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STOCHASTIC SOLUTIONS OF THE TWO-DIMENSIONAL PRIMITIVE EQUATIONS OF THE OCEAN AND ATMOSPHERE WITH AN ADDITIVE NOISE

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The aim of this article is to establish the existence and uniqueness of stochastic solutions of the two-dimensional equations of the ocean and atmosphere. White noise is additive, and the solutions are strong in the probabilistic sense. Finally, from the point of view of partial differential equations, they are of the type z-weak, that is bounded in $L^\infty(L^2)$ together with their derivative in z.

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1. Introduction

The mathematical theory of the Primitive Equations (PEs) for the ocean and the atmosphere has made substantial progress since the early articles [10, 11]. For the most recent developments, see the review article [17] and the subsequent articles by Cao and Titi [4] and by Kobelkov [9]. The object of the present article and of the companion article [7], is to study the existence and uniqueness of stochastic solutions to these equations driven by an additive white noise; the space dimension two is considered in this article and the space dimension three in [7]. Note that these two articles are devoted to the concept of strong solutions, strong in the probabilistic sense, that is, the solutions defined pathwise. The concept of weak solutions defined by martingales will be investigated elsewhere. As we explain below, the white noise is additive, i.e. of the form dW/dt, and strong solutions are obtained by requiring enough spatial regularity on W (as in, e.g., [1, 2]).

We start, in Sec. 2, by presenting the two-dimensional Primitive Equations with periodic boundary conditions as in [12] and give their functional formulation. We also state a result of existence and uniqueness of (semi) weak solutions which will be the starting point for the stochastic case. This result of existence and uniqueness of solutions is an unpublished result of Ziane which will appear in [13], but we prove here, in Sec. 3, a slightly more general version of it. We proceed in Sec. 2 by introducing the probability spaces and the driving white noise. Finally, in Sec. 3, we consider the actual two-dimensional PEs driven by a white noise as well as prove and state the main result of existence and uniqueness of solution.

We consider the equations for the ocean; the equations would be the same for the atmosphere if we use the potential temperature instead of the usual temperature, (see, e.g., [8] or [6]) and if the vertical coordinate is the pressure. The coupled ocean-atmosphere pertain to the same methods. Furthermore, we consider only the space periodic case. All the other cases (ocean with different boundary conditions, atmosphere or coupled ocean-atmosphere) are treated in the same way at the price of some modifications in the notations which are described briefly below and with full details in [17].

2. The Two-Dimensional Space Periodic Primitive Equations

For the sake of simplicity and to follow [12], we do not consider the salinity; introducing the salinity would not produce any additional technical difficulty. In this case, the density ρ is a linear function of the temperature T.

Because of the hydrostatic equation, it is not possible to produce a solution that is space periodic in all variables without restriction. For that reason, ρ, p (the pressure) and T below represent the deviation from a stratified solution. In what follows, $\bar{\rho}$ is the stratification profile for which $N^2 = -(g/\rho_0)(d\bar{\rho}/dz)$ is a constant, and, as usual, by the hydrostatic equation and the equation of state, $d\bar{p}/dz = -g\bar{\rho}$ and $\bar{\rho} = \rho_0(1 - \alpha(\bar{T} - T_0)), \rho_0, T_0$ being reference values of ρ and T (of the same order as $\bar{\rho}$ and $\bar{T}$). Furthermore, the periodic (disturbance) solutions that we consider present certain symmetries that are described below (see (2.2) below). We refer the reader to [12, 17] for more details on the physical background. The PEs that we consider here are written in nondimensional form (see [12]), and they read:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{Ro}v + \frac{1}{Ro}\frac{\partial p}{\partial x} = \nu_v \Delta u + F_u, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + \frac{1}{Ro}u = \nu_v \Delta v + F_v, \tag{2.1b}$$

$$\frac{\partial p}{\partial z} = -\rho, \tag{2.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2.1d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} - \frac{N^2}{Ro} w = \nu_\rho \Delta \rho + F_\rho. \tag{2.1e}$$

All the independent variables (t, x, z) and the dependent variables (u, v, w, ρ, p) are dimensionless, as are the forcing and source terms (F_u, F_v, F_ρ). Here, (u, v, w) are the three components of the velocity vector and, as we have mentioned, p and ρ denote the pressure and density deviations, respectively, from the prescribed stratified state. The (dimensionless) parameters are the Rossby number Ro; N, which is related to the Burger number; and the (eddy) Reynolds numbers ν_v and ν_ρ.

Some motivations on the physical background and the derivation of these equations are given in [12]. The two spatial directions are $0x$ and $0z$, corresponding to the west-east and vertical directions in the so-called f-plane approximation for geophysical flows (see [12]); $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$.

We work in a limited domain $\mathcal{M} = (0, L_1) \times (-L_3/2, L_3/2)$, and we assume space periodicity with period $\mathcal{M}$, that is, all functions are taken to satisfy

$$f(x + L_1, z, t) = f(x, z, t) = f(x, z + L_3, t) \tag{2.2}$$

when extended to $\mathbb{R}^2$.

Moreover, we assume that the following symmetries hold:

$$\begin{aligned} u(x,z,t) &= u(x,-z,t), & F_u(x,z,t) &= F_u(x,-z,t),\\ v(x,z,t) &= v(x,-z,t), & F_v(x,z,t) &= F_v(x,-z,t),\\ \rho(x,z,t) &= -\rho(x,-z,t), & F_\rho(x,z,t) &= -F_\rho(x,-z,t),\\ w(x,z,t) &= -w(x,-z,t), & p(x,z,t) &= p(x,-z,t). \end{aligned} \tag{2.3}$$

Here, u, v and p are said to be even in z, and w and ρ odd in z.

