\Delta t^2 + \tfrac{3}{2} N^{\Delta t}(t_n)\Delta t + M^{\Delta t}(t_n). \end{aligned} \tag{64}$$

Hence,

$$\begin{aligned} U_{n+2} &= U_{n+1} + (U_{n+2} - U_n) - (U_{n+1} - U_n) \\ &= U_{n+1} + [\tfrac{3}{2} a(t_{n+1}, U_{n+1}) - \tfrac{1}{2} a(t_n, U_n)]\Delta t \\ &- \tfrac{3}{2}\Delta t N^{\Delta t}(t_n) + (M^{2\Delta t}(t_n) - M^{\Delta t}(t_n)). \end{aligned} \tag{65}$$

So, we will consider the following version of a SAB scheme:

$$Y_{n+2} = Y_{n+1} + \left[\frac{3}{2}a(t_{n+1}, Y_{n+1}) - \frac{1}{2}a(t_n, Y_n)\right]\Delta t + B_{n+1}(t_{n+1}, Y_{n+1}), \tag{66}$$

in which

$$B_{n+1}(t,x) = \sum_j b^j(t,x)\Delta W_t^j + \sum_j L^0 b^j(t,x) I_{(0,j)} + \sum_j L^j a(t,x) I_{(j,0)} + \sum_{j,k} L^j b^k(t,x) I_{(j,k)}, \tag{67}$$

where the random intervals are evaluated over the interval from t_{n+1} to t_{n+2}. This was proved to be convergent by [15]. The exact scheme for CAM model is:

$$\begin{aligned} Y_{t+2\Delta} = Y_{t+\Delta} &+ \left[\tfrac{3}{2}a(t+\Delta, Y_{t+\Delta}) - \tfrac{1}{2}a(t, Y_t)\right]\Delta + b_1 \Delta W_t^* + b_2 \Delta Z_t^{(1)*} \\ &+ b_3 \Delta Z_t^{(2)*} + L^0 b_1 I_{(0,1)} + L^0 b_2 I_{(0,2)} + L^0 b_3 I_{(0,3)} + L^1 a I_{(1,0)} + L^2 a I_{(2,0)} \\ &+ L^3 a I_{(3,0)} + L^1 b_1 I_{(1,1)} + L^1 b_2 I_{(1,2)} + L^1 b_3 I_{(1,3)} + L^2 b_1 I_{(2,1)} + L^2 b_2 I_{(2,2)} \\ &+ L^2 b_3 I_{(2,3)} + L^3 b_1 I_{(3,1)} + L^3 b_2 I_{(3,2)} + L^3 b_3 I_{(3,3)}, \end{aligned} \tag{68}$$

$$\begin{aligned} X_{t+2\Delta} = X_{t+\Delta} &+ \left[\tfrac{3}{2}a_0(t+\Delta, X_{t+\Delta}) - \tfrac{1}{2}a_0(t, X_t)\right]\Delta + b_0 \Delta W_t^* + L^0 b_0 I_{(0,1)} \\ &+ L^1 a_0 I_{(1,0)} + L^2 a_0 I_{(2,0)} + L^3 a_0 I_{(3,0)} + L^1 b_0 I_{(1,1)} + L^2 b_0 I_{(2,1)} + L^3 b_0 I_{(3,1)}. \end{aligned} \tag{69}$$

Here ΔW_t^* and $\Delta Z_t^{(j)*}$ can also be replaced by $\Delta \tilde{Z}_t^{(i)}$, and the multiple Itô integrals can also be replaced by the simple three-point distributed random variables as mentioned before.

4. The Martingale Control Variate Method for Option Pricing Under CAM Model

Under a risk-neutral pricing probability $\mathbb{P}^*$ parametrized by the combined volatility risk related terms $\Lambda_1(y)$ and $\Lambda_2(y)$, we consider the following CAM model:

$$dX_t = rX_t\,dt + \sigma_t X_t\,dW_t^*, \;\; \sigma_t = f(Y_t), \tag{70}$$

$$dY_t = \left[\frac{1}{\varepsilon}c_1(Y_t) - \frac{g_1(Y_t)}{\sqrt{\varepsilon}}\Lambda_1(Y_t) - \frac{g_2(Y_t)}{\sqrt{\varepsilon}}\Lambda_2(Y_t)\right]dt + \frac{g_1(Y_t)}{\sqrt{\varepsilon}}\,d\hat{Z}_t^{(1)*} + \frac{g_2(Y_t)}{\sqrt{\varepsilon}}Y_t\,d\hat{Z}_t^{(2)*}, \tag{71}$$

where

$$\hat{Z}_t^{(1)*} = \rho_1 W_t^* + \sqrt{1-\rho_1^2}\,Z_t^{(1)*}, \tag{72}$$

$$\hat{Z}_t^{(2)*} = \rho_2 W_t^* + \frac{\rho_3 - \rho_1\rho_2}{\sqrt{1-\rho_1^2}}Z_t^{(1)*} + \sqrt{1 - \rho_2^2 - \frac{(\rho_3-\rho_1\rho_2)^2}{1-\rho_1^2}}\,Z_t^{(2)*}. \tag{73}$$

Here X_t is the underlying asset price process with a constant risk-free interest rate r as explained before. And $c_1(Y_t) = (m - y)$, $\alpha = \frac{1}{\varepsilon}$, $\beta = \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}$, $\gamma = \frac{\chi}{\sqrt{\varepsilon}}$, $g_1(y) = \nu\sqrt{2}$, $g_2(y) = \chi$.

Given the CAM model, the price of a plain European option with the integrable payoff function H and expiry T is given by

$$P^\varepsilon(t, x, y) = \mathbb{E}^*_{t,x,y}\{e^{-r(T-t)}H(X_T)\}, \tag{74}$$

where $\mathbb{E}^*_{t,x,y}$ denotes the expectation with respect to $\mathbb{P}^*$ conditioned on the current states $X_t = x, Y_t = y$. A basic Monte Carlo simulation estimates the option price $P(0, S_0, Y_0)$ at time 0 by

$$\frac{1}{N}\sum_{i=1}^{N} e^{-rT}H(X_T^{(i)}), \tag{75}$$

where N is the total number of independent sample paths and $X_T^{(i)}$ denotes the i-th simulated stock price at time T.

