DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **7**, Number **2**, April **2001**

Website: http://AIMsciences.org

pp. **343–362**

MAXIMUM PRINCIPLES FOR THE PRIMITIVE EQUATIONS OF THE ATMOSPHERE

Brian D. Ewald† and Roger Témam†,‡

†Institute for Scientific Computing and Applied Mathematics
Indiana University
Rawles Hall
831 E. Third Street
Bloomington, IN 47405

‡Université de Paris-Sud
et Centre National de la Recherche Scientifique
Laboratoire d'Analyse Numérique d'Orsay
Bâtiment 425
91405 Orsay Cedex, France

Abstract. In this article, maximum principles are derived for a suitably modified form of the equation of temperature for the primitive equations of the atmosphere; we consider both the limited domain case in Cartesian coordinates and the flow of the whole atmosphere in spherical coordinates.

Introduction. The primitive equations of the atmosphere and of the ocean have been studied, from the mathematical viewpoint, in Lions, Temam, and Wang ([6], [7]; see also [8], [9]) for the coupling of the atmosphere and the ocean. Concerning the equations of temperature, the classical methods used for the maximum principle apply to the temperature equation for the ocean but they seemingly do not apply to the temperature equation for the atmosphere.

This does not appear to be due to a mathematical technicality but rather to a difference of structure of the equations, the water being incompressible and the air being compressible. On physical grounds, it was suggested to us to consider a modified temperature equation for the atmosphere, namely the equation for the *potential temperature* θ, whose definition is recalled below. We have been able, in this way, to derive the desired estimates.

The aim of this article is to introduce the potential temperature equation and to derive L^∞ estimates for θ (which provide, afterwards, positivity and L^∞ estimates for the classical temperature T). The article is organized as follows: in Section 1, we recall the primitive equations of the atmosphere (PEs) and introduce the potential temperature and the corresponding equations. In Section 2, we provide the weak formulation of the PEs in a limited domain (with suitable, physically reasonable boundary conditions), and we establish the existence of weak solutions. Then, maximum principles are established in Section 3 by a combination of the truncation (Stampacchia) method for the positivity and classical methods for the L^∞ bound. In Section 4, we describe the similar results for the whole atmosphere and present the changes in the proofs which are necessary in this case.

2000 *Mathematics Subject Classification.* 35Q35, 86A10, 37N10, 76U05.

This work was supported in part by the grant NSF-DMS-0074334 and by the Research Fund of Indiana University.

The second author (RT) wishes to thank Joe Tribbia for suggesting that he consider the potential temperature and for introducing him to the thermodynamics of the atmosphere.

1. **The Primitive Equations of the Atmosphere.** In this section, we recall the primitive equations and the associated boundary conditions for a limited region of the atmosphere. We also introduce the potential temperature and the corresponding equation.

1.1. **The primitive equations.** We first recall the equation of temperature for the atmosphere (see [10], beginning on p.71):

$$c_{\mathrm{p}} \frac{dT}{dt} - \frac{RT}{p}\omega = \frac{dQ}{dt} + D. \tag{1.1}$$

In this equation, T is the actual atmospheric temperature; p, the pressure, is used as the vertical coordinate; and $\omega = \dfrac{dp}{dt}$ is the vertical velocity in the pressure coordinate system. Both c_{p} and R are constants (the specific heat of the atmosphere at constant pressure and the gas law constant, respectively), and Q is the heat applied to the system.

In (1.1), D represents a dissipation term. As in [6], we choose $D = L_{\mathrm{T}} T$ (see below), and we arrive at the following modification of equation (1.1):

$$c_{\mathrm{p}} \frac{dT}{dt} - \frac{RT}{p}\frac{dp}{dt} - L_{\mathrm{T}} T = Q_E, \tag{1.2}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla + \omega \frac{\partial}{\partial p},$$

$$L_{\mathrm{T}} = \mu_{\mathrm{T}} \Delta + \nu_{\mathrm{T}} \frac{\partial}{\partial p}\left[\left(\frac{gp}{R\bar{T}}\right)^2 \frac{\partial}{\partial p}\right].$$

Here, $\bar{T}(p)$ is the average temperature over the isobar with pressure p; see footnote 2 following equation (1.12) of [8]. We will need to make the assumption that there exist positive constants $\bar{T}^*$ and $\bar{T}_*$ such that $0 < \bar{T}_* \le \bar{T} \le \bar{T}^*$. Both ∇ and Δ are applied only in the horizontal direction. We consider p to have values from p_0 somewhere high in the atmosphere to p_1 near the earth's surface; i.e., $p = p_0$ is an isobar in the high atmosphere, $p = p_1$ is an isobar slightly above the earth and the oceans, and we study the dynamics of the atmosphere in the region $0 < p_0 \le p \le p_1$.

We then introduce the potential temperature defined as

$$\theta = T \left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}}. \tag{1.3}$$

It is important here to observe that

$$c_{\mathrm{p}} \frac{dT}{dt} - \frac{RT}{p}\frac{dp}{dt} = c_{\mathrm{p}} \left(\frac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \frac{d\theta}{dt}. \tag{1.4}$$

Substituting θ for T, we transform (1.2) into

$$c_{\mathrm{p}} \frac{d\theta}{dt} - \mu_{\mathrm{T}} \Delta\theta - \nu_{\mathrm{T}} \left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}} \frac{\partial}{\partial p}\left[\left(\frac{gp}{R\bar{T}}\right)^2 \frac{\partial}{\partial p}\left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta\right)\right] = Q_E. \tag{1.5}$$

In this article, we will primarily be concerned with the temperature equation in this form.

Using θ, we arrive at the following formulation for the PEs (compare to (1.33) in [6]):

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v + \omega\dfrac{\partial v}{\partial t} + f(k\times v) + \nabla\Phi \\ \qquad -\mu_{\mathrm{v}}\Delta v - \nu_{\mathrm{v}}\dfrac{\partial}{\partial p}\left[\left(\dfrac{gp}{R\bar{T}}\right)^2\dfrac{\partial v}{\partial p}\right] = 0 \\ \dfrac{\partial\Phi}{\partial p} + \dfrac{R}{p}\left(\dfrac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}}\theta = 0 \\ \nabla\cdot v + \dfrac{\partial\omega}{\partial p} = 0 \\ c_{\mathrm{p}}\dfrac{\partial\theta}{\partial t} + c_{\mathrm{p}}(v\cdot\nabla)\theta + c_{\mathrm{p}}\omega\dfrac{\partial\theta}{\partial p} - \mu_{\mathrm{T}}\Delta\theta \\ \qquad -\nu_{\mathrm{T}}\left(\dfrac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}}\dfrac{\partial}{\partial p}\left[\left(\dfrac{gp}{R\bar{T}}\right)^2\dfrac{\partial}{\partial p}\left(\left(\dfrac{p}{p_0}\right)\theta\right)\right] = Q_E \end{cases} \tag{1.6}$$

