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Analysis of Stochastic Numerical Schemes for the Evolution Equations of Geophysics

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Abstract—We present and study the stability and convergence, and order of convergence of a numerical scheme used in geophysics, namely, the stochastic version of a deterministic "implicit leapfrog" scheme which has been developed for the approximation of the so-called barotropic vorticity model. Two other schemes which might be useful in the context of geophysical applications are also introduced and discussed.

Keywords—Numerical methods, Stochastic differential equations, Leapfrog scheme, Adams-Bashforth scheme, Geophysical fluid dynamics.

1. INTRODUCTION

Much effort has been invested in studying numerical schemes for stochastic differential equations of the form

$$dU_t = a(U_t)\,dt + b(U_t)\,dW_t, \tag{1.1}$$

where $U_t \in \mathbb{R}^d$, a is a function from $\mathbb{R}^d$ into itself, W is a Wiener process on $\mathbb{R}^m$, and b is a function from $\mathbb{R}^d$ into $\mathbb{R}^{d\times m}$.

For the so-called weak approximation of (1.1), in which the approximation of the expectation of functions of U is considered, extensive work is due, for example, to Talay and his collaborators, work relying on probabilistic methods more involved than those used in this article (see, e.g., [1–3] and the references therein).

The question of strong approximation of (1.1), in which the approximation of sample paths of U is desired, has also been much studied. See, for example, the paper by Mil'shtein [4] for a scheme of order $O(\Delta t)$, and that by Rümelin [5] for an investigation of Runge-Kutta schemes.

The authors are very grateful to C. Penland for bringing these issues to their attention and they acknowledge very useful discussions with her and with P. Sardeshmukh. We are also indebted to A. Debussche for several improvements on an earlier draft, and to S. Faure, who provided the numerical simulations of Section 6.
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Especially, see the text by Kloeden and Platen [6], and the companion volume by Kloeden, Platen and Schurz [7], which are a systematic investigation of numerical schemes for (1.1) in both the sense of Itô and of Stratonovich, the two stochastic calculi which are in applications by far the most useful. Their methods are analytic and are applicable to proving the convergence of a wide range of numerical schemes, and they derive a very general scheme [6, formula (12.6.2)] which, for various choices of parameters, includes stochastic analogues of such deterministic schemes as the explicit and implicit Euler schemes, the Crank-Nicholson scheme, and the leapfrog scheme.

In the geophysics community, an enormous amount of work has been spent in developing large, complex numerical models of the oceans and atmosphere. The questions therefore arise: is it possible to add stochastic numerical noise to these already existing models in such a way that it is known to what the scheme converges (e.g., to the Itô or Stratonovich solution of some stochastic differential equation), to what order they may be expected to converge, etc.? While we certainly do not answer these complex questions here, we consider a simple "implicit leapfrog" scheme for a barotropic model (supplied to us by C. Penland and P. Sardeshmukh), and demonstrate one way of adding stochastic noise to it so that these questions can be answered for the resulting stochastic scheme (Section 3).

We also examine the derivatives of a and b which occur naturally in the above schemes, and which can prove to be troublesome in certain applications in which these functions, especially b, are given by physical parametrizations (i.e., by "tables") and not by analytic expressions. We consider how these derivatives can be replaced by finite differences derived from space-discretization while still maintaining the existing rate of convergence (Section 4).

Last, we propose a stochastic analogue for the deterministic Adams-Bashforth scheme, using methods similar to those of [6], as an attempt to produce alternate schemes which are higher order in time (Section 5).

2. PRELIMINARY RESULTS

We consider a stochastic differential equation

$$dU_t = a(t, U_t)\, dt + b(t, U_t)\, dW_t, \tag{2.1}$$

for $U = (u_1, \dots, u_d) \in \mathbb{R}^d$, where $a : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$, $b : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$, and W is a Wiener process in $\mathbb{R}^m$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

We then have the Itô formula, which states that, if $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{\hat{d}}$, then $F_t = F(t, U_t)$ satisfies the stochastic differential equation

$$dF_t = \left[\frac{\partial F}{\partial t} + a^k(F_t)\frac{\partial F}{\partial u^k} + \frac{1}{2} b^{ij}(F_t) b^{kj}(F_t) \frac{\partial^2 F}{\partial u^i \partial u^k}\right] dt + b^{ij}(F_t)\frac{\partial F}{\partial u^i} dW_t^j; \tag{2.2}$$

here we use the Einstein convention for repeated indices.