Assuming that the European option price $P(t, X_t, Y_t)$ is smooth enough, we apply Itô's lemma to its discounted price $e^{-rt}P$, and then integrate from time 0 to the maturity T. The following martingale representation is obtained

$$P^\varepsilon(0, X_0, Y_0) = e^{-rT}H(X_T) - \mathcal{M}_0(P) - \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_1(P) - \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_2(P), \tag{76}$$

where centered martingales are defined by

$$\mathcal{M}_0(P) = \int_0^T e^{-rs}\frac{\partial P^\varepsilon}{\partial x}\sigma_t X_t\, dW_t^*, \tag{77}$$

$$\mathcal{M}_1(P) = \int_0^T e^{-rs}\frac{\partial P^\varepsilon}{\partial y}\nu\sqrt{2}\, d\hat{Z}_t^{(1)*}, \tag{78}$$

$$\mathcal{M}_2(P) = \int_0^T e^{-rs}\frac{\partial P^\varepsilon}{\partial x}\chi Y_t\, d\hat{Z}_t^{(2)*}. \tag{79}$$

The martingales play the role of "perfect" control variates for Monte Carlo simulations and their integrands would be the perfect Delta hedges if P were known and volatility factors were traded. Like mentioned in [17], neither P nor its gradient at any time $0 \le s \le T$ are in any analytic form even though all the parameters of the model have been calibrated as we suppose here. We can approximate the option price and substitute for P in the martingales above and still retain martingale properties. The approximation of the Black-Scholes type is derived in[18] for continuous payoffs:

$$P^\varepsilon(t, x, y) \approx P_{BS}(t, x; \bar{\sigma}). \tag{80}$$

We denote by $P_{BS}(t,x;\bar{\sigma})$ the solution of the Black-Scholes partial differential equation with the terminal condition $P_{BS}(T,x) = H(x)$. The average volatility $\bar{\sigma}$ is defined by

$$\bar{\sigma} = \exp(-m). \tag{81}$$

Note that the approximate option price $P_{BS}(t,x;\bar{\sigma})$ is independent of the variable y. A martingale variate estimator is formulated as

$$\frac{1}{N}\sum_{i=1}^{N}[e^{-rT}H(X_T^{(i)}) - \mathcal{M}_0^{(i)}(P_{BS})], \tag{82}$$

where

$$\mathcal{M}_0(P_{BS}) = \int_0^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(x, X_s;\bar{\sigma})f(Y_s)X_s\,dW_s^*.$$

5. Variance Analysis of Martingale Control Variates

For the sake of simplicity, we first assume that the instant correlation coefficients, ρ_1, ρ_2 and ρ_3 in (71), (72) and (73), are zero. From (76), the variance of the controlled payoff

$$e^{-rT}H(S_T) - \mathcal{M}_0(P_{BS}) \tag{83}$$

is simply the sum of quadratic variations of martingales:

$$\begin{aligned} &Var(e^{-rT}H(S_T) - \mathcal{M}_0(P_{BS})) \\ &= \mathbb{E}^*_{0,t,x,y}\{\textstyle\int_0^T e^{-2rs}(\frac{\partial P}{\partial x} - \frac{\partial P_{BS}}{\partial x})^2(s,S_s,Y_s)f^2(Y_s)S_s^2\,ds \\ &\quad+\tfrac{1}{\varepsilon}\textstyle\int_0^T e^{-2rs}(\frac{\partial P}{\partial y})^2 2\nu^2\,ds \\ &\quad+\tfrac{1}{\varepsilon}\textstyle\int_0^T e^{-2rs}(\frac{\partial P}{\partial y})^2\chi^2Y_s^2\,ds\}. \end{aligned} \tag{84}$$

Theorem 1.1. *Under the assumptions made above and the payoff function H being continuous piecewise smooth as a call (or a put), for any fixed initial state (0,x,y), there exists a constant $C > 0$ such that for $\varepsilon \leq 1$,*

$$Var(e^{-rT}H(S_T - \mathcal{M}_0(P_{BS})) \leq C\varepsilon.$$

The proof of Theorem 1.1 is given in the Appendix B. The proof is from the similar procedure given in Fouque's paper [17].

6. Numerical Results

The numerical experiments are implemented to illustrate that the martingale control variate method is efficient and robust for European option problems under CAM model with its relevant parameters and initial values specified in Table 1 and Table 2.

TABLE 1. Parameters used in the CAM model.

r	m	β	ρ_1	ρ_2	ρ_3	Φ	$f(y)$
0.1	-2.6	1	-0.5	-0.7	0.5	0	$\exp(y)$

TABLE 2. Initial conditions and call option parameters.

$\$X_0$	Y_0	$\$K$	T years
110	-2.32	100	1

Compared to plain Monte Carlo simulations, significant variance reduction ratios for European options are obtained. These results confirm the robustness of our method based on martingale control variates constructed as in delta hedging strategies. The effectiveness of our method depends on option price approximations to the pricing problem considered. Results of variance reduction under the four different schemes are illustrated in Table 3 – Table 6 with various parameters α and γ. The time step size for all the schemes is $\Delta t = 10^{-3}$ and the number of realizations is $N = 10,000$. Figure 1 – Figure 4 present sampled European option prices with respect to the number of realizations. The dash line correponds to basic Monte Carlo simulations, while the dot line corresponds to the same Monte Carlo simulations using the martingale control variate $\mathcal{M}_0(P_{BS})$. Figure 5 – Figure 8 present standard deviation of simulated option prices with respect to the number of realizations. The dash line corresponds to basic Monte Carlo simulations, while the dot line corresponds to the same Monte Carlo simulations using the martingale control variate method. The results confirm that the standard deviation under control variate method converges faster.

