WEAK VERSIONS OF STOCHASTIC ADAMS-BASHFORTH AND SEMI-IMPLICIT LEAPFROG SCHEMES FOR SDES

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Abstract. We consider the weak analogues of certain strong stochastic numerical schemes considered in [10], namely a Adams-Bashforth scheme and a semi-implicit leapfrog scheme. We show that the weak version of the Adams-Bashforth scheme converges weakly with order 2, and the weak version of the semi-implicit leapfrog scheme converges weakly with order 1. We also note that the weak schemes are computationally simpler and easier to implement than the corresponding strong schemes, resulting in savings in both programming and computational effort.

1. Introduction

A great deal of effort over the last several decades has been spent in studying numerical schemes to approximate the solutions of stochastic differential equations (SDEs)

$$dU_t = a(U_t)\,dt + b(U_t)\,dW_t \tag{1.1}$$

for $U_t \in \mathcal{R}^d$, a a function from $\mathcal{R}^d$ into itself, W a Wiener process on $\mathcal{R}^m$ and b a function from $\mathcal{R}^d$ into $\mathcal{R}^{d\times m}$.

There has been considerable research into strong stochastic multistep schemes; see, for example, [3], [4], [5], [6], [7]. On the other hand, weak multistep schemes seem to have attracted little attention. Weak schemes are generally simpler than corresponding strong schemes, requiring the generation of fewer stochastic increments and the calculation of fewer terms involving the drift and diffusion coefficients of the stochastic differential equation or their derivatives. Further, the terms that can be dropped when truncating from a strong to a weak scheme are usually among the more problematic terms computationally. Therefore, when the application only requires the approximation of moments of the solution of the SDE, it is appropriate and generally advantageous to consider weak schemes. This is often the case in certain applications, such as pricing of options

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in finance; or in geophysics, where in any case certain quantities are only to be predicted in a statistical sense.

In [10], the author along with Roger Temam investigated certain strong multistep schemes, in which the sample paths of solutions are approximated, that are related to certain multistep numerical methods that have been used for the approximation of deterministic equations of use in geophysics, specifically an Adams-Bashforth scheme and a semi-implicit leapfrog scheme. Here we consider corresponding weak schemes.

2. Formulations and preliminary results

We consider the stochastic differential equation (1.1). The stochastic version of the change of variables formula is the so-called Ito formula:

$$dF_t = \left[\frac{\partial F}{\partial t} + a^k(F_t)\frac{\partial F}{\partial u^k} + \frac{1}{2}b^{ij}(F_t)b^{kj}(F_t)\frac{\partial^2 F}{\partial u^i \partial u^k}\right] dt + b^{ij}(F_t)\frac{\partial F}{\partial u^i}\, dW_t, \tag{2.1}$$

for $F : \mathcal{R}^+ \times \mathcal{R}^d \to \mathcal{R}^d$; here we sum over repeated indices.

We will use the following notations from [12]: A multiindex α of length $\ell = \ell(\alpha)$ is a row vector $\alpha = (j_1, \ldots, j_\ell)$, where each $j_i \in \{0, 1, \ldots, m\}$. The multiindex of length 0 is denoted by ν. For adapted, right-continuous functions f, and stopping times ρ, τ with $0 \leq \rho \leq \tau \leq T$ almost surely, we define:

$$I_\alpha[f(\cdot)]_{\rho,\tau} = \begin{cases} f(\tau) & \text{if } \ell(\alpha) = 0, \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s}\, ds & \text{if } \ell(\alpha) \geq 1, j_{\ell(\alpha)} = 0, \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s}\, dW_s^{j_{\ell(\alpha)}} & \text{if } \ell(\alpha) \geq 1, j_{\ell(\alpha)} \neq 0. \end{cases} \tag{2.2}$$

By $\alpha-$, we mean α with its final component removed.

We next define the function spaces $\mathcal{H}_\alpha$ as follows:

The space $\mathcal{H}_\nu$ is the space of adapted, right-continuous stochastic processes f with left-hand limits such that $|f(t)|$ is almost surely finite, for each $t \geq 0$. Then $\mathcal{H}_{(0)}$ is the subspace of $\mathcal{H}_\nu$ consisting of those f which additionally satisfy

$$\int_0^t |f(s)|\, ds < \infty \tag{2.3}$$

almost surely, for every $t \geq 0$, and $\mathcal{H}_{(j)}$, for $j \neq 0$, is the subspace of $\mathcal{H}_\nu$ consisting of those f which additionally satisfy

$$\int_0^t |f(s)|^2 \, ds < \infty \tag{2.4}$$

almost surely, for every $t \geq 0$. And, finally, we recursively define $\mathcal{H}_\alpha$ as the subspace of $\mathcal{H}_\nu$ consisting of those f which satisfy

$$I_{\alpha-}[f(\cdot)]_{0,t} \in \mathcal{H}_{(j_{\ell(\alpha)})} \tag{2.5}$$

almost surely, for every $t \geq 0$.