Our aim is to solve the problem (2.1a)–(2.1e) with initial data

$$u = u_0, \quad v = v_0, \quad \rho = \rho_0, \quad \text{at } t = 0. \tag{2.4}$$

Hence the natural function spaces for this problem are as follows:

$$V = \Big\{ U = (u, v, \rho) \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3, \\ u, v \text{ even in } z, \rho \text{ odd in } z, \int_{-L_3/2}^{L_3/2} u(x, z')\mathrm{d}z' = 0 \Big\}, \tag{2.5}$$

$$H = \text{ closure of } V \text{ in } (\dot{L}^2(\mathcal{M}))^3. \tag{2.6}$$

Here the dot above $\dot{H}^1_{\text{per}}$ or $\dot{L}^2$ denotes the functions with average in $\mathcal{M}$ equal to zero. These spaces are endowed with Hilbert scalar products; in H the scalar product is

$$(U, \tilde{U})_H = (u, \tilde{u})_{L^2} + (v, \tilde{v})_{L^2} + \kappa(\rho, \tilde{\rho})_{L^2}, \tag{2.7}$$

and in $\dot{H}^1_{\text{per}}$ and V the scalar product is (using the same notation when there is no ambiguity):

$$((U, \tilde{U})) = ((u, \tilde{u})) + ((v, \tilde{v})) + \kappa((\rho, \tilde{\rho})). \tag{2.8}$$

Here, we have written $\mathrm{d}\mathcal{M}$ for $\mathrm{d}x\mathrm{d}z$, and

$$((\phi, \tilde{\phi})) = \int_{\mathcal{M}} \left(\frac{\partial \phi}{\partial x} \frac{\partial \tilde{\phi}}{\partial x} + \frac{\partial \phi}{\partial z} \frac{\partial \tilde{\phi}}{\partial z} \right) \mathrm{d}\mathcal{M}; \tag{2.9}$$

the positive constant κ is defined below. We have

$$|U|_H \leq c_0 \|U\|, \quad \forall\, U \in V, \tag{2.10}$$

where $c_0 > 0$ is a positive constant related to κ and the Poincaré constant in $\dot{H}^1_{\text{per}}(\mathcal{M})$. More generally, the c_i, c_i', c_i'' will denote various positive constants. Inequality (2.10) implies that $\|U\| = ((U, U))^{1/2}$ is indeed a norm on V.

Let us recall that we can express the diagnostic variables w and p in terms of the prognostic variables u, v and ρ. For each $U = (u, v, \rho) \in V$, we can determine uniquely $w = w(U)$ from (2.1d):

$$w(U) = w(x, z, t) = -\int_0^z u_x(x, z', t)\, \mathrm{d}z', \tag{2.11}$$

since $w(x, 0) = 0, w$ being odd in z. Furthermore, writing that, by periodicity and antisymmetry, $w(x, -L_3/2, t) = \pm w(x, L_3/2, t) = 0$, we also have

$$\int_{-L_3/2}^{L_3/2} u_x(x, z', t)\, \mathrm{d}z' = 0. \tag{2.12}$$

As for the pressure, we obtain from (2.1d),

$$p(x, z, t) = p_s(x, t) - \int_0^z \rho(x, z', t)\, \mathrm{d}z', \tag{2.13}$$

where $p_s = p(x, 0, t)$ is the surface pressure. Thus, we can uniquely determine the pressure p in terms of ρ up to p_s.

We then derive the variational formulation of problem (2.1a)–(2.1e). For that purpose, we consider a test function $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\rho}) \in V$ and we multiply (2.1a), (2.1b) and (2.1e), respectively by $\tilde{u}, \tilde{v}$ and $\kappa\tilde{\rho}$, where the constant κ (which was already introduced in (2.7) and (2.8)) will be chosen later. We add the resulting equations and integrate over $\mathcal{M}$. We use (2.1c) and (2.1d) for the term involving ρ, and we arrive at:

$$\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_H + a(U,\tilde{U}) + b(U,U,\tilde{U}) + \frac{1}{Ro}e(U,\tilde{U}) = (F,\tilde{U})_H, \quad \forall \tilde{U} \in V. \tag{2.14}$$

Here, we have set

$$a(U,\tilde{U}) = \nu_v((u,\tilde{u})) + \nu_v((v,\tilde{v})) + \kappa\nu_\rho((\rho,\tilde{\rho})),$$

$$e(U,\tilde{U}) = \frac{1}{Ro}\int_{\mathcal{M}}(u\tilde{v} - v\tilde{u})\,\mathrm{d}\mathcal{M} + \frac{1}{Ro}\int_{\mathcal{M}}(\rho\tilde{w} - \kappa N^2 w\tilde{\rho})\,\mathrm{d}\mathcal{M},$$

$$b(U,U^\sharp,\tilde{U}) = \int_{\mathcal{M}}\left(u\frac{\partial u^\sharp}{\partial x} + w(U)\frac{\partial u^\sharp}{\partial z}\right)\tilde{u}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}}\left(u\frac{\partial v^\sharp}{\partial x} + w(U)\frac{\partial v^\sharp}{\partial z}\right)\tilde{v}\,\mathrm{d}\mathcal{M}$$
$$+ \int_{\mathcal{M}}\left(u\frac{\partial \rho^\sharp}{\partial x} + w(U)\frac{\partial \rho^\sharp}{\partial z}\right)\tilde{\rho}\,\mathrm{d}\mathcal{M}.$$

We now choose $\kappa = 1/N^2$ and it can easily be seen that:

$$\begin{aligned}
&a : V \times V \to \mathbb{R} \text{ is bilinear, continuous,}\\
&e : V \times V \to \mathbb{R} \text{ is bilinear, continuous,}\\
&a + e \text{ is coercive, } a(U,U) + e(U,U) \geq c_1\|U\|^2, \forall U \in V, c_1 > 0,\\
&b \text{ is trilinear, continuous from } V \times V_2 \times V \text{ into } \mathbb{R},\\
&\qquad\qquad \text{and from } V \times V \times V_2 \text{ into } \mathbb{R},
\end{aligned} \tag{2.15}$$

where V_2 is the space $V \cap (H^2_{\mathrm{per}}(\mathcal{M}))^3$ (which is closed in $(H^2_{\mathrm{per}}(\mathcal{M}))^3$). Furthermore,

$$\begin{aligned} b(U,\tilde{U},U^\sharp) &= -b(U,U^\sharp,\tilde{U}),\\ b(U,\tilde{U},\tilde{U}) &= 0, \end{aligned} \tag{2.16}$$

when $U, \tilde{U}, U^\sharp \in V$ with $\tilde{U}$ or $U^\sharp$ in V_2. We also have the following (see [10–12]):