1.2. **Boundary conditions.** We will consider equations (1.6) in a parallelepiped $\mathcal{M}$ over the earth's surface, $\mathcal{M} = \mathfrak{M}\times(p_0,p_1)$, where $\mathfrak{M}$ is a horizontal isobar in $\mathcal{M}$. We partition the boundary of $\mathcal{M}$ as:

$$\partial\mathcal{M} = \Gamma_{\mathrm{i}}\cup\Gamma_{\mathrm{u}}\cup\Gamma_\ell,$$

where $\Gamma_{\mathrm{i}} = \mathfrak{M}\times\{p=p_1\}$ corresponds to the "interface" with the ocean or the earth's surface, $\Gamma_{\mathrm{u}} = \mathfrak{M}\times\{p=p_0\}$ (the "upper surface") is the boundary in the high atmosphere, and $\Gamma_\ell = \partial\mathfrak{M}\times(p_0,p_1)$ (the "lateral surface") is the vertical boundary.

We will use the following physically reasonable boundary conditions:

$$\begin{cases} \omega = 0 & \text{on } \Gamma_{\mathrm{u}}\cup\Gamma_{\mathrm{i}}, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_{\mathrm{u}}, \\ \omega = g_\omega & \text{on } \Gamma_\ell, \\ \dfrac{\partial v}{\partial n} = \alpha_{\mathrm{v}}(v_* - v) & \text{on } \Gamma_{\mathrm{i}}, \\ v = g_v & \text{on } \Gamma_\ell, \\ \theta = g_\theta & \text{on } \Gamma_\ell, \\ \dfrac{\partial\theta}{\partial n} = \alpha_{\mathrm{T}}(\theta_* - \theta) - \dfrac{R}{pc_{\mathrm{p}}}\theta & \text{on } \Gamma_{\mathrm{u}}\cup\Gamma_{\mathrm{i}}. \end{cases} \tag{1.7}$$

Here, g_v, g_θ, g_ω, v_*, and θ_* are given functions. Note also that the boundary equation on $\Gamma_{\mathrm{u}}\cup\Gamma_{\mathrm{i}}$, when written in terms of T rather than θ, is the classical Robin type equation $\dfrac{\partial T}{\partial n} = \alpha_{\mathrm{T}}(T_* - T)$, with θ_* and T_* related by (1.3).

2. **The Existence of Solutions.** In this section, we provide the weak formulation of equations (1.6) supplemented by the boundary conditions (1.7) and show how to prove the existence of weak solutions to these equations.

2.1. **Mathematical setting and weak formulation.** We define the function spaces

$$V_v = \left\{ v = (v_1,v_2)\in H^1(\mathcal{M})^2 \;\middle|\; \gamma_0 v|_{\Gamma_\ell} = 0, \int_{p_0}^{p_1}\nabla\cdot v\,dp = 0 \right\},$$
$$V_\theta = \left\{\theta\in H^1(\mathcal{M}) \mid \gamma_0\theta|_{\Gamma_\ell} = 0\right\},$$
$$V = V_v\times V_\theta.$$

Here, $\gamma_0\colon H^1(\mathcal{M}) \to H^{\frac{1}{2}}(\partial\mathcal{M})$ is the trace operator. Notationally, we will write $u = (v, \theta)$, $\hat{u} = (\hat{v}, \hat{\theta})$, $\tilde{u} = (\tilde{v}, \tilde{\theta})$, etc., for elements $u, \hat{u}, \tilde{u}$ of V.

We define the inner products on V_v, V_θ, V, as follows:

$$(\!(v, \hat{v})\!) = \int_{\mathcal{M}} \nabla v \cdot \nabla \hat{v}\, d\mathcal{M} + \int_{\mathcal{M}} \frac{\partial v}{\partial p} \cdot \frac{\partial \hat{v}}{\partial p}\, d\mathcal{M},$$

$$(\!(\theta, \hat{\theta})\!) = \int_{\mathcal{M}} \nabla \theta \cdot \nabla \hat{\theta}\, d\mathcal{M} + \int_{\mathcal{M}} \frac{\partial \theta}{\partial p} \cdot \frac{\partial \hat{\theta}}{\partial p}\, d\mathcal{M},$$

$$(\!(u, \hat{u})\!) = (\!(v, \hat{v})\!) + (\!(\theta, \hat{\theta})\!).$$

These products are equivalent to the standard H^1-inner products on these spaces, and with the norms

$$\|v\| = (\!(v, v)\!)^{\frac{1}{2}}, \quad \|\theta\| = (\!(\theta, \theta)\!)^{\frac{1}{2}}, \quad \|u\| = (\!(u, u)\!)^{\frac{1}{2}},$$

the spaces V_v and V_θ are closed subspaces of $H^1(\mathcal{M})^2$ and $H^1(\mathcal{M})$.

We use single parentheses and norms for the L^2-inner products and norms, e.g.,

$$(v, \hat{v}) = \int_{\mathcal{M}} v \cdot \hat{v}\, d\mathcal{M}, \quad |\theta| = \left(\int_{\mathcal{M}} \theta^2\, d\mathcal{M} \right)^{\frac{1}{2}}.$$

To obtain the weak formulation of (1.6) and (1.7), we make the following observations:

In view of the first equation of (1.7) and the third equation of (1.6), we have

$$\omega(p) = -\int_p^{p_1} \frac{\partial \omega}{\partial p}(p')\, dp' = \int_p^{p_1} \nabla \cdot v(p')\, dp', \tag{2.1}$$

for any $v \in V_v$, whence the condition

$$\int_{p_0}^{p_1} \nabla \cdot v\, dp = \omega(p_1) - \omega(p_0) = 0$$

appearing in the definition of V_v.