We define the differential operators related to the equation (1.1)

$$L^0 = \frac{\partial}{\partial t} + a^k \frac{\partial}{\partial u^k} + \frac{1}{2} b^{kj} b^{lj} \frac{\partial^2}{\partial u^k \partial u^l} \tag{2.6}$$

and

$$L^j = b^{kj} \frac{\partial}{\partial u^k}, \tag{2.7}$$

and for notational compactness, if we have a function $f : \mathcal{R}^+ \times \mathcal{R}^d \to \mathcal{R}$ which has sufficient derivatives, we set $f_n u = f$, and, if $\ell(\alpha) \leq 1$, we define recursively

$$f_\alpha = L^{j_1} f_{\alpha'}, \tag{2.8}$$

where $\alpha' = (j_2, \ldots, j_\ell)$ denotes α with its first component j_1 removed.

We now define a hierarchical set $\mathcal{A}$ as a nonempty set of multiindices such that $\sup_{\alpha \in \mathcal{A}} \ell(\alpha)$ is finite, and α' is in $\mathcal{A}$ whenever $\alpha \neq \nu$ is in $\mathcal{A}$. The remainder set $\mathcal{B}(\mathcal{A})$ consists of those α not in $\mathcal{A}$ such that α' is in $\mathcal{A}$. Then for any hierarchical set $\mathcal{A}$, we will have a stochastic Taylor expansion of a function $f : \mathcal{R}^+ \times \mathcal{R}^d \to \mathcal{R}$ applied to a solution U of (1.1):

$$f(\tau, U_\tau) = \sum_{\alpha \in \mathcal{A}} I_\alpha[f_\alpha(\rho, U_\rho)]_{\rho,\tau} + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} i_\alpha[f_\alpha(\cdot, u.)]_{\rho,\tau}. \tag{2.9}$$

Below, when we use a specific case of this expansion, we will write out the first summation explicitly, and we will write simply R_j^Δ for the second summation, the remainder term, where Δ is the length of the time interval over which we are expanding, and j is just a number to distinguish different remainders.

Finally for $\gamma = 1, 2, \ldots$, we denote by $\mathcal{A}_\gamma$ the hierarchical set consisting of all αs of length at most γ, and we call stochastic Taylor expansion with $\mathcal{A} = \mathcal{A}_\gamma$ the (weak) stochastic Taylor expansion to order γ.

3. A stochastic Adams-Bashforth scheme

In this section, we will consider a weak stochastic analog of the deterministic Adams-Bashforth scheme. This scheme for the ordinary differential equation $\phi' = F(\phi)$ takes the form

$$\phi_{n+1} = \phi_n + \frac{\Delta t}{2}[3F(\phi_n) - F(\phi_{n-1})], \tag{3.1}$$

and is of order Δt^2.

To derive our stochastic version of this scheme, we consider the stochastic Taylor expansion to order 2:

$$\begin{aligned} U_{t+\Delta} &= U_t + b^j \Delta W^j + a\Delta + L^j b^k I_{(j,k)} + L^0 b^k I_{(0,k)} \\ &\quad + L^j a I_{(j,0)} + \frac{1}{2} L^0 a \Delta^2 + R_2^\Delta(t) \\ &= U_t + a\Delta + \frac{1}{2} L^0 a \Delta^2 + M^\Delta(t), \end{aligned}$$

where each coefficient is evaluated at the point (t, U_t), each stochastic integral is from t to $t + \Delta$, $\Delta = \Delta t$, and we sum over repeated indices.

Also, using the stochastic Taylor expansion for orders 1 and 0, we have that

$$a(t + \Delta, U_{t+\Delta}) = a + L^0 a \Delta + N^\Delta(t), \tag{3.2}$$

where $N^\Delta(t) = L^j a \Delta W^j + R_1^\Delta(t)$, where R_1^Δ is the remainder in the order 1 expansion, and

$$L^0 a(t + \Delta, U_{t+\Delta}) = L^0 a + P^\Delta(t), \tag{3.3}$$

where $P^\Delta(t) = R_0^\Delta(t)$, the remainder in the order 0 expansion.