Lemma 2.1. *There exists a constant $c_2 > 0$ such that, for all $U \in V$, $\tilde{U} \in V_2$ and $U^\sharp \in V$:*

$$|b(U,U^\sharp,\tilde{U})| \leq c_2|U|_{L^2}^{1/2}\|U\|^{1/2}\|U^\sharp\|\,|\tilde{U}|_{L^2}^{1/2}\|\tilde{U}\|^{1/2}$$
$$+ c_2\|U\|\|U^\sharp\|^{1/2}|U^\sharp|_{V_2}^{1/2}|\tilde{U}|_{L^2}^{1/2}\|\tilde{U}\|^{1/2}. \tag{2.17}$$

Alternatively, we can introduce the linear and bilinear operators A, B, E from V into V', defined by

$$\begin{aligned}
\langle AU,\tilde{U}\rangle &= a(U,\tilde{U}), & &\forall U,\tilde{U} \in V,\\
\langle EU,\tilde{U}\rangle &= e(U,\tilde{U}), & &\forall U,\tilde{U} \in V,\\
\langle B(U,\tilde{U}),U^\sharp\rangle &= b(U,\tilde{U},U^\sharp), & &\forall U,\tilde{U} \in V,\ U^\sharp \in V_2,
\end{aligned}$$

and we then write (2.14) as a functional differential equation:

$$\frac{dU}{dt} + AU + B(U,U) + EU = F, \tag{2.18}$$

which we supplement with the initial condition

$$U(0) = U_0. \tag{2.19}$$

The usual terminology in PDEs and fluid mechanics is to call weak solutions, the solutions of (2.18)–(2.19) which belong to $L^\infty(0,t_1;H) \cap L^2(0,t_1;V), \forall t_1 > 0$, and strong solutions, the solutions belonging to $L^\infty(0,t_1;V) \cap L^2(0,t_1;V_2), \forall t_1 > 0$. It was shown (see [17] and the references therein) that, for the incompressible Navier–Stokes equations in space dimension two, (2.18) and (2.19) possess a unique strong solution defined for all time (with suitable hypotheses on the data). Concerning the weak solutions, existence for all time has been shown (see, e.g., [10, 11, 17]), but, unlike the Navier–Stokes, the uniqueness of the two-dimensional weak solutions has not been proven. Instead, we have a result of existence and uniqueness of (semi) weak solutions which we now recall.

Theorem 2.2. *Given* $U_0 \in H$, *with* $U_{0z} = \partial U_0/\partial z \in L^2(\mathcal{M})^3$, *and* $F \in L^\infty(\mathbb{R}_+;H)$ *with* $F_z = \partial F/\partial z \in L^\infty(\mathbb{R}_+;L^2(\mathcal{M})^3))$, *there exists a unique solution of* $(2.18) - (2.19)$, *defined for all* $t > 0$ *and satisfying*:

$$U \in \mathcal{C}([0,t_1];H) \cap L^2(0,t_1;V), \quad \forall t_1 > 0,$$
$$U_z \in \mathcal{C}([0,t_1];L^2(\mathcal{M})^3) \cap L^2(0,t_1;H^1(\mathcal{M})^3), \quad \forall t_1 > 0.$$

As indicated before, this unpublished result of Ziane will be included in [13]. However, we show below in Sec. 3 a result slightly more general than Theorem 2.2.

Remark 2.3. Before we proceed, we would like to explain how Eq. (2.18) relates to the initial equations (2.1a)–(2.1e). For that purpose, we introduce the orthogonal projector P from $L^2(\mathcal{M})^3$ onto H. It is easy to see that if $U = (u,v,\rho) \in L^2(\mathcal{M})^3$, then

$$PU = (u - \bar{u}, v, \rho), \tag{2.20}$$

where $\bar{u}$ is the average

$$\bar{u}(x) = \frac{1}{L_3}\int_{-L_3/2}^{L_3/2} u(x,z')\mathrm{d}z'. \tag{2.21}$$

The domain $D(A)$ of A in H is the same as the space denoted V_2 before, and for $U \in D(A)$,

$$AU = -(\nu_v\Delta(u-\bar{u}), \nu_v\Delta v, \kappa\nu_\rho\Delta\rho).$$

Hence, with $w = w(U)$ and $p = p(U)$, defined as explained in (2.11)–(2.13), the second and third components of Eq. (2.18) are the same as (2.1b) and (2.1e), whereas the first component of (2.18) expresses the fact that the projection P of Eq. (2.1a) is satisfied:

$$P\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{Ro}v + \frac{1}{Ro}\frac{\partial p}{\partial x}\right) = P(\nu_v \Delta u + F_u). \tag{2.22}$$

Alternatively,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{Ro}v + \frac{1}{Ro}\frac{\partial p}{\partial x} = \nu_v \Delta u + F_u + \phi, \tag{2.23}$$

where $\phi = \phi(x,t) \in (I-P)(L^2(\mathcal{M})^3)$. According to (2.13), p is not fully determined by the knowledge of ρ, as $p_s = p_s(x,t)$ remains unknown. Hence, by changing p (that is, p_s), we can in fact choose $\phi = 0$ in (2.23) and rewrite this equation as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial x} - \frac{1}{Ro}v - \frac{1}{Ro}\frac{\partial}{\partial x}\int_0^z \rho(z')\,\mathrm{d}z' + \frac{1}{\varepsilon}\frac{\partial}{\partial x}p_s = \nu_v \Delta u + F_u. \tag{2.24}$$

Here, $p_s = p_s(x,t)$ is defined up to a function of t which could be determined if we impose, e.g., $\int_{-L_3/2}^{L_3/2} p_s(x,t)\,dx = 0$. Furthermore, once u, v, ρ, w are determined by Eqs. (2.18), (2.19) and (2.11), Eq. (2.24) precisely determines $p_s = p_s(x,t)$. This remark will be useful in the understanding of the component of the white noise on the orthogonal of H in $L^2(\mathcal{M})^3$, that is $(I-P)(L^2(\mathcal{M})^3)$; see Remark 3.4.