We use the following notations from [6]. A multi-index is a row vector $\alpha = (j_1, j_2, \dots, j_\ell)$ (each $j_i \in \{0, 1, \dots, m\}$) of length $\ell = \ell(\alpha) \in \{0, 1, \dots\}$. We define $n(\alpha)$ to be the number of entries of α which are 0. For adapted, right-continuous functions f, and stopping times ρ, τ such that $0 \leq \rho \leq \tau \leq T$ a.s., we define (where $\alpha-$ is α with its final component removed)

$$I_\alpha[f(\cdot)]_{\rho,\tau} = \begin{cases} f(\tau), & \text{if } \ell(\alpha) = 0, \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s}\, ds, & \text{if } \ell(\alpha) \geq 1,\ j_{\ell(\alpha)} = 0, \\ \int_p^\tau I_{\alpha-}[f(\cdot)]_{\rho,s}\, dW_s^{j_{\ell(\alpha)}}, & \text{if } \ell(\alpha) \geq 1,\ j_{\ell(\alpha)} \neq 0. \end{cases} \tag{2.3}$$

We also define the operators

$$L^0 = \frac{\partial}{\partial t} + a^k \frac{\partial}{\partial u^k} + \frac{1}{2} b^{kj} b^{lj} \frac{\partial^2}{\partial u^k\, \partial u^l}, \qquad L^j = b^{kj}\frac{\partial}{\partial u^k}, \tag{2.4}$$

and, if $f \in C^h(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R})$, where $h \geq \ell(\alpha) + n(\alpha)$, we set

$$f_\alpha = \begin{cases} f, & \text{if } \ell(\alpha) = 0, \\ L^{j_1} f_{-\alpha}, & \text{if } \ell(\alpha) \geq 1. \end{cases} \tag{2.5}$$

Here $-\alpha$ is α with its first component removed.

We note that if $f(t,u) \equiv u$, then $f_{(0)} = a$, $f_{(j)} = b^j$, etc. In what follows, unless explicitly stated otherwise, we will assume that f is this identity function.

A set, $\mathcal{A}$, of multi-indices is said to be a hierarchical set if $\mathcal{A} \neq \emptyset$, $\sup_{\alpha \in \mathcal{A}} \ell(\alpha) < \infty$, and $-\alpha \in \mathcal{A}$ whenever $\alpha \in \mathcal{A} - \{v\}$. We then define the remainder set $\mathcal{B}(\mathcal{A})$ of $\mathcal{A}$ by $\mathcal{B}(\mathcal{A}) = \{\alpha \mid \alpha \notin \mathcal{A}$ and $-\alpha \in \mathcal{A}\}$. We can now provide a stochastic Taylor expansion for U satisfying (2.1): if $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$, then, provided the derivatives and integrals exist,

$$f(\tau, U_\tau) = \sum_{\alpha \in \mathcal{A}} I_\alpha[f_\alpha(\rho, U_\rho)]_{\rho,\tau} + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(\cdot, U_\cdot)]_{\rho,\tau}, \tag{2.6}$$

where $\mathcal{A}$ is some hierarchical set.

Now, for $\gamma = 0.5, 1.0, 1.5, \dots$, we set

$$\mathcal{A}_\gamma = \left\{ \alpha \;\middle|\; \ell(\alpha) + n(\alpha) \leq 2\gamma \text{ or } \ell(\alpha) = n(\alpha) = \gamma + \frac{1}{2} \right\}. \tag{2.7}$$

We call the stochastic Taylor expansion with $\mathcal{A} = \mathcal{A}_\gamma$ the expansion to order γ.

3. A STOCHASTIC "IMPLICIT LEAPFROG" SCHEME

The barotropic vorticity model supplied to us by C. Penland and P. Sardeshmukh of the National Oceanic and Atmospheric Administration in Boulder, Colorado (see [8]), takes the form

$$\frac{\partial \zeta}{\partial t} = -\nabla \cdot (v\zeta) + S - r\xi - \kappa \nabla^4 \xi, \tag{3.1}$$

where $\zeta = \nabla^2 \psi + f = \xi + f$ and $v = \hat{k} \times \nabla \psi$. Here, ζ is the total vorticity, v is the velocity vector, f is the Coriolis term, S is a (deterministic) forcing, r and κ are constants, and ξ is the local vorticity.