Also, writing $\Phi_{\mathrm{s}} = \Phi|_{p=p_1}$ for the value of Φ at the earth's surface, we have

$$\begin{aligned} \Phi(p) &= \Phi_{\mathrm{s}} - \int_p^{p_1} \frac{\partial \Phi}{\partial p}\, dp' \\ &= \Phi_{\mathrm{s}} + \int_p^{p_1} \frac{R}{p'} \left(\frac{p'}{p_0} \right)^{\frac{R}{c_{\mathrm{p}}}} \nabla \theta\, dp', \qquad \text{(from the second equation of (1.6)).} \end{aligned}$$

Thus,

$$\nabla \Phi(p) = \nabla \Phi_{\mathrm{s}} + \int_p^{p_1} \frac{R}{p'} \left(\frac{p'}{p_0} \right)^{\frac{R}{c_{\mathrm{p}}}} \nabla \theta\, dp',$$

and, for $\hat{v} \in V_v$

$$\begin{aligned} \int_{\mathcal{M}} \nabla \Phi_{\mathrm{s}} \cdot \hat{v}\, d\mathcal{M} &= \int_{\Gamma_\ell} \Phi_{\mathrm{s}} n_{\mathrm{h}} \cdot \hat{v}\, d\Gamma_\ell - \int_{\mathcal{M}} \Phi_{\mathrm{s}} \nabla \cdot \hat{v}\, d\mathcal{M} \\ &= \int_{\mathcal{M}} \Phi_{\mathrm{s}} \frac{\partial \hat{\omega}}{\partial p}\, d\mathcal{M} \\ &= \int_{\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}} \Phi_{\mathrm{s}} n_{\mathrm{p}} \cdot \hat{\omega}\, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}} - \int_{\mathcal{M}} \frac{\partial \Phi_{\mathrm{s}}}{\partial p} \hat{\omega}\, d\mathcal{M}. \\ &= 0. \end{aligned}$$

Thus, we have

$$\int_{\mathcal{M}} \nabla\Phi \cdot \hat{v}\, d\mathcal{M} = \int_{\mathcal{M}} \left(\int_{p}^{p_1} \frac{R}{p'} \left(\frac{p'}{p_0} \right)^{\frac{R}{c_{\mathrm{p}}}} \nabla\theta\, dp' \right) \cdot \hat{v}\, d\mathcal{M}. \tag{2.2}$$

Now, we note that, for $v, \hat{v} \in V_v$,

$$\begin{aligned}
-\mu_{\mathrm{V}} \int_{\mathcal{M}} \Delta v \cdot \hat{v}\, d\mathcal{M} &- \nu_{\mathrm{V}} \int_{\mathcal{M}} \frac{\partial}{\partial p} \left[\left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial v}{\partial p} \right] \cdot \hat{v}\, d\mathcal{M} \\
&= \mu_{\mathrm{V}} \int_{\mathcal{M}} \nabla v \cdot \nabla \hat{v}\, d\mathcal{M} + \nu_{\mathrm{V}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial v}{\partial p} \cdot \frac{\partial \hat{v}}{\partial p}\, d\mathcal{M} \\
&\quad - \mu_{\mathrm{V}} \int_{\Gamma_\ell} \nabla v n_{\mathrm{h}} \cdot \hat{v}\, d\Gamma_\ell - \nu_{\mathrm{V}} \int_{\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial v}{\partial p} \cdot n_{\mathrm{p}} \hat{v}\, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}} \\
&= \mu_{\mathrm{V}} \int_{\mathcal{M}} \nabla v \cdot \nabla \hat{v}\, d\mathcal{M} + \nu_{\mathrm{V}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial v}{\partial p} \cdot \frac{\partial \hat{v}}{\partial p}\, d\mathcal{M} \\
&\quad - \nu_{\mathrm{V}} \int_{\Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{V}} (v_* - v) \cdot \hat{v}\, d\Gamma_{\mathrm{i}}.
\end{aligned} \tag{2.3}$$

Similarly, for $\theta, \hat{\theta} \in V_\theta$,

$$\begin{aligned}
-\mu_{\mathrm{T}} \int_{\mathcal{M}} \Delta\theta \cdot \hat{\theta}\, d\mathcal{M} &- \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{p_0}{p} \right)^{\frac{R}{c_{\mathrm{p}}}} \frac{\partial}{\partial p} \left[\left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial}{\partial p} \left(\left(\frac{p}{p_0} \right)^{\frac{R}{c_{\mathrm{p}}}} \theta \right) \right] \hat{\theta}\, d\mathcal{M} \\
&= \mu_{\mathrm{T}} \int_{\mathcal{M}} \nabla\theta \cdot \nabla\hat{\theta}\, d\mathcal{M} - \mu_{\mathrm{T}} \int_{\Gamma_\ell} n_{\mathrm{h}} \cdot \nabla\theta\, \hat{\theta}\, d\Gamma_\ell \\
&\quad + \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial}{\partial p} \left(\left(\frac{p}{p_0} \right)^{\frac{R}{c_{\mathrm{p}}}} \theta \right) \frac{\partial}{\partial p} \left(\left(\frac{p_0}{p} \right)^{\frac{R}{c_{\mathrm{p}}}} \hat{\theta} \right) d\mathcal{M} \\
&\quad - \nu_{\mathrm{T}} \int_{\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 n_{\mathrm{p}} \cdot \frac{\partial}{\partial p} \left(\left(\frac{p}{p_0} \right)^{\frac{R}{c_{\mathrm{p}}}} \theta \right) \left(\frac{p_0}{p} \right)^{\frac{R}{c_{\mathrm{p}}}} \hat{\theta}\, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}} \\
&= \mu_{\mathrm{T}} \int_{\mathcal{M}} \nabla\theta \cdot \nabla\hat{\theta}\, d\mathcal{M} + \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial\theta}{\partial p} \frac{\partial\hat{\theta}}{\partial p}\, d\mathcal{M} \\
&\quad - \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{g}{c_{\mathrm{p}}\bar{T}} \right)^2 \theta\hat{\theta}\, d\mathcal{M} - \nu_{\mathrm{T}} \int_{\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{T}} (\theta_* - \theta) \hat{\theta}\, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}.
\end{aligned} \tag{2.4}$$

In view of equations (2.1)-(2.4), we make the following definition:

Definition 2.1. For $u = (v, \theta), \hat{u} = (\hat{v}, \hat{\theta}), \tilde{u} = (\tilde{v}, \tilde{\theta}) \in V$, we set

$$\begin{cases}
a_v(v, \hat{v}) = \mu_{\mathrm{V}} \int_{\mathcal{M}} \nabla v \cdot \nabla \hat{v}\, d\mathcal{M} + \nu_{\mathrm{V}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial v}{\partial p} \cdot \frac{\partial \hat{v}}{\partial p}\, d\mathcal{M}, \\
a_\theta(\theta, \hat{\theta}) = \mu_{\mathrm{T}} \int_{\mathcal{M}} \nabla\theta \cdot \nabla\hat{\theta}\, d\mathcal{M} + \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}} \right)^2 \frac{\partial\theta}{\partial p} \frac{\partial\hat{\theta}}{\partial p}\, d\mathcal{M} \\
\qquad\qquad -\nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{g}{c_{\mathrm{p}}\bar{T}} \right)^2 \theta\hat{\theta}\, d\mathcal{M}, \\
a(u, \hat{u}) = a_v(v, \hat{v}) + a_\theta(\theta, \hat{\theta}).
\end{cases} \tag{2.5}$$

$$\begin{cases}
b_v(v, \hat{v}, \tilde{v}) = \int_{\mathcal{M}} \left((v \cdot \nabla)\hat{v} + \omega \frac{\partial \hat{v}}{\partial p} \right) \cdot \tilde{v}\, d\mathcal{M}, \\
b_\theta(v, \hat{\theta}, \tilde{\theta}) = c_{\mathrm{p}} \int_{\mathcal{M}} \left((v \cdot \nabla)\hat{\theta} + \omega \frac{\partial \hat{\theta}}{\partial p} \right) \tilde{\theta}\, d\mathcal{M}, \\
b(u, \hat{u}, \tilde{u}) = b_v(v, \hat{v}, \tilde{v}) + b_\theta(v, \hat{\theta}, \tilde{\theta}),
\end{cases} \tag{2.6}$$

where $\omega = \int_p^{p_1} \nabla \cdot v \, dp'$.