7. CONCLUSION

In this paper we have presented the application of a Coupled Additive-Multiplicative Noise model in option pricing. We have focused our attention on four different schemes: Euler scheme, simplified weak Euler scheme, order 2.0 weak Taylor scheme and SAB scheme. The effectiveness of the four schemes is presented. A martingale control variate method is proposed to price European options by Monte Carlo simulations. The size of the variance reduction by this generic control variate method has been characterized by a theoretical variance analysis. We also obtain the significant variance reduction ratio by comparing to the results from plain Monte Carlo simulations. The results confirm the practical application of the CAM model and the robustness of the martingale control variates method constructed as in delta hedging strategies.

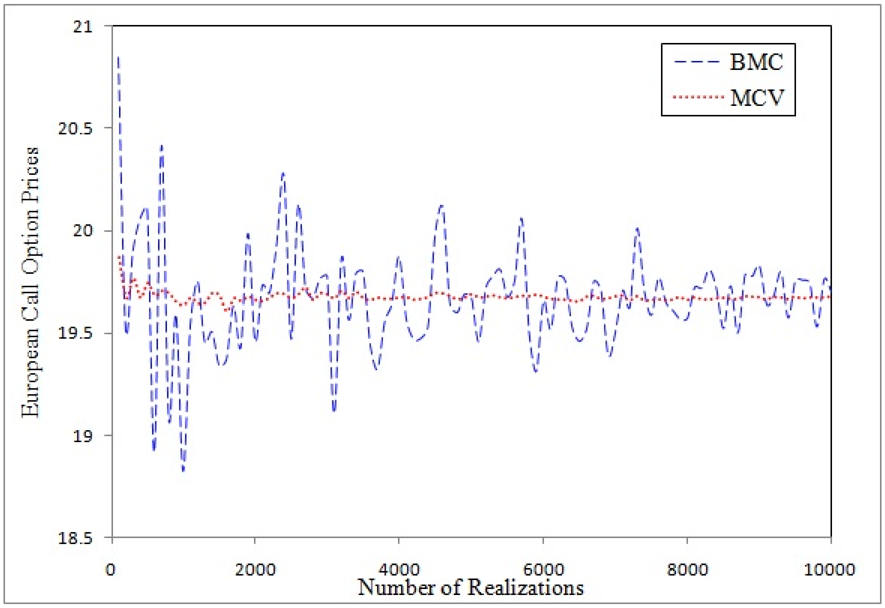

FIGURE 1. Implicit Euler scheme: Monte Carlo simulations under implicit Euler scheme for a European call option price when $\alpha = 3$ and $\gamma = 1$. Sampled prices are obtained along the number of realizations.