3. The Stochastic Primitive Equations

Our aim is now to consider the stochastic version of Theorem 2.2. We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space[a] with expectation E. The process $W = W(t,\omega), t \geq 0$, $\omega \in \Omega$, is an H-valued stochastic process defined on the probability space (for instance, a Wiener process, cf. [5]), subject to the following regularity in space and time: for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$W(\cdot,\omega) \in \mathcal{C}(\mathbb{R}_+; V), \tag{3.1}$$

and

$$\frac{\partial}{\partial z}W(\cdot,\omega) \in \mathcal{C}(\mathbb{R}_+; V). \tag{3.2}$$

Furthermore, the mapping $\omega \mapsto W(\cdot,\omega)$ is measurable with respect to the Borel measures generated by the corresponding spaces.

We also have a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, that is, the $\mathcal{F}_t$ are σ-subalgebras of $\mathcal{A}$ which increase in t and are right-continuous in t. The Wiener process W will be adapted to the filtration, and the initial condition U_0 must be measurable with respect to $\mathcal{F}_0$.

[a]In this article, Ω denotes the probability space and not the angular velocity of the earth, as is usual in geophysical fluid mechanics.

We are now interested in solving the Ito differential equation:

$$dU = -(AU + B(U,U) + EU - F)dt + dW, \tag{3.3}$$

or, in short, with obvious notations,

$$dU = -N(U)dt + dW, \tag{3.4}$$

with

$$U(0) = U_0. \tag{3.5}$$

For that purpose, we will perform the change of unknown function $\tilde{U} = U - W$. In this way, the white noise dW/dt disappears and the equation for $\tilde{U}$ is a statistical equation, that is an equation similar to (2.18) with the probabilistic parameter ω; namely

$$\frac{d\tilde{U}}{dt} + A\tilde{U} + B(W,\tilde{U}) + B(\tilde{U},W) + B(\tilde{U},\tilde{U}) + B(W,W) + E\tilde{U} = \tilde{F}, \tag{3.6}$$

with

$$\tilde{U}(0) = \tilde{U}_0 = U_0 - W(0), \tag{3.7}$$

and

$$\tilde{F} = -(AW + EW) + F. \tag{3.8}$$

The resolution of (3.6)–(3.8) will be similar to that of (2.18), (2.19), provided we assume enough regularity on W. In fact, the only difference between (2.18) and (3.3) is the occurrence of the terms (linear in $\tilde{U}$) $B(W,\tilde{U})$ and $B(\tilde{U},W)$.

Due to (3.8) and the hypotheses (3.1)–(3.2) on W, we can prove the following lemma:

Lemma 3.1.

$$\tilde{U}_0 \in H,\ \tilde{U}_{0z} \in L^2(\mathcal{M})^3, \tag{3.9}$$

$$\tilde{F}, \tilde{F}_z \in L^\infty(\mathbb{R}_+; V'). \tag{3.10}$$

Proof. For (3.9) and due to (3.7), it suffices to notice that $W(0) \in H$ and $W_z(0) \in L^2(\mathcal{M})^3$.

For (3.10), due to (3.8), it suffices to show that each of the following terms separately belong to $L^\infty(\mathbb{R}_+; L^2(\mathcal{M})^3)$, as well as their derivatives in z: AW, EW, $B(W,W)$. The result follows promptly for AW and AW_z (for the latter, it suffices to observe that ΔW_z is in $L^\infty(\mathbb{R}_+; L^2(\mathcal{M})^3)$). The result is also easy for EW and $(EW)_z$. The lemma is proven. □

Having established the properties of $\tilde{F}$ and $\tilde{V}_0$, we now show the existence and uniqueness of solutions of (3.6)–(3.8) which will imply the main result, the existence and uniqueness of solutions of (3.3)–(3.5). Note also that, when $W = 0$, Eqs. (3.6)–(3.8) are the same as (3.3)–(3.5) and (2.18)–(2.19) so that, in fact, the following theorem is a generalization of Theorem 2.2.

Theorem 3.2. *There exists a unique solution* $\tilde{U}$ *of* (3.6)–(3.8) *such that*

$$\tilde{U} \text{ and } \tilde{U}_z \in \mathcal{C}([0,t_1]);H) \cap L^2(0,t_1;V), \quad \forall t_1 > 0. \tag{3.11}$$

Proof. For the existence of solutions, the proof of this theorem as well as that of Theorem 2.2 is based on the obtention of formal *a priori* estimates which are established by assuming enough regularity on $\tilde{U}$.

(a) We start with the *a priori* estimates concerning $\tilde{U}$ and continue in point (b) with the *a priori* estimates concerning $\tilde{U}_z$.

We take the scalar product of (3.6) with $\tilde{U}$ in the duality between V and V' and, taking (2.16) into account we find:

$$\frac{1}{2}\frac{d}{dt}|\tilde{U}|_H^2 - a(\tilde{U},\tilde{U}) + b(\tilde{U},W,\tilde{U}) + b(W,W,\tilde{U}) + e(\tilde{U},\tilde{U}) = (\tilde{F},\tilde{U})_H.$$

We now take into account the coercivity of $a + e$ (see (2.15)) and this yields:

$$\frac{1}{2}\frac{d}{dt}|\tilde{U}|_H^2 + c_1\|\tilde{U}\|^2 \le |\tilde{F}|_{V'}\|\tilde{U}\| + |b(\tilde{U},W,\tilde{U})| + |b(W,W,\tilde{U})|.$$

Since

$$b(\tilde{U},W,\tilde{U}) = \int_{\mathcal{M}} \tilde{u}\frac{\partial W}{\partial x}\cdot\tilde{U}d\mathcal{M} + \int_{\mathcal{M}} w\frac{\partial W}{\partial z}\cdot\tilde{U}d\mathcal{M},$$

the first term of $b(\tilde{U},W,\tilde{U})$ can be estimated as:

$$\begin{aligned}\left|\int_{\mathcal{M}} \tilde{u}\frac{\partial W}{\partial x}\cdot\tilde{U}d\mathcal{M}\right| &\le |\tilde{u}|_{L^4}\cdot\left|\frac{\partial W}{\partial x}\right|_{L^2}\cdot|\tilde{U}|_{L^4} \le c\left|\frac{\partial W}{\partial x}\right|_{L^2(\mathcal{M})}|\tilde{U}|_H\|\tilde{U}\| \\ &\le \frac{c_1}{4}\|\tilde{U}\|^2 + c_1'\|W\|^2|\tilde{U}|^2_{L^2(\mathcal{M})},\end{aligned}$$

where the c_i' continue to denote various positive constants.