$$\begin{cases} e_c(v,\hat{v}) = \displaystyle\int_{\mathcal{M}} f(k \times v) \cdot \hat{v} \, d\mathcal{M}, \\ e_\Phi(\theta,\hat{v}) = \displaystyle\int_{\mathcal{M}} \int_p^{p_1} \left(\frac{R}{p'} \left(\frac{p'}{p_0} \right)^{\frac{R}{c_{\mathrm{p}}}} \nabla\theta \right) dp' \cdot \hat{v} \, d\mathcal{M}, \\ e(u,\hat{u}) = e_c(v,\hat{v}) + e_\Phi(\theta,\hat{v}). \end{cases} \tag{2.7}$$

$$\begin{cases} d_v(v,\hat{v}) = \nu_{\mathrm{V}} \displaystyle\int_{\Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{V}} (v_* - v) \cdot \hat{v} \, d\Gamma_{\mathrm{i}}, \\ d_\theta(\theta,\hat{\theta}) = \nu_{\mathrm{T}} \displaystyle\int_{\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{T}} (\theta_* - \theta) \hat{\theta} \, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}, \\ d(u,\hat{u}) = d_v(v,\hat{v}) + d_\theta(\theta,\hat{\theta}). \end{cases} \tag{2.8}$$

Thus, we arrive at the weak formulation of (1.6) supplemented with the boundary conditions (1.7), namely:

$$\begin{cases} \text{To find } u(t) \in V + g_u, \text{ where } g_u = (g_v, g_\theta), \text{ such that} \\ \left(\dfrac{\partial v}{\partial t}, \hat{v} \right) + c_{\mathrm{p}} \left(\dfrac{\partial \theta}{\partial t}, \hat{\theta} \right) + a(u,\hat{u}) + b(u,u,\hat{u}) + e(u,\hat{u}) \\ \qquad\qquad - d(u,\hat{u}) = (Q_E, \hat{\theta}), \quad \text{for all } \hat{u} \in V. \end{cases} \tag{2.9}$$

2.2. **Some bounds and inequalities.** In order to establish the existence of solutions u to (2.9), we need some basic bounds and inequalities.

Lemma 2.1. *For $v, \hat{v}, \tilde{v} \in H^1(\mathcal{M})^2$, $\theta, \hat{\theta} \in H^1(\mathcal{M})$, we have*

i) $|a_v(v,\hat{v})| \le \mu_{\mathrm{V}} |\nabla v| \cdot |\nabla \hat{v}| + \nu_{\mathrm{V}} c_1 \left| \dfrac{\partial v}{\partial p} \right| \cdot \left| \dfrac{\partial \hat{v}}{\partial p} \right|$,

ii) $a_v(v,v) \ge \mu_{\mathrm{V}} |\nabla v|^2 + \nu_{\mathrm{V}} c_2 \left| \dfrac{\partial v}{\partial p} \right|^2$,

iii) $|a_\theta(\theta,\hat{\theta})| \le \mu_{\mathrm{T}} |\nabla \theta| \cdot |\nabla \hat{\theta}| + \nu_{\mathrm{T}} c_1 \left| \dfrac{\partial \theta}{\partial p} \right| \cdot \left| \dfrac{\partial \hat{\theta}}{\partial p} \right| + \nu_{\mathrm{T}} c_3 |\theta| \cdot |\hat{\theta}|$,

iv) $a_\theta(\theta,\theta) \ge \mu_{\mathrm{T}} |\nabla \theta|^2 + \nu_{\mathrm{T}} c_2 \left| \dfrac{\partial \theta}{\partial p} \right|^2 - \nu_{\mathrm{T}} c_3 |\theta|^2$,

v) $b_v(v,\hat{v},\hat{v}) = 0$, *if $v \in V_v$*,

vi) $|b_v(v,\hat{v},\tilde{v})| \le |v| \cdot \|\nabla \hat{v}\|_{L^\infty(\mathcal{M})} \cdot |\tilde{v}|$,

vii) $b_\theta(v,\theta,\theta) = 0$, *if $v \in V_v$*,

viii) $|b_\theta(v,\theta,\hat{\theta})| \le |v| \cdot \|\nabla \theta\|_{L^\infty(\mathcal{M})} \cdot |\hat{\theta}|$,

ix) $|d_v(v,\hat{v})| \le \nu_{\mathrm{V}} \alpha_{\mathrm{V}} c_1 |v_* - v|_{L^2(\partial\mathcal{M})} \cdot \|\hat{v}\|$,

x) $d_v(v,v) = \nu_{\mathrm{V}} \displaystyle\int_{\Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{V}} v_* \cdot v \, d\Gamma_{\mathrm{i}} - \nu_{\mathrm{V}} \int_{\Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{V}} |v|^2 \, d\Gamma_{\mathrm{i}}$,

where $\nu_{\mathrm{V}} \left| \displaystyle\int_{\Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{V}} v_* \cdot v \, d\Gamma_{\mathrm{i}} \right| \le \nu_{\mathrm{V}} \alpha_{\mathrm{V}} c_1 |v_*|_{L^2(\partial\mathcal{M})} \cdot \|v\|$,

and $\nu_{\mathrm{V}} \displaystyle\int_{\Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{V}} |v|^2 \, d\Gamma_{\mathrm{i}} \ge \nu_{\mathrm{V}} \alpha_{\mathrm{V}} c_2 \|v\|^2$,

xi) $|d_\theta(\theta,\hat{\theta})| \le \nu_{\mathrm{T}} \alpha_{\mathrm{T}} c_1 |\theta_* - \theta|_{L^2(\partial\mathcal{M})} \cdot \|\hat{\theta}\|$,

xii) $d_\theta(\theta,\theta) = \nu_{\mathrm{T}} \displaystyle\int_{\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{T}} \theta_* \theta \, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}} - \nu_{\mathrm{T}} \int_{\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{T}} \theta^2 \, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}$,