The numerical scheme they provided for this uses spherical harmonics, and, writing F for $-\nabla \cdot (v\zeta)$, the equation becomes

$$\frac{d}{dt} \zeta_n^m = F_n^m + S_n^m - r\xi_n^m - \kappa \left[\frac{n(n+1)}{a^2} \right]^2 \zeta_n^m. \tag{3.2}$$

Then the scheme has two steps. First, a leapfrog step:

$$\tilde{\zeta}_n^m(t + \Delta t) = \zeta_n^m(t - \Delta t) + 2\Delta t [F_n^m(t) + S_n^m(t)], \tag{3.3}$$

followed by an implicit step:

$$\zeta_n^m(t + \Delta t) = \frac{\tilde{\zeta}_n^m(t + \Delta t)}{1 + 2\Delta t \left[r + \kappa \left[(n(n+1))/a^2 \right]^2 \right]}. \tag{3.4}$$

If we simplify notation and write a_1 for $F + S$ and a_2 for $-r\xi - \kappa \nabla^4 \xi$, we see that this is just an "implicit leapfrog" scheme

$$\begin{aligned} \tilde{Y}(t + \Delta t) &= Y(t - \Delta t) + 2\Delta t a_1(t, Y(t)), \\ Y(t + \Delta t) &= \tilde{Y}(t + \Delta t) + 2\Delta t a_2(t + \Delta t, Y(t + \Delta t)), \end{aligned} \tag{3.5}$$

for the equation

$$dU(t) = [a_1(t, U(t)) + a_2(t, U(t))]\, dt. \tag{3.6}$$

Therefore, we consider a stochastic differential equation of the form

$$dU_t = (a_1(t, U_t) + a_2(t, U_t))\, dt + b(t, U_t)\, dW_t. \tag{3.7}$$

Note that we have simply added a general diffusion term to the deterministic differential equation (3.6).

We will consider the scheme

$$\begin{aligned} \tilde{Y}_{n+2} &= Y_n + 2a_1(t_{n+1}, Y_{n+1})\Delta t + M_n(Y_n) + M_{n+1}(Y_{n+1}), \\ Y_{n+2} &= \tilde{Y}_{n+2} + 2a_2(t_{n+2}, Y_{n+2})\Delta t, \end{aligned} \tag{3.8}$$

where

$$M_n(y) = b(t_n, y)\Delta W_n + bb'(t_n, y) I_{(1,1),n}. \tag{3.9}$$

THEOREM 3.1. *Suppose that the coefficient functions* f_α *satisfy*

$$|f_\alpha(t, x) - f_\alpha(t, y)| \le K|x - y|, \tag{3.10}$$

for all $\alpha \in \mathcal{A}_{1.0}$, $t \in [0, T]$, *and* $x, y \in \mathbb{R}^d$;

$$f_{-\alpha} \in C^{1,2} \qquad \text{and} \qquad f_\alpha \in \mathcal{H}_\alpha, \tag{3.11}$$

for all $\alpha \in \mathcal{A}_{1.0} \cup \mathcal{B}(\mathcal{A}_{1.0})$; *and*

$$|f_\alpha(t, x)| \le K(1 + |x|), \tag{3.12}$$

for all $\alpha \in \mathcal{A}_{1.0} \cup \mathcal{B}(\mathcal{A}_{1.0})$, $t \in [0, T]$, *and* $x \in \mathbb{R}^d$. *Choose* $\Delta t \le 1$ *and set* $N = T/\Delta t$, *and define* $t_n = n\Delta t$ *for* $n = 1, \ldots, N$. *Suppose that some appropriate numerical scheme is used to generate* Y_1 *such that* $\mathbb{E}[|U_{t_1} - Y_1|^2 \mid \mathcal{F}_0]^{1/2} \le C\Delta t$. *Then,*

$$\mathbb{E}\left[\sup_{0 \le n \le N} |U_{t_n} - Y_n|^2 \;\Big|\; \mathcal{F}_0\right]^{1/2} \le C\Delta t. \tag{3.13}$$

4. SPACE DISCRETIZATION

It sometimes happens in applications that the functions a and b may only be known empirically (i.e., in tables) rather than analytically. In such cases, analytic derivatives of these functions can be difficult to obtain. It is, therefore, useful to replace these derivatives by discrete approximations. As a first example, consider this modification of Mil'shtein's scheme

$$\begin{aligned} \hat{Y}^k_{n+1} = \hat{Y}^k_n &+ \sum_{j=1}^{d} b^{k,j}\left(\hat{Y}_n\right)\Delta W^j_n + a^k\left(\hat{Y}_n\right)\Delta t \\ &+ \sum_{j_1, j_2, \ell = 1}^{d} \frac{1}{\Delta x} b^{\ell, j_1}\left(\hat{Y}_n\right)\left(b^{k, j_2}\left(\hat{Y}_n + \Delta x e^\ell\right) - b^{k, j_2}\left(\hat{Y}_n\right)\right) I_{(j_1, j_2), n}, \end{aligned} \tag{4.1}$$

where e^ℓ is the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ℓ^{th} position, and we have chosen $\Delta x > 0$. We have also suppressed the dependence of a and b on time to simplify notation.