We combine these results to yield, for any η and θ,

$$\begin{aligned} U_{t+\Delta} &= U_t + [\eta a(t+\Delta, U_{t+\Delta}) + (1-\eta)a]\Delta \\ &\quad + \left(\frac{1}{2}-\eta\right)[\theta L^0 a(t+\Delta, U_{t+\Delta}) + (1-\theta)L^0 a]\Delta^2 \\ &\quad -\eta\Delta N^\Delta(t) - \left(\frac{1}{2}-\eta\right)\theta\Delta^2 P^\Delta(t) + M^\Delta(t). \end{aligned}$$

So, if $t = t_n$, $\Delta = 2\Delta t$, $\eta = \theta = 0$, and writing U_n for U_{t_n},

$$U_{n+2} = U_n + 2a(t_n, U_n)\Delta t + 2L^0 a(t_n, U_n)\Delta t^2 + M^{2\Delta t}(t_n), \tag{3.4}$$

and if $t = t_n$, $\Delta = \Delta t$, $\eta = -\frac{3}{2}$, and $\theta = 0$,

$$\begin{aligned} U_{n+1} &= U_n - \frac{3}{2}a(t_{n+1}, U_{n+1})\Delta t + \frac{5}{2}a(t_n, U_n)\Delta t \\ &\quad +2L^0 a(t_n, U_n)\Delta t^2 + \frac{3}{2}N^{\Delta t}(t_n)\Delta t + M^{\Delta t}(t_n). \end{aligned}$$

Hence,

$$\begin{aligned} U_{n+2} &= U_{n+1} + (U_{n+1} - U_n) - (U_{n+1} - U_n) \\ &= U_{n+1} + \left[\frac{3}{2}a(t_{n+1}, U_{n+1}) - \frac{1}{2}a(t_n, U_n)\right]\Delta t \\ &\quad -\frac{3}{2}\Delta t N^{\Delta t}(t_n) + (M^{2\Delta t}(t_n) - M^{\Delta t}(t_n)). \end{aligned}$$

So, we will consider the following version of a stochastic Adams-Bashforth scheme:

$$Y_{n+2} = Y_{n+1} + \left[\frac{3}{2}a(t_{n+1}, Y_{n+1}) - \frac{1}{2}a(t_n, Y_n)\right]\Delta t + B_{n+1}(t_{n+1}, Y_{n+1}), \tag{3.5}$$

in which

$$B_{n+1}(t,x) = b^j(t,x)\Delta W^j + L^0 b^j(t,x)I_{(0,j)} + L^j a(t,x)I_{(j,0)} + L^j b^k(t,x)I_{(j,k)}, \tag{3.6}$$

where the random intervals are evaluated over the interval from t_{n+1} to t_{n+2}.

Theorem 1. *Suppose that the coefficient functions f_α, defined as in (2.8) with $f(x) = x$ satisfy the conditions $|f_\alpha(t,x) - f_\alpha(t,y)| \leq K|x-y|$, $f_\alpha \in \mathcal{H}_\alpha$, and $|f_\alpha(t,x)| \leq K(1+|x|)$, for all $\alpha \in \mathcal{A}_4$. Then if Y_1 is chosen such that $|Eg(U_1) - Eg(Y_1)| = O(\Delta t^2)$ holds for every polynomial g, then we also have*

$$|Eg(U_N) - Eg(Y_N)| = O(\Delta t^2). \tag{3.7}$$

Remark 1. *Note that, in general, the constant C implied by the notation $O(\Delta t)$ in (3.7) will depend on the polynomial g.*

Proof:

For the proof, we will use Theorem 14.5.2 of [12]. For this, we will need to check that (3.5) satisfies the following three conditions (3.8)–(3.10):

For $q = 1, 2, \ldots$, there exists a constant C and integer r such that

$$E\Big[\max_{0\le n\le N} |Y_n|^{2q}\Big|\mathcal{F}_0\Big] \le C(1 + |Y_0|^{2r}); \tag{3.8}$$

$$E[|Y_{n+1} - Y_n|^{2q} \mid \mathcal{F}_n] \le C\Big(1 + \max_{0\le k\le n} |Y_k|^{2r}\Big)\Delta t^q, \tag{3.9}$$

where $\mathcal{F}_n$ is the σ-algebra to time n in the standard filtration; and, for $\ell = 1, 2, 3, 4, 5$,

$$\big|E(Y_{n+1} - Y_n)^\ell - E(Y^*_{n+1} - y^*_n)^\ell\big| \le C\Big(1 + \max_{0\le n\le N}\Big)|Y_n|^{2r}\Delta t^3. \tag{3.10}$$

In this last condition, Y^* denotes the second order weak Taylor scheme.