where $\nu_{\mathrm{T}} \left| \displaystyle\int_{\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}} \right)^2 \alpha_{\mathrm{T}} \theta_* \theta \, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}} \right| \le \nu_{\mathrm{T}} \alpha_{\mathrm{T}} c_1 |\theta_*|_{L^2(\partial\mathcal{M})} \cdot \|\theta\|$, *and* $\nu_{\mathrm{T}} \displaystyle\int_{\Gamma_{\mathrm{u}}\cup\Gamma_{\mathrm{i}}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\mathrm{T}}\theta^2 \, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}} \geq \nu_{\mathrm{T}}\alpha_{\mathrm{T}} c_2 \|\theta\|^2$,

xiii) $|e_c(v, \hat{v})| \leq c_4 |v| \cdot |\hat{v}|$,

xiv) $e_c(v, v) = 0$,

xv) $|e_\Phi(\theta, v)| \leq c_5 |\theta| \cdot |\nabla v|$, *if* $v \in V_v$,

where $c_1 = \left(\dfrac{gp_1}{R\bar{T}_*}\right)^2$, $c_2 = \left(\dfrac{gp_0}{R\bar{T}^*}\right)^2$, $c_3 = \left(\dfrac{gp_1}{p_0 c_{\mathrm{p}} \bar{T}_*}\right)^2$, $c_4 = \|f\|_{L^\infty}$, *and* $c_5 = 2(p_1 - p_0)\dfrac{R}{p_0}\left(\dfrac{p_1}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}}$.

Proof. Since (i)-(iv), (vi), (viii)-(xiii) are clear, we omit the proofs of these.

For (v), we see that

$$
\begin{aligned}
b(v, \hat{v}, \hat{v}) &= \int_{\mathcal{M}} \left((v \cdot \nabla)\hat{v} + \omega \frac{\partial \hat{v}}{\partial p}\right) \cdot \hat{v} \, d\mathcal{M} \\
&= \frac{1}{2} \int_{\mathcal{M}} (v \cdot \nabla)|\hat{v}|^2 + \omega \frac{\partial}{\partial p}|\hat{v}|^2 \, d\mathcal{M} \\
&= \frac{1}{2} \int_{\mathcal{M}} \nabla \cdot (|\hat{v}|^2 v) - |\hat{v}|^2 (\nabla \cdot v) + \omega \frac{\partial}{\partial p}|\hat{v}|^2 \, d\mathcal{M} \\
&= \frac{1}{2} \int_{\Gamma_\ell} n_{\mathrm{h}} \cdot (|\hat{v}|^2 v) \, d\Gamma_\ell - \frac{1}{2} \int_{\mathcal{M}} |\hat{v}|^2 (\nabla \cdot v) \, d\mathcal{M} \\
&\quad + \frac{1}{2} \int_{\Gamma_{\mathrm{u}}\cup\Gamma_{\mathrm{i}}} \omega |\hat{v}|^2 \, d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}} - \frac{1}{2} \int_{\mathcal{M}} \frac{\partial \omega}{\partial p} |\hat{v}|^2 \, d\mathcal{M} \\
&= 0,
\end{aligned}
$$

since the boundary terms are zero and $\nabla \cdot v + \dfrac{\partial \omega}{\partial p} = 0$; the proof of (vii) is similar.

For (xiv), $k \times v \perp v$, so that $(k \times v) \cdot v = 0$.

For (xv), we see that

$$
\begin{aligned}
e_\Phi(\theta, v) &= \int_{\mathcal{M}} \left(\int_p^{p_1} \frac{R}{p'} \left(\frac{p'}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \nabla\theta \, dp'\right) \cdot v \, d\mathcal{M} \\
&= \int_{\mathcal{M}} \nabla \left(\int_p^{p_1} \frac{R}{p'} \left(\frac{p'}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta \, dp'\right) \cdot v \, d\mathcal{M} \\
&= \int_{\Gamma_\ell} \left(\int_p^{p_1} \frac{R}{p'} \left(\frac{p'}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta \, dp'\right) n_{\mathrm{h}} \cdot v \, d\Gamma_\ell \\
&\quad - \int_{\mathcal{M}} \left(\int_p^{p_1} \frac{R}{p'} \left(\frac{p'}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta \, dp'\right) \nabla \cdot v \, d\mathcal{M} \\
&= - \int_{\mathcal{M}} \left(\int_p^{p_1} \frac{R}{p'} \left(\frac{p'}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta \, dp'\right) \nabla \cdot v \, d\mathcal{M},
\end{aligned}
$$

since the boundary term is zero. Now,

$$
\left(\int_p^{p_1} \frac{R}{p'} \left(\frac{p'}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta \, dp'\right)^2 \leq (p_1 - p_0) \left(\frac{R}{p_0} \left(\frac{p_1}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}}\right)^2 \int_{p_0}^{p_1} \theta^2 \, dp',
$$

and so

$$\begin{aligned}
|e_\Phi(\theta, v)| &\le (p_1 - p_0)^{\frac{1}{2}} \frac{R}{p_0} \left(\frac{p_1}{p_0}\right)^{\frac{R}{c_p}} \int_{\mathcal{M}} \left(\int_{p_0}^{p_1} \theta^2 \, dp'\right)^{\frac{1}{2}} \nabla \cdot v \, d\mathcal{M} \\
&\le 2(p_1 - p_0)^{\frac{1}{2}} \frac{R}{p_0} \left(\frac{p_1}{p_0}\right)^{\frac{R}{c_p}} \left(\int_{\mathcal{M}} \left(\int_{p_0}^{p_1} \theta^2 \, dp'\right) d\mathcal{M}\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} |\nabla v|^2 \, d\mathcal{M}\right)^{\frac{1}{2}} \\
&\le 2(p_1 - p_0) \frac{R}{p_0} \left(\frac{p_1}{p_0}\right)^{\frac{R}{c_p}} |\theta| \cdot |\nabla v|.
\end{aligned}$$

□

2.3. **Existence of solutions.** We prove the following theorem about the existence of solutions to (2.9).