We then have the following theorem.

THEOREM 4.1. *Suppose that* a *and* b *have the regularity required for Mil'shtein's scheme to converge to the solution* U *to order* Δt. *Then,*

$$\mathbb{E}\left[\sup_{0 \le n \le N} \left|U_n - \hat{Y}_n\right|^2\right]^{1/2} = O\left(\max\left\{\Delta t, \Delta x \Delta t^{1/2}\right\}\right). \tag{4.2}$$

Note that if we want to maintain the order of convergence of Mil'shtein's scheme, we need that $\Delta x = O(\Delta t^{1/2})$.

5. A STOCHASTIC ADAMS-BASHFORTH SCHEME

The following is a stochastic version of a scheme which is very effective and commonly used in computational fluid dynamics. The deterministic Adams-Bashforth scheme for the ordinary differential equation $\phi' = F(\phi)$ takes the form

$$\phi_{n+1} = \phi_n + \frac{\Delta t}{2}[3F(\phi_n) - F(\phi_{n-1})]. \tag{5.1}$$

This scheme is order Δt^2 in the deterministic case.

We consider the following stochastic Adams-Bashforth (SAB) scheme:

$$Y_{n+2} = Y_{n+1} + \left[\frac{3}{2}a(t_{n+1}, Y_{n+1}) - \frac{1}{2}a(t_n, Y_n)\right]\Delta t - \frac{3}{2}\Delta t A_n(t_n, Y_n) + B_n(t_n, Y_n), \tag{5.2}$$

in which

$$A_n(t,x) = L^j a(t,x)\Delta W^j + L^{j_1}L^{j_2}a(t,x)I_{(j_1,j_2)}, \tag{5.3}$$

where the random intervals are from time t_n to t_{n+1}, and

$$\begin{aligned} B_n(t,x) = {} & b^j(t,x)\Delta W^j + L^0 b^j(t,x)I_{(0,j)} + L^j a(t,x)I_{(j,0)} \\ & + L^{j_1}b^{j_2}(t,x)I_{(j_1,j_2)} + L^0 L^{j_1}b^{j_2}(t,x)I_{(0,j_1,j_2)} \\ & + L^{j_1}L^0 b^{j_2}(t,x)I_{(j_1,0,j_2)} + L^{j_1}L^{j_2}a(t,x)I_{(j_1,j_2,0)} \\ & + L^{j_1}L^{j_2}b^{j_3}(t,x)I_{(j_1,j_2,j_3)} + L^{j_1}L^{j_2}L^{j_3}b^{j_4}(t,x)I_{(j_1,j_2,j_3,j_4)}, \end{aligned} \tag{5.4}$$

where the random intervals are those from time t_n to t_{n+2} minus those from time t_n to t_{n+1}.

We then have the following theorem.

THEOREM 5.1. *Suppose that the coefficient functions f_α satisfy*

$$|f_\alpha(t,x) - f_\alpha(t,y)| \le K|x-y|, \tag{5.5}$$

for all $\alpha \in \mathcal{A}_{2.0}$, $t \in [0,T]$, and $x, y \in \mathbb{R}^d$;

$$f_{-\alpha} \in C^{1,2} \qquad \text{and} \qquad f_\alpha \in \mathcal{H}_\alpha, \tag{5.6}$$