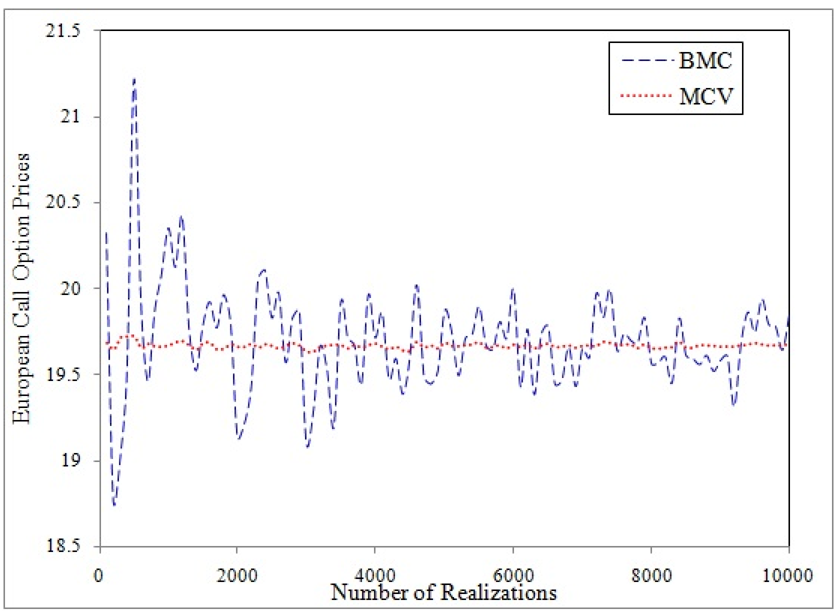

FIGURE 2. Implicit Euler scheme with coin flips

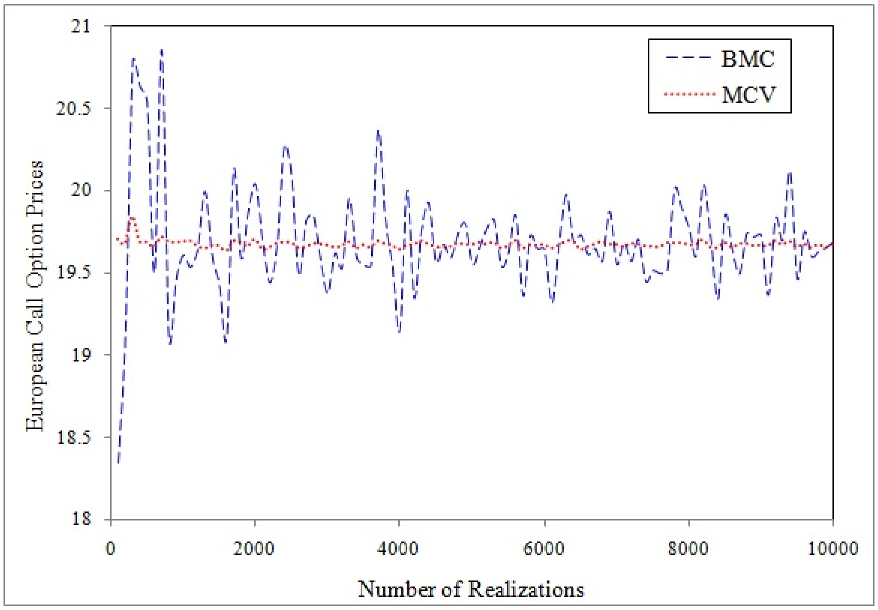

Figure 3. Weak order 2.0 scheme

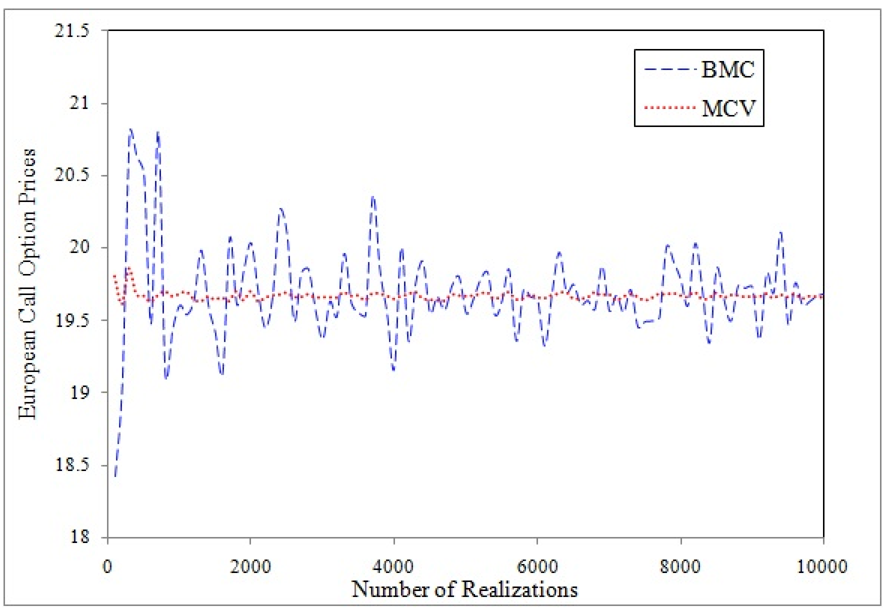

Figure 4. SAB scheme

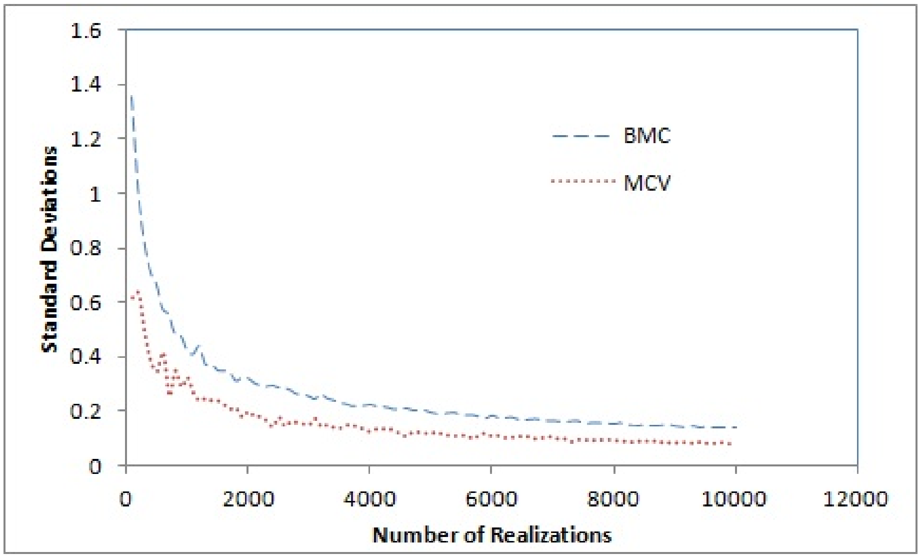

FIGURE 5. Implicit Euler scheme: Monte Carlo simulations under implicit Euler scheme for a European call option price when $\alpha = 3$ and $\gamma = 1$. Standerd error are obtained along the number of realizations.

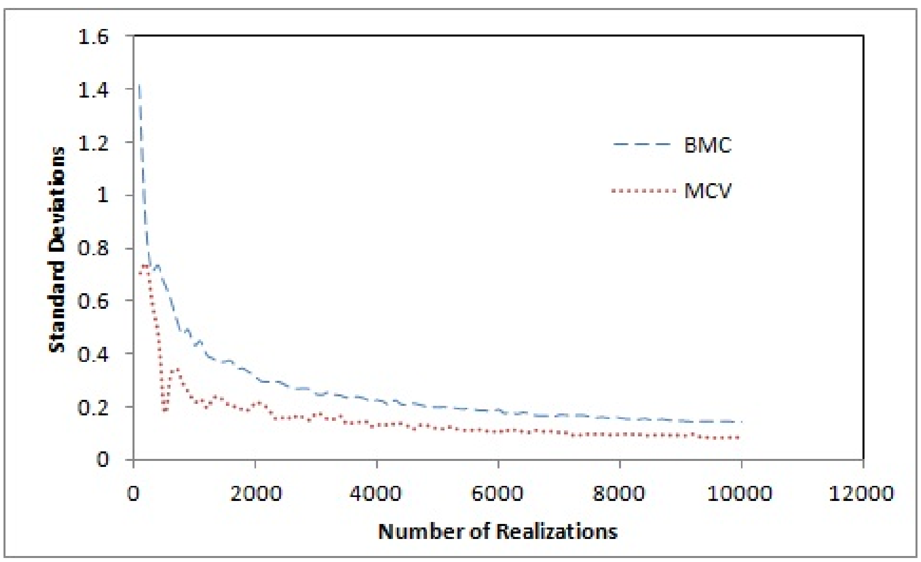

FIGURE 6. Implicit Euler scheme with coin flips

TABLE 3. Implicit Euler Scheme: Comparison of standard errors with various α and γ. The notation Std^{BMC} stands for the standard error estimated from basic Implicit Euler Scheme Monte Carlo simulations, and Std^{MCV} the standard error from the same Monte Carlo simulations but using the martingale control variate. Numbers within parenthesis in the third and fourth columns are sample means estimated from the two Monte Carlo methods, respectively. The fifth column records the variance reduction ratio, which is calculated by $(Std^{BMC}/Std^{MCV})^2$.