Theorem 2.2. *For any u_0 sufficiently regular, there exists a solution u to* (2.9) *satisfying $u(0) = u_0$.*

Proof. We will have use for an à priori estimate obtained by setting $\hat{u} = u - g_u$ in (2.9), where u is a certain solution, so that $u = \hat{u} + g_u$:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\hat{v}|^2 &+ \frac{c_p}{2}\frac{d}{dt}|\hat{\theta}|^2 + a_v(\hat{v}, \hat{v}) + a_\theta(\hat{\theta}, \hat{\theta}) + e_\Phi(\hat{\theta}, \hat{v}) \\
&= (Q_E, \hat{\theta}) + d_v(\hat{v}, \hat{v}) + d_\theta(\hat{\theta}, \hat{\theta}) - a_v(g_v, \hat{v}) - a_\theta(g_\theta, \hat{\theta}) \\
&- b_v(\hat{v}, g_v, \hat{v}) - b_v(g_v, g_v, \hat{v}) - b_\theta(\hat{v}, g_\theta, \hat{\theta}) - b_\theta(g_v, g_\theta, \hat{\theta}) \\
&- e_c(g_v, \hat{v}) - e_\Phi(g_\theta, \hat{v}) + d_v(g_v, \hat{v}) + d_\theta(g_\theta, \hat{\theta}) - \Big(\frac{\partial g_v}{\partial t}, \hat{v}\Big) - \Big(\frac{\partial g_\theta}{\partial t}, \hat{\theta}\Big).
\end{aligned} \tag{2.10}$$

Therefore, by using Lemma 2.1, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\hat{v}|^2 + \frac{c_p}{2}\frac{d}{dt}|\hat{\theta}|^2 + \mu_V|\nabla\hat{v}|^2 + \nu_V c_2\Big|\frac{\partial \hat{v}}{\partial p}\Big|^2 \\
&\quad + \mu_T|\nabla\hat{\theta}|^2 + \nu_T c_2\Big|\frac{\partial\hat{\theta}}{\partial p}\Big|^2 + \nu_V\alpha_V c_2\|\hat{v}\|^2 + \nu_T\alpha_T c_2\|\hat{\theta}\|^2 \\
&\qquad \le \xi,
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
\xi = \nu_T c_3|\hat{\theta}|^2 &+ c_5|\hat{\theta}| \cdot |\nabla\hat{v}| + |Q_E| \cdot |\hat{\theta}| + \nu_V\alpha_V c_1|v_*|_{L^2(\partial\mathcal{M})} \cdot \|\hat{v}\| \\
&+ \nu_T\alpha_T c_1|\theta_*|_{L^2(\partial\mathcal{M})} \cdot \|\hat{\theta}\| + \mu_V|\nabla g_v| \cdot |\nabla\hat{v}| + \nu_V c_1\Big|\frac{\partial g_v}{\partial p}\Big| \cdot \Big|\frac{\partial\hat{v}}{\partial p}\Big| \\
&+ \mu_T|\nabla g_\theta| \cdot |\nabla\hat{\theta}| + \nu_T c_1\Big|\frac{\partial g_\theta}{\partial p}\Big| \cdot \Big|\frac{\partial\hat{\theta}}{\partial p}\Big| + \nu_T c_3|g_\theta| \cdot |\hat{\theta}| \\
&+ \|\nabla g_v\|_{L^\infty} \cdot |\hat{v}|^2 + \|\nabla g_v\|_{L^\infty} \cdot |g_v| \cdot |\hat{v}| + c_p\|\nabla g_\theta\|_{L^\infty} \cdot |\hat{v}| \cdot |\hat{\theta}| \\
&+ c_p\|\nabla g_\theta\|_{L^\infty} \cdot |g_v| \cdot |\hat{\theta}| + c_5|g_\theta| \cdot |\nabla\hat{v}| + c_4|g_v| \cdot |\hat{v}| \\
&+ \nu_V\alpha_V c_1|v_* - g_v|_{L^2(\partial\mathcal{M})} \cdot \|\hat{v}\| + \nu_T\alpha_T c_1|\theta_* - g_\theta|_{L^2(\partial\mathcal{M})} \cdot \|\hat{\theta}\|
\end{aligned} \tag{2.12}$$

$$
\begin{aligned}
&\leq \nu_{\mathrm{T}} c_3 |\hat{\theta}|^2 + \frac{3c_5^2}{2\mu_{\mathrm{V}}}|\hat{\theta}|^2 + \frac{\mu_{\mathrm{V}}}{6}|\nabla\hat{v}|^2 + \frac{1}{2}|Q_E|^2 + \frac{1}{2}|\hat{\theta}|^2 \\
&\quad + \frac{\nu_{\mathrm{V}}\alpha_{\mathrm{V}}c_1^2}{c_2}|v_*|^2_{L^2(\partial\mathcal{M})} + \frac{\nu_{\mathrm{V}}\alpha_{\mathrm{V}}c_2}{4}\|\hat{v}\|^2 + \frac{\nu_{\mathrm{T}}\alpha_{\mathrm{T}}c_1^2}{c_2}|\theta_*|^2_{L^2(\partial\mathcal{M})} \\
&\quad + \frac{\nu_{\mathrm{T}}\alpha_{\mathrm{T}}c_2}{4}\|\hat{\theta}\|^2 + \frac{3\mu_{\mathrm{V}}}{2}|\nabla g_v|^2 + \frac{\mu_{\mathrm{V}}}{6}|\nabla\hat{v}|^2 + \frac{\nu_{\mathrm{V}}c_1^2}{2c_2}\Big|\frac{\partial g_v}{\partial p}\Big|^2 \\
&\quad + \frac{\nu_{\mathrm{V}}c_2}{2}\Big|\frac{\partial \hat{v}}{\partial p}\Big|^2 + \frac{\mu_{\mathrm{T}}}{2}|\nabla g_\theta|^2 + \frac{\mu_{\mathrm{T}}}{2}|\nabla\hat{\theta}|^2 + \frac{\nu_{\mathrm{T}}c_1^2}{2c_2}\Big|\frac{\partial g_\theta}{\partial p}\Big|^2 \\
&\quad + \frac{\nu_{\mathrm{T}}c_2}{2}\Big|\frac{\partial \hat{\theta}}{\partial p}\Big|^2 + \frac{\nu_{\mathrm{T}}c_3}{2}|g_\theta|^2 + \frac{\nu_{\mathrm{T}}c_3}{2}|\hat{\theta}|^2 + \|\nabla g_v\|^2_{L^\infty}\cdot|\hat{v}|^2 \\
&\quad + \frac{1}{2}\|\nabla g_v\|^2_{L^\infty}\cdot|g_v|^2 + \frac{1}{2}|\hat{v}|^2 + \frac{1}{2}|\hat{v}|^2 + \frac{{c_{\mathrm{p}}}^2}{2}\|\nabla g_\theta\|^2_{L^\infty}\cdot|\hat{\theta}|^2 \\
&\quad + \frac{1}{2}|g_v|^2 + \frac{{c_{\mathrm{p}}}^2}{2}\|\nabla g_\theta\|^2_{L^\infty}\cdot|\hat{\theta}|^2 + \frac{3c_5^2}{2\mu_{\mathrm{V}}}|g_\theta|^2 + \frac{\mu_{\mathrm{V}}}{6}|\nabla\hat{v}|^2 \\
&\quad + \frac{c_4^2}{2}|g_v|^2 + \frac{1}{2}|\hat{v}|^2 + \frac{\nu_{\mathrm{V}}\alpha_{\mathrm{V}}c_1^2}{c_2}|v_* - g_v|^2_{L^2(\partial\mathcal{M})} + \frac{\nu_{\mathrm{V}}\alpha_{\mathrm{V}}c_2}{4}\|\hat{v}\|^2 \\
&\quad + \frac{\nu_{\mathrm{T}}\alpha_{\mathrm{T}}c_1^2}{c_2}|\theta_* - g_\theta|^2_{L^2(\partial\mathcal{M})} + \frac{\nu_{\mathrm{T}}\alpha_{\mathrm{T}}c_2}{4}\|\hat{\theta}\|^2.
\end{aligned}
$$