for all $\alpha \in \mathcal{A}_{2.0} \cup \mathcal{B}(\mathcal{A}_{2.0})$; and

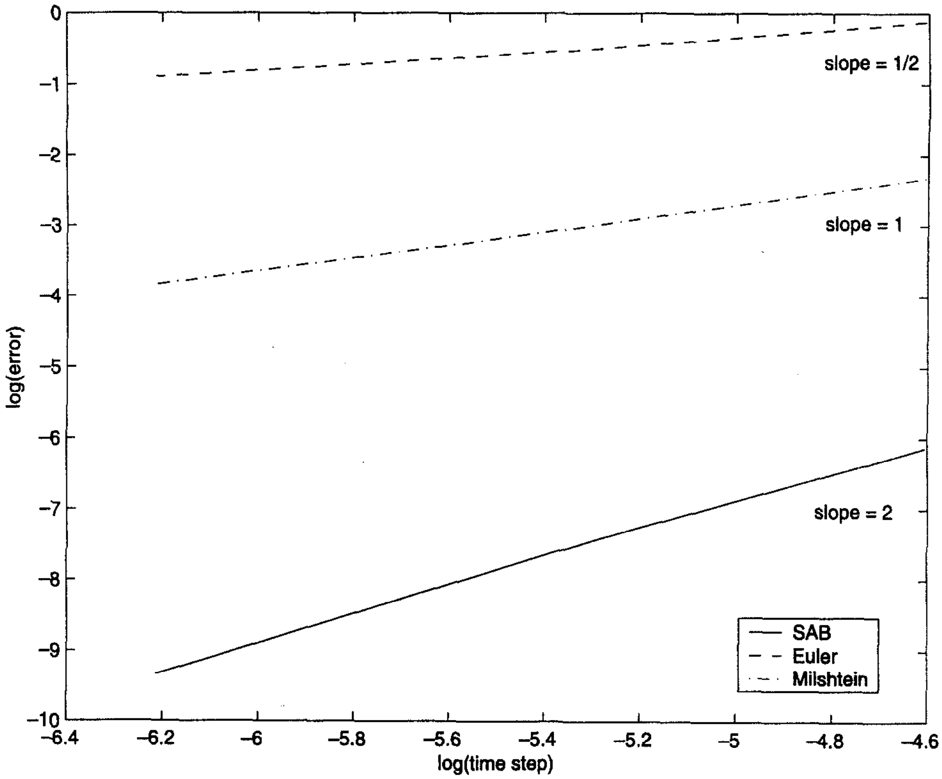
$$|f_\alpha(t,x)| \le K(1+|x|), \tag{5.7}$$

for all $\alpha \in \mathcal{A}_{2.0} \cup \mathcal{B}(\mathcal{A}_{2.0})$, $t \in [0,T]$, and $x \in \mathbb{R}^d$. Choose $\Delta t \le 1$ and set $N = T/\Delta t$, and define $t_n = n\Delta t$ for $n = 1, \dots, N$. Suppose that some appropriate numerical scheme is used to generate Y_1 such that $\mathbb{E}[|U_{t_1} - Y_1|^2 \mid \mathcal{F}_0]^{1/2} \le C\Delta t^2$. Then,

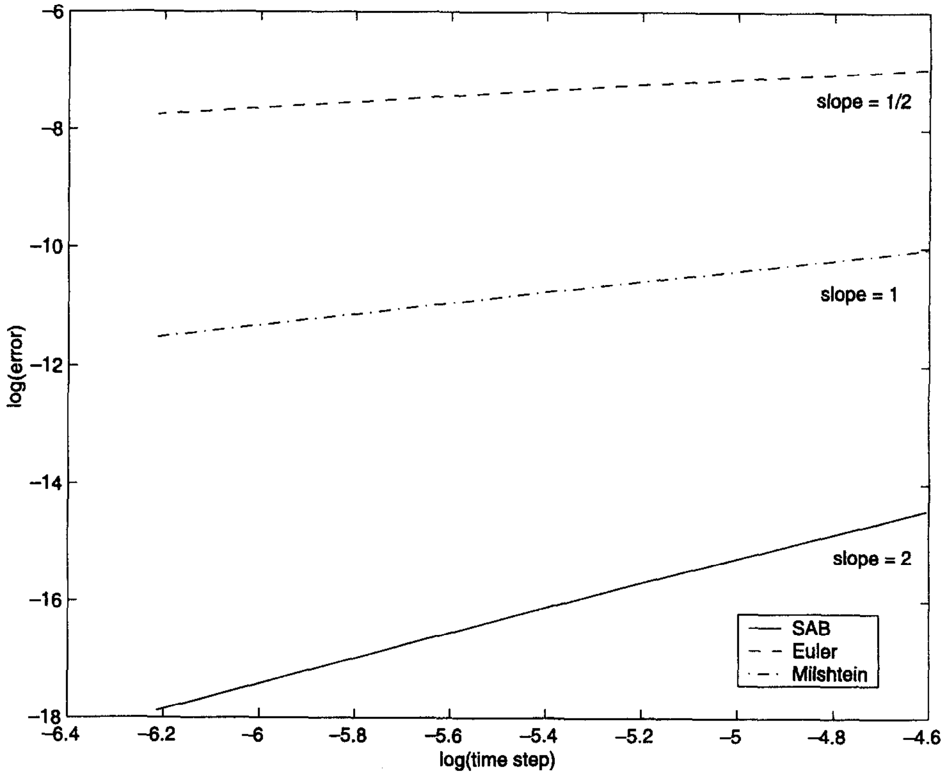
$$\mathbb{E}\left[\sup_{0\le n\le N} |U_{t_n} - Y_n|^2 \mid \mathcal{F}_0\right]^{1/2} \le C\Delta t^2. \tag{5.8}$$