The second term of $b(\tilde{U},W,\tilde{U})$ is estimated as follows:

$$\begin{aligned}\left|\int_{\mathcal{M}} w(\tilde{U})\frac{\partial W}{\partial z}\cdot\tilde{U}d\mathcal{M}\right| &\le |w(\tilde{U})|_{L^2(\mathcal{M})}\left|\frac{\partial W}{\partial z}\right|_{L^4(\mathcal{M})}|\tilde{U}|_{L^4(\mathcal{M})} \\ &\le c_2'\|\tilde{U}\|^{3/2}\cdot\left|\frac{\partial W}{\partial z}\right|_{L^2}^{1/2}\left\|\frac{\partial W}{\partial z}\right\|^{1/2}|\tilde{U}|_H^{1/2} \\ &\le \frac{c_1}{4}\|\tilde{U}\|^2 + c_3'\left|\frac{\partial W}{\partial z}\right|_{L^2}^2\left\|\frac{\partial W}{\partial z}\right\|^2|\tilde{U}|_H^2.\end{aligned}$$

We have now to estimate $b(W,W,\tilde{U})$ as follows:

$$\begin{aligned}|b(W,W,\tilde{U})| &= \left|\int_{\mathcal{M}} u^\flat \frac{\partial W}{\partial x}\cdot \tilde{U} d\mathcal{M} + \int_{\mathcal{M}} w^\flat \frac{\partial W}{\partial z}\cdot \tilde{U} d\mathcal{M}\right| \\ &\leq c_4' |u^\flat|_{L^4} \left|\frac{\partial W}{\partial x}\right|_{L^2} |\tilde{U}|_{L^4} + |w^\flat|_{L^4} \left|\frac{\partial W}{\partial z}\right|_{L^2} |\tilde{U}|_{L^4} \\ &\leq c_5' \|W\|^2 |\tilde{U}|_H^{1/2} \|\tilde{U}\|^{1/2} \\ &\leq \frac{c_1}{4}\|\tilde{U}\|^2 + c_6' \|W\|^{8/3} |\tilde{U}|_H^{2/3},\end{aligned}$$

where $W = (u^\flat, v^\flat, w^\flat)$.

Taking into account all the above estimates, we find:

$$\frac{d}{dt}|\tilde{U}|_H^2 + c_1\|\tilde{U}\|^2 \leq g_1 |\tilde{U}|_H^2 + g_2, \tag{3.12}$$

where g_1 and g_2 are the following functions:

$$\begin{aligned} g_1 &= g_1(t) = c_3' \left|\frac{\partial W}{\partial z}\right|_{L^2}^2 \left\|\frac{\partial W}{\partial z}\right\|^2 + c_6' \|W\|^{8/3}, \\ g_2 &= g_2(t) = c_6' \|W\|^{8/3} + |\tilde{F}|_{V'}^2. \end{aligned}$$

We classically derive from (3.12) that

The norms of $\tilde{U}$ in $L^\infty(0,t_1;H)$ and $L^2(0,t_1;V)$ are bounded in terms of the data, $\forall\, t_1 > 0$. (3.13)

(b) We now continue with the *a priori* estimates concerning $\tilde{U}_z$. For that purpose, we differentiate (2.1a), (2.1b) and (2.1e) with respect to z and then multiply these equations by u_z, w_z and $\kappa\rho_z$ respectively, and integrate over $\mathcal{M}$. By adding the resulting equations, we find (compare to (3.6)):

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\tilde{U}_z|_H^2 + a(\tilde{U}_z,\tilde{U}_z) + e(\tilde{U}_z,\tilde{U}_z) + b(W_z,\tilde{U},\tilde{U}_z) \\ &\quad + b(\tilde{U},W_z,\tilde{U}_z) + b(\tilde{U}_z,W,\tilde{U}_z) + b(\tilde{U}_z,\tilde{U},\tilde{U}_z) \\ &\quad + b(W_z,W,\tilde{U}_z) + b(W,W_z,\tilde{U}_z) = (\tilde{F}_z,\tilde{U}_z)_H. \end{aligned}$$

Using (2.10), (2.15) (coercivity of $a+e$), (2.16), and the Schwarz inequality, we find:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\tilde{U}_z|_H^2 + c_1\|\tilde{U}_z\|^2 &\leq c_0|\tilde{F}_z|_{V'}\|\tilde{U}_z\| + |b(W_z,\tilde{U},\tilde{U}_z)| + |b(\tilde{U},W_z,\tilde{U}_z)| \\ &\quad + |b(\tilde{U}_z,W,\tilde{U}_z)| + |b(\tilde{U}_z,\tilde{U},\tilde{U}_z)| + |b(W_z,W,\tilde{U}_z)| \\ &\quad + |b(W,W_z,\tilde{U}_z)|. \end{aligned} \tag{3.14}$$

We can now estimate the trilinear terms from the right-hand side of (3.14) as follows:

$$\begin{aligned}\bullet|b(W_z,\tilde{U},\tilde{U}_z)| &= \left|\int_{\mathcal{M}} u_z^\flat \frac{\partial \tilde{U}}{\partial x}\cdot \tilde{U}_z\, d\mathcal{M} + w_z^\flat \frac{\partial \tilde{U}}{\partial z}\tilde{U}_z\, d\mathcal{M}\right| \\ &= \left|\int_{\mathcal{M}} u_z^\flat \tilde{U}_x\tilde{U}_z d\mathcal{M} - \int_{\mathcal{M}} u_x^\flat \tilde{U}_z\tilde{U}_z\, d\mathcal{M}\right| \\ &\le c_7'\int_{\mathcal{M}} |W_z||\tilde{U}_x||\tilde{U}_z|\, d\mathcal{M} + c_7'\int_{\mathcal{M}} |W_x||\tilde{U}_z|^2 d\mathcal{M} \\ &\le c_8'|W_z|_{L^4}|\tilde{U}_x|_{L^2}|\tilde{U}_z|_{L^4} + c_8'|W_x|_{L^2}|\tilde{U}_z|_{L^4}^2 \\ &\le c_9'\|W_z\||\tilde{U}_x|_{L^2}|\tilde{U}_z|_{L^2}^{1/2}\|\tilde{U}_z\|^{1/2} + c_9'|W_x|_{L^2}|\tilde{U}_z|_{L^2}\|\tilde{U}_z\| \\ &\le \frac{c_1}{12}\|\tilde{U}_z\|^2 + c_{10}'\|W_z\|^{4/3}|\tilde{U}_x|_{L^2}^{4/3}|\tilde{U}_z|_{L^2}^{2/3} + c_{11}'|W_x|_{L^2}^2|\tilde{U}_z|_{L^2}^2.\end{aligned}$$