Then (2.11) and (2.12) give us

$$
\begin{aligned}
&\frac{d}{dt}|\hat{v}|^2 + c_{\mathrm{p}}\frac{d}{dt}|\hat{\theta}|^2 + \mu_{\mathrm{V}}|\nabla\hat{v}|^2 + \nu_{\mathrm{V}}c_2\Big|\frac{\partial\hat{v}}{\partial p}\Big|^2 + \\
&\mu_{\mathrm{T}}|\nabla\hat{\theta}|^2 + \nu_{\mathrm{T}}c_2\Big|\frac{\partial\hat{\theta}}{\partial p}\Big|^2 + \nu_{\mathrm{V}}\alpha_{\mathrm{V}}c_2\|\hat{v}\|^2 + \nu_{\mathrm{T}}\alpha_{\mathrm{T}}c_2\|\hat{\theta}\|^2 \\
&\qquad \leq K_1 + K_2|\hat{\theta}|^2 + K_3|\hat{v}|^2,
\end{aligned}
\tag{2.13}
$$

where the K_i's are constants depending on the data.

Hence,

$$
\frac{d}{dt}(|\hat{v}|^2 + c_{\mathrm{p}}|\hat{\theta}|^2) \leq K_1 + K_4(|\hat{v}|^2 + c_{\mathrm{p}}|\hat{\theta}|^2),
$$

and

$$
\frac{d}{dt}\big(e^{-K_4 t}(|\hat{v}|^2 + |\hat{\theta}|^2)\big) \leq K_1 e^{-K_4 t}.
$$

Thus,

$$
|\hat{v}(t)|^2 + c_{\mathrm{p}}|\hat{\theta}(t)|^2 \leq e^{K_4 t}(|\hat{v}(0)|^2 + c_{\mathrm{p}}|\hat{\theta}(0)|^2) + K_1\int_0^t e^{K_4(t-s)}\,ds. \tag{2.14}
$$

Then, by integrating (2.14) from 0 to t_1, we obtain

$$
\begin{aligned}
&\|\hat{v}\|^2_{L^2(0,t_1;L^2(\mathcal{M}))} + c_{\mathrm{p}}\|\hat{\theta}\|^2_{L^2(0,t_1;L^2(\mathcal{M}))} \\
&\qquad \leq (|\hat{v}_0|^2 + |\hat{\theta}_0|^2)\int_0^{t_1} e^{K_4 t}\,dt + K_1\int_0^{t_1}\!\!\int_0^t e^{K_4(t-s)}\,ds\,dt.
\end{aligned}
\tag{2.15}
$$

Also, by integrating (2.13) from 0 to t_1, we see that

$$\begin{aligned}
|\hat{v}(t_1)|^2 + c_{\rm P}|\hat{\theta}(t_1)|^2 + \mu_{\rm V}\int_0^{t_1} |\nabla\hat{v}(s)|^2\,ds + \nu_{\rm V}c_2\int_0^{t_1}\Big|\frac{\partial\hat{v}(s)}{\partial p}\Big|^2\,ds \\
+ \mu_{\rm T}\int_0^{t_1}|\nabla\hat{\theta}(s)|^2\,ds + \nu_{\rm T}c_2\int_0^{t_1}\Big|\frac{\partial\hat{\theta}(s)}{\partial p}\Big|^2\,ds \\
+ \nu_{\rm V}\alpha_{\rm V}c_2\int_0^{t_1}\|\hat{v}(s)\|^2\,ds + \nu_{\rm T}\alpha_{\rm T}c_2\int_0^{t_1}\|\hat{\theta}(s)\|^2\,ds \\
\le |\hat{v}_0|^2 + c_{\rm P}|\hat{\theta}_0|^2 + tK_1 + K_2\|\hat{v}\|^2_{L^2(0,t_1;L^2(\mathcal{M}))} + K_3\|\hat{\theta}\|^2_{L^2(0,t_1;L^2(\mathcal{M}))}.
\end{aligned} \tag{2.16}$$

Thus, in view of (2.15) and (2.16), we can bound each of $\|\nabla\hat{v}\|_{L^2(0,t_1;L^2(\mathcal{M}))}$, $\Big\|\frac{\partial\hat{v}}{\partial p}\Big\|_{L^2(0,t_1;L^2(\mathcal{M}))}$, $\|\nabla\hat{\theta}\|_{L^2(0,t_1;L^2(\mathcal{M}))}$, $\Big\|\frac{\partial\hat{\theta}}{\partial p}\Big\|_{L^2(0,t_1;L^2(\mathcal{M}))}$, $\|\hat{v}\|_{L^2(0,t_1;V)}$, $\|\hat{\theta}\|_{L^2(0,t_1;V)}$, $\|\hat{v}\|_{L^\infty(0,t_1;L^2(\mathcal{M}))}$, and $\|\hat{\theta}\|_{L^\infty(0,t_1;L^2(\mathcal{M}))}$ by quantities depending only on the data.

Now that we have the above à priori estimates, we can proceed with the proof of the theorem. The proof uses the Galerkin method, and since this method is standard, we will just present an outline of the proof.

Let $\psi_j = (\psi_j^v, \psi_j^\theta)$, for $j = 1, 2, \dots$ be a complete orthonormal basis in V. Then we look for approximate solutions $u_m(t) = \sum_{j=1}^m g_{jm}(t)\psi_j$ which satisfy, with $u_m = (v_m, \theta_m)$:

$$\begin{cases}
\Big(\frac{\partial v_m}{\partial t}, \psi_j^v\Big) + c_{\rm P}\Big(\frac{\partial\theta_m}{\partial t}, \psi_j^\theta\Big) + a(u_m,\psi_j) + b(u_m,u_m,\psi_j) + e(u_m,\psi_j) \\
\quad -d(u_m,\psi_j) = (Q_E, \psi_j^\theta), \qquad \text{for } j = 1, 2, \dots, m, \\
(u_m(0), \psi_j) = (u_0, \psi_j), \qquad \text{for } j = 1, 2, \dots, m.
\end{cases} \tag{2.17}$$

Solving for u_m is then just solving a system of differential equations for the g_{jm}.