α	γ	Std^{BMC}	Std^{MCV}	Variance Reduction Ratio
0.03	0.01	0.1317(20.7965)	0.1002(20.8989)	1.73
0.3	0.1	0.1218(20.2950)	0.0941(20.3741)	1.68
3	1	0.1415(19.5385)	0.0893(19.6697)	2.51

TABLE 4. Implicit Euler Scheme with coin flips

α	γ	Std^{BMC}	Std^{MCV}	Variance Reduction Ratio
0.03	0.01	0.1329(21.2100)	0.0895(20.8610)	2.21
0.3	0.1	0.1226(20.6810)	0.0843(20.3546)	2.12
3	1	0.1417(20.0061)	0.0786(19.6817)	3.25

TABLE 5. Weak order 2.0 scheme

α	γ	Std^{BMC}	Std^{MCV}	Variance Reduction Ratio
0.03	0.01	0.1319(20.8082)	0.1027(20.9511)	1.65
0.3	0.1	0.1214(20.2783)	0.0962(20.4159)	1.59
3	1	0.1446(19.7554)	0.0850(19.6829)	2.89

TABLE 6. SAB Scheme

α	γ	Std^{BMC}	Std^{MCV}	Variance Reduction Ratio
0.03	0.01	0.1318(20.8002)	0.1027(20.9431)	1.65
0.3	0.1	0.1213(20.2724)	0.0962(20.4097)	1.59
3	1	0.1445(19.7457)	0.0852(19.6734)	2.88

APPENDIX A. WHEN DOES $\mathbb{E}Y_t^n$ STAY BOUNDED AS $t \to \infty$?

For $n = 1$, the solution of the CAM process (19) is explicitly given in terms of its (assumed known) starting value y by

$$Y_t = y + \int_0^t \alpha(m - Y_s)\, ds + \int_0^t \beta\, d\hat{Z}^{(1)} + \int_0^t \gamma Y_s\, dW^{(2)}. \tag{85}$$

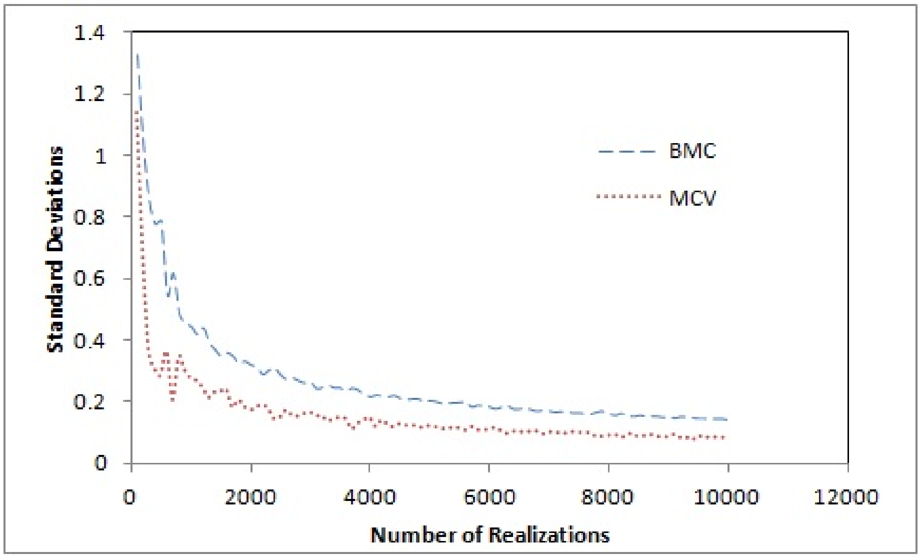

FIGURE 7. Weak order 2.0 scheme

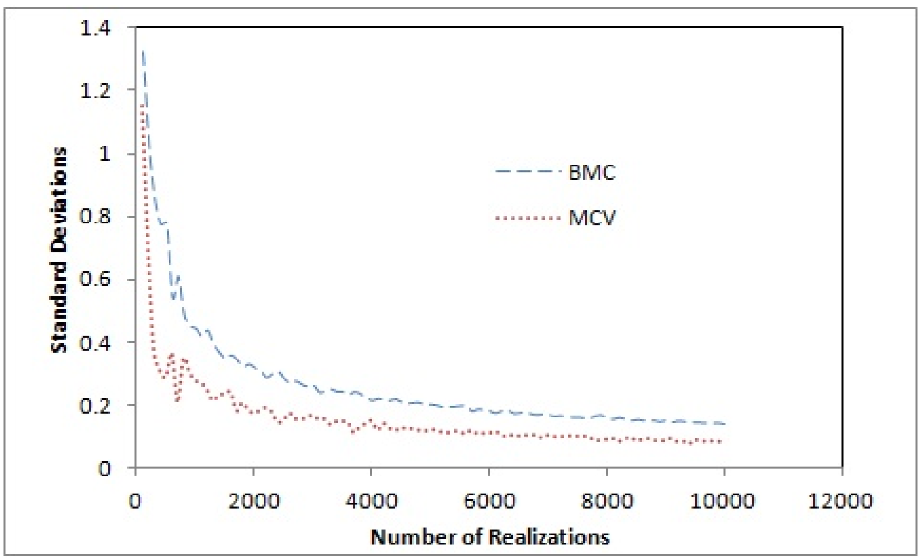

FIGURE 8. SAB scheme

Taking expectations for this solution (85) will give

$$\mathbb{E}Y_t = \mathbb{E}y + \mathbb{E}\int_0^t \alpha(m - Y_s)\, ds, \tag{86}$$

so

$$\frac{\mathbb{E}Y_t}{dt} = \mathbb{E}[\alpha(m - Y_t)] \tag{87}$$

then

$$\mathbb{E}Y_t = m + Ce^{-\alpha t}. \tag{88}$$

The condition for $\mathbb{E}Y_t$ stay bounded is $\alpha \geq 0$.