We then estimate the following term:

$$\bullet|b(\tilde{U},W_z,\tilde{U}_z)| = \left|\int_{\mathcal{M}} \tilde{u}\frac{\partial W_z}{\partial x}\cdot\tilde{U}_z\, d\mathcal{M} + \int_{\mathcal{M}} w(\tilde{U})\frac{\partial W_z}{\partial z}\cdot\tilde{U}_z\, dU\right|. \qquad (3.15)$$

The first term of (3.15) can be bounded in the following way:

$$\begin{aligned}\left|\int_{\mathcal{M}} \tilde{u}\frac{\partial W_z}{\partial x}\cdot\tilde{U}_z\, d\mathcal{M}\right| &\le \int_{\mathcal{M}} |\tilde{u}|\cdot|W_{xz}|\cdot|\tilde{U}_z|\, d\mathcal{M} \le |\tilde{u}|_{L^4}|W_{xz}|_{L^2}|\tilde{U}_z|_{L^4} \\ &\le c_{12}'|\tilde{U}|_H^{1/2}\|\tilde{U}\|^{1/2}\|W_z\|\cdot|\tilde{U}_z|^{1/2}\|\tilde{U}_z\|^{1/2} \\ &\le \frac{c_1}{12}\|\tilde{U}_z\|^2 + c_{13}'|\tilde{U}|_H^{2/3}\|\tilde{U}\|^{2/3}\|W_z\|^{4/3}|\tilde{U}_z|^{2/3}.\end{aligned}$$

The second term of (3.15) requires a different treatment for the integrals in the vertical and, respectively, the horizontal direction:

$$\begin{aligned}\left|\int_{\mathcal{M}} w(\tilde{U})W_{zz}\tilde{U}_z\, d\mathcal{M}\right| &\le \int_0^{L_1} |w(\tilde{U})|_{L_z^\infty}|W_{zz}|_{L_z^2}|\tilde{U}_z|_{L_z^2}\, dx \\ &\le c_{14}'\int_0^{L_1} |\tilde{U}_x|_{L_z^2}|W_{zz}|_{L_z^2}|\tilde{U}_z|_{L_z^2}\, dx \\ &\le c_{15}'|\tilde{U}_x|_{L^2(\mathcal{M})}|W_{zz}|_{L^2(\mathcal{M})}||\tilde{U}_z|_{L_z^2}|_{L_x^\infty} \\ &\le c_{16}'|\tilde{U}_x|_{L^2(\mathcal{M})}\,\|W_z\|\,\|\tilde{U}_z\| \\ &\le \frac{c_1}{12}\|\tilde{U}_z\|^2 + c_{17}'\|\tilde{U}\|^2\|W_z\|^2.\end{aligned}$$

Here and below, L_x^2 is $L^2(0,L_1)$ and L_z^2 is $L^2(-L_{3/2},L_{3/2})$. We also used the fact that in dimension one, we have the Sobolev embedding $H_x^1 \subset L_x^\infty$, which implies:

$$|\tilde{U}|_{L_x^\infty(L_z^2)} \le c|\tilde{U}|_{H_x^1(L_z^2)} \le c\|\tilde{U}\|.$$

The third and fourth trilinear functional forms from the right-hand side of (3.14) are now estimated as follows:

- $$\begin{aligned} |b(\tilde{U}_z, W, \tilde{U}_z)| &= \left| \int_{\mathcal{M}} \tilde{u}_z \frac{\partial W}{\partial x} \cdot \tilde{U}_z \, dU + \int_{\mathcal{M}} \tilde{w}_z \frac{\partial W}{\partial z} \cdot \tilde{U}_z \, d\mathcal{M} \right| \\ &\le c|W_x|_{L^2} |\tilde{U}_z|_{L^2} \|\tilde{U}_z\| + c|\tilde{U}_x|_{L^2} |W_z|^{1/2} \|W_z\|^{1/2} |\tilde{U}_z|_{L^2}^{1/2} \|\tilde{U}_z\|^{1/2} \\ &\le \frac{c_1}{12} \|\tilde{U}_z\|^2 + c'_{18} |W_x|_{L^2}^2 |\tilde{U}_z|_{L^2}^2 + c'_{19} |\tilde{U}_x|_{L^2}^{4/3} |W_z|_{L^2}^{2/3} \|W_z\|^{2/3} \\ &\quad + c'_{20} |\tilde{U}_x|_{L^2}^{4/3} |W_z|_{L^2}^{2/3} \|W_z\|^{2/3} |\tilde{U}_z|_{L^2}^2, \end{aligned}$$

and

- $$\begin{aligned} |b(\tilde{U}_z, \tilde{U}, \tilde{U}_z)| &= \left| \int_{\mathcal{M}} \tilde{u}_z \frac{\partial \tilde{U}}{\partial x} \cdot \tilde{U}_z \, d\mathcal{M} + \int \tilde{w}_z \frac{\partial \tilde{U}}{\partial z} \cdot \tilde{U}_z \, d\mathcal{M} \right| \\ &\le c'_{21} |\tilde{U}_x|_{L^2} |\tilde{U}_z|_{L^2} \|\tilde{U}_z\| \\ &\le \frac{c_1}{12} \|\tilde{U}_z\|^2 + c'_{22} |\tilde{U}_x|_{L^2}^2 |\tilde{U}_z|_{L^2}^2. \end{aligned}$$