Note that, roughly speaking, (3.8) is necessary for the numerical scheme not to "blow up". In particular, it makes the definition of weak convergence meaningful. The second condition (3.9) requires that we have some control on oscillations in the scheme. And the final condition (3.10) tells us that our scheme converges at the correct order to the correct distribution, namely, the same as the second order weak Taylor scheme, which is known to be what we want.

For (3.8), we use the linear growth conditions on the coefficient functions a and b as well as the fact that the expected value of any power of any stochastic increment is either 0 or a constant times a power of Δt to find that

$$E|Y_{n+2}|^{2q} \le E\big[|Y_{n+1}|^{2q} + C(1 + |Y_{n+1}|^{2q} + |Y_n|^{2q})\Delta t\big]. \tag{3.11}$$

(Note that here, the expectations are really conditional expectations given $\mathcal{F}_0$, which we leave understood.) Therefore, we can define $\eta_n = E|Y_n|^{2q}$, and this is just

$$\eta_{n+2} \le \eta_{n+1} + C(1 + \eta_{n+1} + \eta_n)\Delta t. \tag{3.12}$$

Now define $\epsilon_0 = \frac{1}{2} + \eta_0$, $\epsilon_1 = \frac{1}{2} + \eta_1$, and recursively,

$$\epsilon_{n+2} = \epsilon_{n+1} + C(\epsilon_{n+1} + \epsilon_n)\Delta t, \tag{3.13}$$

so that $\eta_n \le \epsilon_n$, for every n. Note also that the ϵ's increase, and so $\epsilon_{n+2} \le (1+2C\Delta t)\epsilon_{n+1}$, and therefore

$$\eta_N \le \epsilon_N \le (1+2C\Delta t)^N \epsilon_0 \approx e^{2CT}\epsilon_0, \tag{3.14}$$

and so, in view of Doob's L^p-inequality (see, for example, Theorem II.52.6 of [17]), we have (3.8).

For (3.9), we have, as above,

$$\begin{aligned} E[|Y_{n+2} - Y_{n+1}|^{2q} \mid \mathcal{F}_{n+1}] &\le CE[1 + |Y_{n+1}|^{2q} + |Y_n|^{2q} \mid \mathcal{F}_{n+1}]\Delta t^q \\ &\le C(1 + E[\max_{0\le k\le n+1} |Y_k|^{2q} \mid \mathcal{F}_{n+1}])\Delta t^q \\ &= C(1 + E\max_{0\le k\le n+1} |Y_k|^{2q})\Delta t^q. \end{aligned}$$

Finally, for (3.10), we note that here we are comparing moments of the stochastic Adams-Bashforth scheme with moments of the weak Taylor scheme. For this we note the following moments of the weak Taylor scheme, from (6.2) chapter 15 of [12] (here we write simply a for $a(Y_n^*)$, b for $b(Y_n^*)$, etc.):

$$E(Y_{n+1}^* - Y_n^*) = E\left[a\Delta t + \frac{1}{2}(aa' + \frac{1}{2}b^2a'')\Delta t^2\right] + O(\Delta t^3), \tag{3.15}$$

$$E(Y_{n+1}^* - Y_n^*)^2 = E\left[b^2\Delta t + \frac{1}{2}[2a(a+bb') + b^2(2a' + (b')^2 + bb'')]\Delta t^2\right] + O(\Delta t^3), \tag{3.16}$$

$$E(Y_{n+1}^* - Y_n^*)^3 = E[3b^2(a+bb')\Delta t^2] + O(\Delta t^3), \tag{3.17}$$

$$E(Y_{n+1}^* - Y_n^*)^4 = E[3b^4\Delta t^2] + O(\Delta t^3), \tag{3.18}$$

$$E(Y_{n+1}^* - Y_n^*)^5 = O(\Delta t^3), \tag{3.19}$$

We need only show that the corresponding moments of the stochastic Adams-Bashforth scheme agree with these up to $O(\Delta t^3)$.

In what follows, we use a_n for $a(Y_n)$, b_n for $b(Y_n)$, etc.