For $n = 2$, the solution of the stochastic differential equation

$$dY_t^2 = 2Y_t\, dY_t + \tfrac{1}{2}(2) dY_t \cdot dY_t$$

$$= 2Y_t[\alpha(m - Y_t)\, dt + \beta\, d\hat{Z}^{(1)} + \gamma Y_t\, d\hat{Z}^{(1)}] + \beta^2\, dt + \gamma^2 Y_t^2\, dt + 2\beta\gamma Y_t \rho_2\, dt \tag{89}$$

$$= [2\alpha Y_t(m - Y_t) + \beta^2 + \gamma^2 Y_t^2 + 2\beta\gamma Y_t \rho_2]\, dt + 2\beta Y_t\, d\hat{Z}^{(1)} + 2\gamma Y_t^2\, d\hat{Z}^{(2)}$$

is

$$\begin{aligned} Y_t^2 = y^2 + \textstyle\int_0^t [(\gamma^2 - 2\alpha) Y_s^2 + (2\alpha m + 2\beta\rho_2\gamma) Y_s + \beta^2]\, ds \\ + \textstyle\int_0^t 2\beta Y_s\, d\hat{Z}^{(1)} + \int_0^t 2\gamma Y_s^2\, d\hat{Z}^{(2)}. \end{aligned} \tag{90}$$

Taking expectations on both sides of this solution we will get

$$\mathbb{E}Y_t^2 = \mathbb{E}y^2 + \mathbb{E}\int_0^t [(\gamma^2 - 2\alpha) Y_s^2 + (2\alpha m + 2\beta\rho_2\gamma) Y_s + \beta^2]\, ds, \tag{91}$$

so

$$\begin{aligned} \tfrac{d\mathbb{E}Y_t^2}{dt} &= \mathbb{E}[(\gamma^2 - 2\alpha) Y_t^2 + (2\alpha m + 2\beta\rho_2\gamma) Y_t + \beta^2] \\ &= (\gamma^2 - 2\alpha)\mathbb{E}[Y_t^2] + (2\alpha m + 2\beta\rho_2\gamma)\mathbb{E}[Y_t] + \beta^2. \end{aligned} \tag{92}$$

The solution for this ordinary differential equation is

$$\mathbb{E}Y_t^2 = \frac{2\alpha m^2 + 2\beta\rho_2\gamma m + \beta^2}{2\alpha - \gamma^2} + \frac{(2\alpha m + 2\beta\rho_2\gamma)C}{\alpha - \gamma^2} \cdot e^{-\alpha t} + C' \cdot e^{-(2\alpha - \gamma^2)t}. \tag{93}$$

The condition for the moment to stay bounded as $t \to \infty$ is $\alpha \geq \frac{\gamma^2}{2}$.

For $n = 3$, the solution of the stochastic differential equation

(94)

$$dY_t^3 = 3Y_t^2\, dY_t + \tfrac{1}{2}(6Y_t) dY_t \cdot dY_t$$

$$= 3Y_t^2[\alpha(m - Y_t)\, dt + \beta\, d\hat{Z}^{(1)} + \gamma Y_t\, d\hat{Z}^{(2)}] + 3Y_t[\beta^2\, dt + \gamma^2 Y_t^2\, dt + 2\beta\rho_2\gamma Y_t\, dt]$$

$$= [(3\gamma^2 - 3\alpha) Y_t^3 + (3m\alpha + 6\beta\rho_2\gamma) Y_t^2 + 3\beta^2 Y_t]\, dt + 3\beta Y_t^2\, d\hat{Z}^{(1)} + 3\gamma Y_t^3\, d\hat{Z}^{(2)}$$

is

$$Y_t^3 = y^3 + \int_0^t \left[(3\gamma^2 - 3\alpha)Y_s^3 + (3m\alpha + 6\beta\rho_2\gamma)Y_s^2 + 3\beta^2 Y_s\right] ds + \int_0^t 3\beta Y_s^2 \, d\hat{Z}^{(1)} + \int_0^t 3\gamma Y_s^3 \, d\hat{Z}^{(2)}. \tag{95}$$

Taking expectations on both sides of the solution gives

$$\mathbb{E}Y_t^3 = \mathbb{E}y^3 + \mathbb{E}\int_0^t \left[(3\gamma^2 - 3\alpha)Y_s^3 + (3m\alpha + 6\beta\rho_2\gamma)Y_s^2 + 3\beta^2 Y_s\right] ds. \tag{96}$$

So the corresponding ordinary differential equation is

$$\frac{d\mathbb{E}Y_t^3}{dt} = 3(\gamma^2 - \alpha)\mathbb{E}[Y_t^3] + 3(m\alpha + 2\beta\rho_2\gamma)\mathbb{E}[Y_t^2] + 3\beta^2\mathbb{E}[Y_t]. \tag{97}$$

From the solution of the ordinary differential equation, the condition for not blowing up is $\alpha \geq \gamma^2$.

Similarly as before, when $n = 4$,

$$\frac{d\mathbb{E}Y_t^4}{dt} = (6\gamma^2 - 4\alpha)\mathbb{E}Y_t^4 + (4m\alpha + 12\beta\rho_2\gamma)\mathbb{E}Y_t^3 + 6\beta^2\mathbb{E}Y_t^2. \tag{98}$$

From the solution of this ordinary differential equation, the condition for not blowing up is $\alpha \geq \frac{3}{2}\gamma^2$.

Similarly, we can conclude that for any positive interger n, the condition under which the nth moment of Y_t does not blow up is $\alpha \geq \frac{(n-1)}{2}\gamma^2$. In our simulation, we want $n = 5$, so we use the relationship $\alpha \geq 2\gamma^2$.

Appendix B. Derivation of the accuracy of the variance analysis

In order to prove Theorem 1.1, we need the following three lemmas.