We also find:

- $$\begin{aligned} |b(W_z, W, \tilde{U}_z)| &= \left| \int_{\mathcal{M}} u_x^\flat \frac{\partial W}{\partial x} \cdot \tilde{U}_z d\mathcal{M} + \int_{\mathcal{M}} w_z^\flat \frac{\partial W}{\partial z} \cdot \tilde{U}_z \, d\mathcal{M} \right| \\ &\le c'_{23} \int_{\mathcal{M}} |W_x| \, |W_z| \, |\tilde{U}_z| d\mathcal{M} \\ &\le c'_{24} |W_x|_{L^2} |W_z|_{L^2}^{1/2} \|W_z\|^{1/2} |\tilde{U}_z|_{L^2}^{1/2} \|\tilde{U}_z\|^{1/2} \\ &\quad + c'_{24} |W_z|_{L^2} |W_z|_{L^2}^{1/2} \|W_z\|^{1/2} |\tilde{U}_z|_{L^2}^{1/2} \|\tilde{U}_z\|^{1/2} \\ &\le \frac{c_1}{12} \|\tilde{U}_z\|^2 + c'_{25} (|W_x|_{L^2}^{4/3} |W_z|_{L^2}^{2/3} + |W_z|_{L^2}^2) \|W_z\|^{2/3} |\tilde{U}_z|_{L^2}^{2/3} \\ &\le \frac{c_1}{12} \|\tilde{U}_z\|^2 + c'_{25} (|W_x|_{L^2}^{4/3} |W_z|_{L^2}^{2/3} + |W_z|_{L^2}^2) \|W_z\|^{2/3} (1 + |\tilde{U}_z|_H^2). \end{aligned}$$

The last trilinear form in (3.12) is estimated as follows, using again a different treatment for the vertical and horizontal directions:

- $$\begin{aligned} |b(W, W_z, \tilde{U}_z)| &= \left| \int_{\mathcal{M}} u^\flat \frac{\partial W_z}{\partial x} \cdot \tilde{U}_z \, d\mathcal{M} + \int_{\mathcal{M}} w^\flat \frac{\partial W_z}{\partial z} \cdot \tilde{U}_z \, d\mathcal{M} \right| \\ &\le \int_{\mathcal{M}} |u^\flat| \, |W_{xz}| \, |\tilde{U}_z| \, d\mathcal{M} + \int_0^{L_1} |w^\flat|_{L_z^\infty} |W_{zz}|_{L_z^2} |\tilde{U}_z|_{L_z^2} \, dx \\ &\le c'_{26} |W|_{L^2}^{1/2} \|W\|^{1/2} \|W_z\| |\tilde{U}_z|_{L^2}^{1/2} \|\tilde{U}_z\|^{1/2} \\ &\quad + c'_{27} |W_x|_{L^2(\mathcal{M})} |W_{zz}|_{L^2(\mathcal{M})} \; || \tilde{U}_z|_{L_z^2}|_{L_x^\infty} \\ &\le \frac{c_1}{12} \|\tilde{U}_z\|^2 + c'_{28} |W|_H^{2/3} \|W\|^{2/3} \|W_z\|^{4/3} (1 + |\tilde{U}_z|_H^2) \\ &\quad + c'_{29} \|W\|^2 \|W_z\|^2. \end{aligned}$$

1 Gathering all the above estimates, we find:

$$\frac{d}{dt} |\tilde{U}_z|_{L^2}^2 + c_1 \|\tilde{U}_z\|^2 \le g_3(t) |\tilde{U}_z|_{L^2}^2 + g_4(t),$$

where

$$\begin{aligned} g_3(t) = c_1''\{&\|W_z\|^{4/3}|\tilde{U}_x|_{L^2}^{4/3} + |W_x|_{L^2}^2 + |\tilde{U}|_H^{2/3}\|\tilde{U}\|^{2/3}\|W_z\|^{4/3} \\ &+ |W_z|_{L^2}^2\|W_z\|^{2/3} + |W_x|_{L^2}^{4/3}|W_z|_{L^2}^{2/3}\|W_z\|^{2/3} + |\tilde{U}_x|_{L^2}^2 \\ &+ |W|_H^{2/3}\|W\|^{2/3}\|W_z\|\}, \end{aligned}$$

and

$$\begin{aligned} g_4(t) = c_2''\{&|\tilde{F}_z|_{V'}^2 + \|W_z\|^{4/3}|\tilde{U}_x|_{L^2}^{4/3} + |\tilde{U}|_H^{2/3}\|\tilde{U}\|^{2/3}\|W_z\|^{4/3} + \|\tilde{U}\|^2\|W_z\|^2 \\ &+ |W_z|_{L^2}^2\|W_z\|^{2/3} + |W_x|_{L_2}^{4/3}|W_z|_{L^2}^{2/3}\|W_z\|^{2/3} + |W|_H^{2/3}\|W\|^{2/3}\|W_z\|^{4/3} \\ &+ \|W\|^2\|W_z\|^2\}. \end{aligned}$$

From the assumption on F and W (see Lemma 3.1) and the previous estimates on $\tilde{U}$, we know that g_3 and g_4 belong to $L^1(0,t_1)$, for every $t_1 > 0$, and that we can bound the norms of these functions in $L^1(0,t_1)$ in terms of the data. Hence, using the Gronwall lemma, we find

$$\begin{aligned} &\textit{The norms of } \tilde{U}_z \textit{ in } L^\infty(0,t_1;H) \textit{ and} \\ &L^2(0,t_1;V) \textit{ are bounded in terms of the data}, \forall\, t_1. \end{aligned} \tag{3.16}$$

(c) *Existence of solutions*
Using the Galerkin method based on the suitable Fourier series expansions (in V and H), and repeating the calculations leading to (3.13) and (3.16), we classically obtain a solution $\tilde{U}$ of (3.6)–(3.8) such that

$$\tilde{U} \quad \text{and} \quad \tilde{U}_z \in L^\infty(0,t_1;H) \cap L^2(0,t_1;V), \quad \forall\, t_1 > 0.$$

Passing then from $L^\infty(0,t_1;H)$ to $\mathcal{C}([0,t_1];H)$ as stated in (3.11) can be made using classical techniques (see e.g. [15, 17]).