6. NUMERICAL SIMULATION

The object of this section is to test numerically the accuracy of the scheme of Section 5 and compare it to the theoretical result above (i.e., $O(\Delta t^2)$ accuracy) and to the accuracy of the Euler and Mil'shtein schemes (respectively, $O(\Delta t^{1/2})$ and $O(\Delta t)$). All the numerical results below are consistent with the theoretical ones.



(a) Stochastic equation (6.1).



(b) Stochastic equation (6.2).

Figure 1. Results obtained with the stochastic equations (6.1) and (6.2).

We consider the following equations:

$$dX_t = \frac{1}{2}\alpha^2 X_t\,dt + \alpha\sqrt{X_t^2 - 1}\,dW_t, \tag{6.1}$$

with $\alpha = 1$ and $X_0 = 10$, and

$$dX_t = \beta^2 \sinh X_t \cosh^2 X_t\,dt + \beta\cosh^2 X_t\,dW_t, \tag{6.2}$$

with $\beta = 1/10$ and $X_0 = 1/2$. These have the exact solutions

$$X_t = \cosh(\alpha W_t + \operatorname{arccosh} X_0) \tag{6.3}$$

and

$$X_t = \operatorname{arctanh}(\beta W_t + \tanh X_0), \tag{6.4}$$

respectively. These can be easily verified using Itô's formula and are just two of many possible examples listed in [6].

We computed approximate solutions Y_n using the Euler and Mil'shtein schemes and the SAB scheme from Section 3. Then we computed the following error:

$$e = \sqrt{\mathbb{E}\left(\sup_{0\le n\le N} |X_n - Y_n|^2\right)}. \tag{6.5}$$

To obtain the mean value needed, we used 500 trajectories.

In the figures, the order of each scheme is given by the slope of the corresponding line. So we can see that the orders are 1/2 for Euler, 1 for Mil'shtein, and 2 for the SAB of Section 3.

Note that for the SAB scheme, the stochastic integral $I_{(0,1,1)}$ (which is difficult to generate) was approximated by a normal law. The results tend to show that this does not affect the accuracy (at least in these two cases). We will try to improve this point, which seems to raise interesting probabilistic questions, as already mentioned in the Introduction.

REFERENCES

1. V. Bally and D. Talay, The law of the Euler scheme for stochastic differential equations I. Convergence rate of the distribution function, *Probab. Theory Relat. Fields* **104**, 43–60, (1996).
2. V. Bally and D. Talay, The law of the Euler scheme for stochastic differential equations II. Convergence rate of the density, *Monte Carlo Methods and Appl.* **2** (2), 93–128, (1996).
3. D. Talay, Simulation of stochastic differential equations, In *Probabilistic Methods in Applied Physics*, (Edited by P. Krée and W. Wedig), pp. 54–96, Springer-Verlag, Berlin, (1995).
4. G.N. Mil'shtein, Approximate integration of stochastic differential equations, *Theory Prob. Appl.* **19**, 557–562, (1974).
5. W. Rümelin, Numerical treatment of stochastic differential equations, *SIAM J. Numer. Anal.* **19** (3), 604–613, (June 1982).
6. P.E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations, Application of Mathematics, Volume 23*, Springer-Verlag, Berlin, (1992).
7. P.E. Kloeden, E. Platen and H. Schurz, *Numerical Solution of SDE through Computer Experiments*, Universitext, Springer-Verlag, Berlin, (1994).
8. P.D. Sardeshmukh and B.J. Hoskins, The generation of global rotational flow by steady idealized tropical divergence, *Journal of the Atmospheric Sciences* **45** (7), 1228–1251, (1988).
9. J.G. Gaines and T.J. Lyons, Random generation of stochastic area integrals, *SIAM Journal on Applied Mathematics* **54** (4), 1132–1146, (August 1994).