Lemma A.1. *Under the assumptions of Theorem 1.1, for any fixed initial state (0,x,y), there exists a positive constant* $C_1 > 0$ *such that for* $\varepsilon \leq 1$, *one has*

$$\mathbb{E}^*_{0,t,x,y}\left\{\int_0^T e^{-2rs}\left(\frac{\partial P^\varepsilon}{\partial x} - \frac{\partial P_{BS}}{\partial x}\right)^2 (s, S_s, Y_s) f^2(Y_s) S_s^2 \, ds\right\} \leq C_1\varepsilon.$$

Proof: By Cauchy-Schwartz inequality

$$\begin{aligned} &\mathbb{E}^*_{0,t,x,y}\{\textstyle\int_0^T e^{-2rs}(\frac{\partial P^\varepsilon}{\partial x} - \frac{\partial P_{BS}}{\partial x})^2(s, S_s, Y_s) f^2(Y_s) S_s^2 ds\} \\ &\leq \sqrt{\mathbb{E}^*\{\textstyle\int_0^T (\frac{\partial P^\varepsilon}{\partial x} - \frac{\partial P_{BS}}{\partial x})^4(s, S_s, Y_s)\} \, ds} \\ &\times \sqrt{\textstyle\int_0^T \mathbb{E}^*\{f^4(Y_s)(e^{-rs}S_s)^4\} \, ds}. \end{aligned} \tag{99}$$

The second factor on the right hand side is bounded by

$$\sqrt{\int_0^T \mathbb{E}^*\{f^4(Y_s)(e^{-rs}S_s)^4\}\,ds} \le C_f^2\sqrt{\int_0^T \mathbb{E}^*\{(e^{-rs}S_s)^4\}\,ds} \tag{100}$$

for some constant C_f, as the volatility function f is bounded. Using the notation $\sigma_t = f(Y_t)$ as in (70), and if $W_t^* = W$ for simplicity, one has

$$e^{-rs}S_s = S_0 e^{\int_0^s \sigma_u dW_u - \frac{1}{2}\int_0^s \sigma_u^2 du},$$

and therefore

$$\begin{aligned}\mathbb{E}^*\{(e^{-rs}S_s)^4\} &= S_0^4\mathbb{E}^*\{e^{\sigma\int_0^s \sigma_u^2 du} e^{\int_0^s 4\sigma_u dW_u - \frac{1}{2}\int_0^s 16\sigma_u^2 du}\}\\ &\le C_f' S_0^4 \mathbb{E}^*\{e^{\int_0^s 4\sigma_u dW_u - \frac{1}{2}\int_0^s 16\sigma_u^2 du}\} = C_f' S_0^4,\end{aligned}$$

where we have used again the boundness of f and the martingale property. Combined with (100) we obtain

$$\sqrt{\int_0^T \mathbb{E}^*\{f^4(Y_s)(e^{-rs}S_s)^4\}ds} \le C_2, \tag{101}$$

for some positive constant C_2.

In order to study the first factor on the right hand side of the inequality (99), we have to control the "delta" approximation, $\frac{\partial P^\varepsilon}{\partial x} \to \frac{\partial P_{BS}}{\partial x}$, as opposed to the option price approximation, $P^\varepsilon \to P_{BS}$, studied in [18] for European options, or in [1] for digital-type options.

By pathwise differentiation (see [19] for instance), the chain rule can be applied and we obtain

$$\frac{\partial P^\varepsilon}{\partial S_t}(t, S_t, Y_t) = \mathbb{E}^*\left\{ e^{-r(T-t)}\mathbb{I}_{\{S_T > K\}} \frac{\partial S_T}{\partial S_t}\middle| S_t, Y_t\right\}.$$

At time $t = 0$,

$$e^{-rT}\frac{\partial S_T}{\partial S_0} = e^{\int_0^T \sigma_t dW_t^* - \frac{1}{2}\int_0^T \sigma_t^2 dt} \tag{102}$$

gives an exponential martingale, and therefore one can construct a $\mathbb{P}^*$-equivalent probability measure $\tilde{\mathbb{P}}$ by Girsanov Theorem. As a result, the delta $\frac{\partial P^\varepsilon}{\partial S_t}(t, S_t, Y_t)$ has a probabilistic representation under the new measure $\tilde{\mathbb{P}}$ corresponding to the digital-type option

$$\frac{\partial P^\varepsilon}{\partial S_t}(t, S_t, Y_t) = \tilde{\mathbb{E}}\{\mathbb{I}_{\{S_T > K\}} \mid S_t, Y_t\},$$

where the dynamics of S_t become

$$dS_t = (r + f^2(Y_t))S_t dt + \sigma_t S_t d\tilde{W}_t,$$

with $\tilde{W}$ being a standard Brownian motion under $\tilde{\mathbb{P}}$. The dynamics of Y_t remain the same because we have assumed here zero correlation between Brownian motions. The one can apply the accuracy result in [1] for digital options to claim that

$$\left|\tilde{\mathbb{E}}\{\mathbb{I}_{\{S_T>K\}} \mid S_t, Y_t\} - \bar{\mathbb{E}}\{\mathbb{I}_{\{\bar{S}_T>K\}} \mid \bar{S}_t = S - t\}\right| \leq C_3(Y_t)\sqrt{\varepsilon},$$

where the constant C_3 may depend on Y_t, and the "homogenized" stock price $\bar{S}_t$ satisfies

$$d\bar{S}_t = (r + \bar{\sigma}^2)\bar{S}_t\, dt + \bar{\sigma}\bar{S}_t\, d\bar{W}_t$$

with $\bar{W}_t$ being a standard Brownian motion [1]. In fact, the homogenized approximation $\bar{E}\{\mathbb{I}_{\{\bar{S}_T>K\}} \mid \bar{S}_t\}$ is a probabilistic representation of the homogenized "delta", $\frac{\partial P_{BS}}{\partial x}$. Consequently, we obtain the accuracy result for delta approximation:

$$\left|\left(\frac{\partial P^\varepsilon}{\partial x} - \frac{\partial P_{BS}}{\partial x}\right)(t, S_t, Y_t)\right| \leq C_3(Y_t)\sqrt{\varepsilon}.$$