Note first that we have the Taylor expansion

$$\begin{aligned} a_{n-1} &= a_n + a_n'(Y_{n-1} - Y_n) + \frac{1}{2}a_n''(Y_{n-1} - Y_n)^2 + O(|Y_{n-1} - Y_n|^3) \\ &= \text{(upon taking expectations)} \\ &= a_n - a_n'\left(\frac{3}{2}a_n - \frac{1}{2}a_{n-1}\right)\Delta t + \frac{1}{2}a_n''b_n^2\Delta t + O(\Delta t^2) \\ &= a_n - a_n a_n'\Delta t + \frac{1}{2}a_n''b_n^2\Delta t^2 + O(\Delta t^2). \end{aligned}$$

Therefore,

$$\begin{aligned} E(Y_{n+1} - Y_n) &= E\left(\frac{3}{2}a_n - \frac{1}{2}a_{n-1}\right)\Delta t \\ &= Ea_n\Delta t + E\left(\frac{1}{2}a_n a_n' + \frac{1}{4}b_n^2 a_n''\right)\Delta t^2 + O(\Delta t^3), \end{aligned}$$

and the first moments agree.

Next, using that $E\Delta W^2 = \Delta t$, $E\Delta W^4 = 3\Delta t^2$, etc., we have that

$$\begin{aligned} E(Y_{n+1} - Y_n)^2 &= E\left[\left(\frac{3}{2}a_n - \frac{1}{2}a_{n-1}\right)\Delta t^2 + b^2\Delta t + \frac{1}{2}b^2(b')^2\Delta t^2 \right. \\ &\quad \left. + \frac{1}{2}b\left(ab' + \frac{1}{2}bb''\right)\Delta t^2 + b^2 a'\Delta t^2\right] + O(\Delta t^3), \end{aligned}$$

and we see that the second moments agree.

For the third moments,

$$E(Y_{n+1} - Y_n)^3 = 3E\left[\left(\frac{3}{2}a_n - \frac{1}{2}a_{n-1}\right)b^2\Delta t^2 + b^3 b'\Delta t^2\right] + O(\Delta t^3),$$

and we have agreement.

The fourth and fifth moments are trivial.

This completes our proof.

Remark 2. *This result is a modification and simplification of the strong second order Adams-Bashforth scheme in* [10]*. In the case of the weak scheme here, we have convergence to the same order as the strong scheme (except, of course, that the convergence is only weak), but the most problematic terms from the strong scheme are absent here, resulting in considerable savings.*

4. A stochastic semi-implicit leapfrog scheme

Similarly to section 3, we want to now find a (weak) stochastic analog of the deterministic scheme

$$\phi_{n+1} = \phi_{n-1} + 2\Delta t a_1(\phi_n) + 2\Delta t a_2(\phi_{n+1}) \tag{4.1}$$

for the differential equation $\phi' = a_1(\phi) + a_2(\phi)$, where a_2 is to be treated implicitly. Therefore we consider the stochastic differential equation

$$dU_t = [a_1(t, U_t) + a_2(t, U_t)]\, dt + b(t, U_t)\, dW_t, \tag{4.2}$$

similar to (1.1).

Using stochastic Taylor schemes to order 0, we have that

$$\begin{aligned}
a_1(t_{n+1}, U_{n+1}) &= a_1(t_n, U_n) + R_1^{\Delta} t(t_n); \\
a_2(t_{n+2}, U_{n+2}) &= a_1(t_n, U_n) + R_2^{\Delta} t(t_n); \\
a_2(t_{n+2}, U_{n+2}) &= a_1(t_{n+1}, U_{n+1}) + R_3^{\Delta} t(t_n).
\end{aligned}$$

Therefore, using stochastic Taylor series to order 1 and the above,

$$\begin{aligned}
U_{n+2} &= U_n + (U_{n+2} - U_{n+1}) + (U_{n+1} - U_n) \\
&= U_n + [b^j(t_{n+1}, U_{n+1})\Delta W^j + a_1(t_{n+1}, U_{n+1})\Delta t + a_2(t_{n+1}, U_{n+1})\Delta t + R_4(t_{n+1})] \\
&\quad + [b^j(t_n, U_n)\Delta W^j + a_1(t_n, U_n)\Delta t + a_2(t_n, U_n)\Delta t + R_5(t_n)].
\end{aligned}$$