(d) To conclude the proof of Theorem 3.2, we need to show the uniqueness of $\tilde{U}$.

For that purpose, let $\tilde{U}_1$ and $\tilde{U}_2$ be two solutions of (3.6)–(3.8) and let $\tilde{U} = \tilde{U}_1 - \tilde{U}_2$. Subtracting the corresponding equations (3.6)–(3.7) from each other, we find

$$\begin{gathered} \tfrac{d\tilde{U}}{dt} + A\tilde{U} + B(W,\tilde{U}) + B(\tilde{U},W) + B(\tilde{U}_1,\tilde{U}) + B(\tilde{U},\tilde{U}_2) + E\tilde{U} = 0, \\ \tilde{U}(0) = 0. \end{gathered} \tag{3.17}$$

We take the scalar product of (3.17) with $\tilde{U}$, and use (2.15)–(2.17). We find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\tilde{U}|_H^2 + c_1\|\tilde{U}\|^2 &= -b(\tilde{U}, W+\tilde{U}_2, \tilde{U}) \\ &\le c_3'' \left|\frac{\partial(W+\tilde{U}_2)}{\partial x}\right|_{L^2} |\tilde{U}|_H\|\tilde{U}\| + c_4''|\tilde{U}_x|_{L^2} \\ &\quad \times \left|\frac{\partial(W+\tilde{U}_2)}{\partial z}\right|_{L^2}^{1/2} \left\|\frac{\partial(W+\tilde{U}_2)}{\partial z}\right\|^{1/2} |\tilde{U}|_H^{1/2}\|\tilde{U}\| \end{aligned}$$

$$\leq \frac{c_1}{2}\|\tilde{U}\|^2 + c_5''\left|\frac{\partial(W+\tilde{U}_2)}{\partial x}\right|_{L^2}^2 |\tilde{U}|_H^2 + c_6'' \times \left|\frac{\partial(W+\tilde{U}_2)}{\partial z}\right|_H \left\|\frac{\partial(W+\tilde{U}_2)}{\partial z}\right\| |\tilde{U}|_H.$$

Hence,

$$\frac{d}{dt}|\tilde{U}|_H^2 \leq c''\left\{\left|\frac{\partial(W+\tilde{U}_2)}{\partial x}\right|_{L^2}^2 + \left|\frac{\partial(W+\tilde{U}_2)}{\partial z}\right|_{L^2}\left\|\frac{\partial(W+\tilde{U}_2)}{\partial z}\right\|\right\}|\tilde{U}|_H^2. \tag{3.18}$$

By the properties of W and $\tilde{U}$, the function $t \to \|W(t)+\tilde{U}_2(t)\|$ is integrable. Then, (3.18), $\tilde{U}(0) = 0$ and the Gronwall lemma imply that $\tilde{U}(t) = 0$, $\forall\, t > 0$.

Theorem 3.2 is thus proved and, as we have mentioned, this gives also a proof of Theorem 2.2. □

We now conclude by restating Theorem 3.2.2 in terms of U for Eqs. (3.3) and (3.5), and this is the main result of this article, namely the existence and uniqueness of solution of the 2D Primitive Equations with an additive white noise.

Theorem 3.3. *We consider the probability space* $(\Omega, \mathcal{F}, \mathcal{P})$ *and the process* $W = W(t,\omega)$ *satisfying the hypotheses* (3.1) *and* (3.2). *We are given* $U_0 \in H$ *with* $U_{0z} = \partial U_0/\partial z \in L^2(\mathcal{M})^3$ *and* $F \in L^\infty(\mathbb{R}_+; H)$ *with* $F_z = \partial F/\partial z \in L^\infty(\mathbb{R}_+; L^2(\mathcal{M})^3)$.

Then, there exists a unique solution U *of* (3.3)–(3.5) (*the 2D Primitive Equations with an additive white noise*), *such that*

$$U \text{ and } U_z \in \mathcal{C}([0,t_1]; H) \cap L^2(0,t_1;V), \quad \forall\, t_1 > 0. \tag{3.19}$$

Remark 3.4. This remark concerns the interpretation of (3.3). First, we observe, from the stochastic point of view, that (3.3) and (3.5) amount to the following,

$$U(t) = U_0 - \int_0^t [AU(s) + B(U(s),U(s)) + EU(s) - F(s)]\,ds + \int_0^t dW(s), \tag{3.20}$$

where the last integral is an Ito integral. From the PDE point of view, (3.20) is valid in V', for every t (or in H if the solution U enjoys additional regularity properties). Hence, as in (2.22)–(2.24), the first component of (3.20) is equivalent in $H^{-1}(\mathcal{M})$ (or $L^2(\mathcal{M})$ with more regularity), to:

$$u(t) + \int_0^t \left[u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{Ro}v + \frac{1}{Ro}\frac{\partial p}{\partial x}\right] ds = \int_0^t (\mu_v \Delta u + F_u)ds + \underline{\int_0^t dW(s)} + \phi, \tag{3.21}$$

with $\phi = \phi(x,t)$; as for (2.24) we can assume that $\phi = 0$, by changing p_s. Also there is no component of p_s related to the underlined Ito integral if W is chosen as in (3.1) and (3.2). If we replace V by $H^{-1}(\mathcal{M})^3$ for the hypotheses on W, in particular, if instead of (3.1) and (3.2), we assume that

$$W(\cdot,\omega), \frac{\partial W}{\partial z}(\cdot,\omega) \in \mathcal{C}(\mathbb{R}_+; H^{-1}(\mathcal{M})^3), \quad \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{3.22}$$

then p $(\partial p_s/\partial x)$ will contain a contribution from the white noise (the Ito integral) and we have (since $(I-P)u = 0$):

$$\frac{\partial p_s}{\partial x} = (I-P)\int_0^t \left[u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{Ro}v - \frac{1}{Ro}\int_0^z \frac{\partial}{\partial x}\rho(x,z',s)\,dz + \mu_v \Delta u + F_u \right] ds + \int_0^t dW(s). \tag{3.23}$$

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