The existence of moments of Y_t ensures the existence of the fourth moment of $C_3(Y_t)$, and therefore the first factor on the right hand side of (99) is bounded by

$$\sqrt{\mathbb{E}^*\left\{\int_0^T \left(\frac{\partial P^\varepsilon}{\partial x} - \frac{\partial P_{BS}}{\partial x}\right)^4 (s, S_s, Y_s)\, ds\right\}} \leq C_4\varepsilon \tag{103}$$

for some positive constant C_4. From (99), (103) and (101), we conclude that

$$\mathbb{E}^*\left\{\int_0^T e^{-2rs}\left(\frac{\partial P^\varepsilon}{\partial x} - \frac{\partial P_{BS}}{\partial x}\right)^2 (s, S_s, Y_s) f^2(Y_s) S_s^2\, ds\right\} \leq C_1\varepsilon$$

for some constant C_1.

Lemma A.2. *Under the assumptions of Theorem 1.1 for any fixed initial state $(0, x, y)$, there exists ε a positive constant C such that for $\varepsilon \leq 1$, one has*

$$\int_0^T e^{-2rs}\left(\frac{\partial P^\varepsilon}{\partial y}\right)^2 (s, S_s, Y_s) g_1^2(Y_s)\, ds \leq C\varepsilon^2.$$

Proof: Conditioning on the path of the volatility process and by iterative expectations, the price of a European option can be expressed as

$$\begin{aligned} P^\varepsilon(t, x, y) &= \mathbb{E}^*_{t,x,y}\{\mathbb{E}^*\{e^{-r(T-t)}(S_T - K)^+ \mid \sigma_s, t \leq s \leq T\}\} \\ &= \mathbb{E}^*_{t,x,y}\{P_{BS}(t, x; K, T; \sqrt{\bar{\sigma}^2})\}, \end{aligned} \tag{104}$$

where the realized variance is denoted by $\bar{\sigma}^2$:

$$\bar{\sigma}^2 = \frac{1}{T - t}\int_t^T f(Y_s)^2 ds. \tag{105}$$

Taking a pathwise derivative for P^ε with respect to the fast varying variable y, we deduce by the chain rule

$$\frac{\partial P^\varepsilon}{\partial y}(t,x,y) = \mathbb{E}^*_{t,x,y}\left\{\frac{\partial P_{BS}}{\partial \sigma}(t,x;K,T;\sqrt{\bar\sigma^2(y)}\frac{\partial\sqrt{\bar\sigma^2}}{\partial y}\right\}. \tag{106}$$

Inside of the expectation the first derivative, known as Vega,

$$\frac{\partial P_{BS}}{\partial \sigma} = \frac{xe^{-d_1^2/2}\sqrt{T-t}}{2\pi},$$

with $d_1 = \frac{\log(x/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, is uniformly bounded in σ. Using the chain rule one obtains

$$\frac{\partial\sqrt{\bar\sigma^2}}{\partial y} = \frac{1}{(T-t)\sqrt{\bar\sigma^2}}\int_t^T\left[\frac{\partial f}{\partial y}(Y_s)\frac{\partial Y_s}{\partial y}\right]f(Y_s)\,ds. \tag{107}$$

In order to control the growth rate of $\frac{\partial Y_s}{\partial y}$ we consider its dynamics:

$$\frac{d}{dS}\left(\frac{\partial Y_s}{\partial y}\right) = \left[-\frac{1}{\varepsilon}+\frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\frac{\partial\Lambda_1}{\partial y}(Y_s)+\frac{\chi}{\sqrt{\varepsilon}}\frac{\partial\Lambda_2}{\partial y}(Y_s)\right]\frac{\partial Y_s}{\partial y} \tag{108}$$

with the initial condition $\frac{\partial Y_t}{\partial y} = 1$.

Rescaling the system (108) by defing $\tilde{Y}_s^\varepsilon = Y_{s\varepsilon}$, we deduce

$$\frac{d}{dS}\left(\frac{\partial\tilde{Y}_s^\varepsilon}{\partial y}\right) = -\frac{\partial\tilde{Y}_s^\varepsilon}{\partial y}+\sqrt{\varepsilon}\left(\nu\sqrt{2}\frac{\partial\tilde\Lambda_1(\tilde{Y}_s^\varepsilon)}{\partial y}+\chi\frac{\partial\tilde\Lambda_2(\tilde{Y}_s^\varepsilon)}{\partial y}\right)\frac{\partial\tilde{Y}_s^\varepsilon}{\partial y}.$$

The functions $\tilde\Lambda_1$ and $\tilde\Lambda_2$ are defined according to the rescaling and they are smooth and bounded as Λ's. By a classical stability result [20], we obtain $\left|\frac{\partial Y_s}{\partial y}\right| < C_5e^{-(s-t)/\varepsilon}$ for some constant C_5. Applying these estimates to (107) and by the smooth boundedness of f, we obtain

$$\frac{\partial\sqrt{\bar\sigma^2}}{\partial y} \le C\varepsilon$$

for some C, and consequently a similar bound for $\frac{\partial P^\varepsilon}{\partial y}(t,x,y)$ in (106). Finally, as $g_1 = \nu\sqrt{2}$, Lemma A.2 follows.

Lemma A.3. *Under the assumptions of Theorem 1.1, for any fixed initial state $(0,x,y)$, there exists a positive constant C such that for $\varepsilon \le 1$, one has*

$$\int_0^T e^{-2rs}\left(\frac{\partial P^\varepsilon}{\partial y}\right)^2 g_2^2(Y_s)\,ds \le C'\varepsilon$$

with $g_2(Y_t) = \mu Y_t$.

Proof: The proof is similar to Lemma A.2.

From the bounds in Lemma A.1, A.2 and A.3, we deduce Theorem 1.1.

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