Therefore, we will consider the scheme

(4.3)

$$Y_{n+2} = Y_n + 2a_1(t_{n+1}, Y_{n+1})\Delta t + 2a_2(t_{n+2}, Y_{n+2})\Delta t + b^j(t_n, Y_n)\Delta W_n^j + b^j(t_{n+1}, Y_{n+1})\Delta W_{n+1}^j.$$

Then we have

Theorem 2. *Suppose that the coefficient functions f_α satisfy the conditions $|f_\alpha(t, x) - f_\alpha(t, y)| \leq K(|x - y|$, $f_\alpha \in \mathcal{H}_\alpha$, and $|f_\alpha(t, x)| \leq K(1 + |x|)$, for all $\alpha \in \mathcal{A}_3$. Then if Y_1 is chosen such that $|Eg(U_1) - Eg(Y_1)| = O(\Delta t)$, for every polynomial g, then*

$$|Eg(U_N) - Eg(Y_N)| = O(\Delta t). \tag{4.4}$$

Proof:

We proceed similarly to the proof of Theorem 1, except that, since this scheme is only order 1, we have the following for (3.10): For $\ell = 1, 2, 3$,

$$\left| E(Y_{n+1} - Y_n)^\ell - E(Y^*_{n+1} - y^*_n)^\ell \right| \le C(1 + \max_{0\le n\le N})|Y_n|^{2r}\Delta t^2. \tag{4.5}$$

To show (3.8), if we set $\eta_n = E|Y_n|^{2q}$ (the conditioning on $\mathcal{F}_0$ is once more left understood), and again using the linear growth condition for a_1, a_2, b, etc., we have that

$$\eta_{n+2} \le \eta_n + C(1 + \eta_{n+2} + \eta_{n+1} + \eta_n)\Delta t. \tag{4.6}$$

Therefore, we set $\epsilon_0 = \frac{1}{3} + \eta_0$, $\epsilon_1 = \frac{1}{3} + \eta_1$, and, recursively,

$$\epsilon_{n+2} = \epsilon_n + C(\epsilon_{n+2} + \epsilon_{n+1} + \epsilon_n)\Delta t. \tag{4.7}$$

That is,

$$\begin{aligned} \epsilon_{n+2} &= \frac{1 + C\Delta t}{1 - C\Delta t}\epsilon_n + \frac{C\Delta t}{1 - C\Delta t}\epsilon_{n+1} \\ &\le \frac{1 + 2C\Delta t}{1 - C\Delta t}\epsilon_{n+1}, \end{aligned}$$

and so

$$\epsilon_N \le \left(\frac{1 + 2C\Delta t}{1 - C\Delta t}\right)^N \epsilon_0 \approx e^{3CT}\epsilon_0, \tag{4.8}$$

and we have (3.8).

The condition (3.9) is easily seen to be satisfied, as for Theorem 1.

Finally, for (4.5), we will need the following moments for the weak Taylor scheme:

$$E(Y^*_{n+2} - Y^*_n) = E[2(a_1 + a_2)\Delta t + O(\Delta t^2)], \tag{4.9}$$

$$E(Y^*_{n+2} - Y^*_n)^2 = E[2b^2\Delta t + O(\Delta t^2)], \tag{4.10}$$

$$E(Y^*_{n+2} - Y^*_n)^3 = O(\Delta t^2)], \tag{4.11}$$

These are easily compared to the comparable moments for the implicit leapfrog scheme, since

$$\begin{aligned} E(Y_{n+2} - Y_n) &= 2(a_{1,n+1} + a_{2,n+2})\Delta t + O(\Delta t^2) \\ &= 2(a_{1,n} + a_{2,n})\Delta t + O(\Delta t^2), \end{aligned}$$

and

$$\begin{aligned} E(Y_{n+2} - Y_n)^2 &= (b_n^2 + b_{n+1}^2)\Delta t + O(\Delta t^2) \\ &= 2b_n^2 \Delta t + O(\Delta t^2). \end{aligned}$$

The final moment is trivial, and our proof is complete.

Remark 3. *We note again that this is a modification and simplification of the strong semi-implicit leapfrog scheme in* [10]*. Again, we have the same convergence rate (order 1), except that the convergence is only weak. We further note that this scheme no longer requires the calculation of derivatives of* b*.*

5. Conclusion

We mainly note the advantages that the weak schemes here have over the corresponding strong schemes in [10]. There are many fewer terms, resulting in computational savings over the strong schemes. So long as we only desire the computation of various moments of the solutions to our differential equations (as is very often the case), there is no point in using the more complicated strong schemes.

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