It is easy to see that the à priori estimates above apply as well to each u_m, so that the u_m are all in a subset of $L^\infty(0,t_1;L^2(M)) \cap L^2(0,t_1;V)$ bounded independently of m. We are then able to pass to the limit using standard compactness methods and we obtain a solution u to (2.9). □

Remark 2.1. Since p is bounded from below away from 0 ($p \ge p_0 > 0$) and from above ($p \le p_1$), the properties of θ are the same as those of T in [6], as one can expect.

3. **Maximum Principles.**

In this section, we will establish maximum principles for T, or, what is the same thing, for θ. First, we will use the truncation method of Stampacchia to establish the positivity of θ. Then we will use a classical method to bound θ from above.

3.1. **Positivity.**

We use Stampacchia's method to show positivity of θ. We make, as is usual, the following additional hypotheses on the appropriate boundary conditions, forcing terms, and initial conditions:

$$\begin{cases}
Q_E \ge 0, \\
g_\theta \ge 0, \\
\theta_* \ge 0, \\
\theta_0 \ge 0.
\end{cases} \tag{3.1}$$

We have the following theorem:

Theorem 3.1. *Under the assumptions* (3.1)*, any solution* $u = (v,\theta)$ *to* (2.9) *satisfies* $\theta(t) \geq 0$*, for all* $t \in [0,t_1]$*. Consequently,* $T(t) \geq 0$*, for all* $t \in [0,t_1]$*.*

Proof. We decompose $\theta = \theta_+ - \theta_-$ into its positive and negative parts. We then multiply the equation for θ in (2.9) by $-\theta_-$, and integrate over $\mathcal{M}$. The following terms appear.

- The first term is

$$c_{\mathrm{p}} \int_{\mathcal{M}} \frac{\partial}{\partial t}\theta \cdot (-\theta_-)\, d\mathcal{M} = \frac{c_{\mathrm{p}}}{2}\frac{d}{dt}\int_{\mathcal{M}} |\theta_-|^2\, d\mathcal{M}.$$

- Then, we have the term

$$\begin{aligned} &c_{\mathrm{p}} \int_{\mathcal{M}} \Big((v\cdot\nabla)\theta + \omega\frac{\partial\theta}{\partial p}\Big)\cdot(-\theta_-)\, d\mathcal{M} \\ &\quad = c_{\mathrm{p}}\int_{\mathcal{M}} (v\cdot\nabla)\Big(\frac{\theta_-^2}{2}\Big) + \omega\frac{\partial}{\partial p}\Big(\frac{\theta_-^2}{2}\Big)\, d\mathcal{M} \\ &\quad = c_{\mathrm{p}}\int_{\partial\mathcal{M}} (v,\omega)\cdot n\Big(\frac{\theta_-^2}{2}\Big)\, d(\partial\mathcal{M}) - c_{\mathrm{p}}\int_{\mathcal{M}}\Big(\nabla\cdot v + \frac{\partial\omega}{\partial p}\Big)\Big(\frac{\theta_-^2}{2}\Big)\, d\mathcal{M}, \end{aligned}$$

  and this term vanishes, since, on $\Gamma_{\mathrm{u}}\cup\Gamma_{\mathrm{i}}$, $(v,\omega)\cdot n = \pm\omega = 0$; on Γ_ℓ, $\theta = \theta_g \geq 0$ so that $\theta_- = 0$; and in all of $\mathcal{M}$, $\nabla\cdot v + \dfrac{\partial\omega}{\partial p} = 0$.
- The horizontal dissipation term for θ reads

$$\begin{aligned} &\mu_{\mathrm{T}}\int_{\mathcal{M}} \Delta\theta\cdot\theta_-\, d\mathcal{M} \\ &\qquad = \mu_{\mathrm{T}}\int_{\Gamma_\ell} n_{\mathrm{h}}\cdot\nabla\theta\,\theta_-\, d\Gamma_\ell - \mu_{\mathrm{T}}\int_{\mathcal{M}}\nabla\theta\cdot\nabla\theta_-\, d\mathcal{M} \\ &\qquad = \mu_{\mathrm{T}}\int_{\mathcal{M}} |\nabla\theta_-|^2\, d\mathcal{M}, \end{aligned}$$

  since, as before, $\theta_- = 0$ on Γ_ℓ.
- The vertical dissipation term for θ reads

$$\begin{aligned} &\nu_{\mathrm{T}}\int_{\mathcal{M}}\Big(\frac{p_0}{p}\Big)^{\frac{R}{c_{\mathrm{p}}}}\frac{\partial}{\partial p}\bigg[\Big(\frac{gp}{R\bar{T}}\Big)^2\frac{\partial}{\partial p}\bigg(\Big(\frac{p}{p_0}\Big)^{\frac{R}{c_{\mathrm{p}}}}\theta\bigg)\bigg]\theta_-\, d\mathcal{M} \\ &\quad = \nu_{\mathrm{T}}\int_{\mathfrak{M}}\Big(\frac{p_0}{p}\Big)^{\frac{R}{c_{\mathrm{p}}}}\theta_-\Big(\frac{gp}{R\bar{T}}\Big)^2\frac{\partial}{\partial p}\bigg(\Big(\frac{p}{p_0}\Big)^{\frac{R}{c_{\mathrm{p}}}}\theta\bigg)\bigg|_{p=p_0}^{p_1} d\mathfrak{M} \\ &\qquad - \nu_{\mathrm{T}}\int_{\mathcal{M}}\Big(\frac{gp}{R\bar{T}}\Big)^2\frac{\partial}{\partial p}\bigg(\Big(\frac{p}{p_0}\Big)^{\frac{R}{c_{\mathrm{p}}}}\theta\bigg)\frac{\partial}{\partial p}\bigg(\Big(\frac{p_0}{p}\Big)^{\frac{R}{c_{\mathrm{p}}}}\theta_-\bigg)\, d\mathcal{M} \\ &\quad = I - J. \end{aligned}$$

Now,

$$
\begin{aligned}
I &= \nu_{\mathrm{T}} \int_{\mathfrak{M}} \left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta_- \left(\frac{gp}{R\bar{T}}\right)^2 \frac{\partial T}{\partial p}\bigg|_{p=p_0}^{p_1} \, d\mathfrak{M} \\
&= \nu_{\mathrm{T}} \int_{\mathfrak{M}} \left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta_- \left(\frac{gp}{R\bar{T}}\right)^2 \frac{\partial T}{\partial p}\bigg|_{p=p_1} - \left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta_- \left(\frac{gp}{R\bar{T}}\right)^2 \frac{\partial T}{\partial p}\bigg|_{p=p_0} \, d\mathfrak{M} \\
&= \nu_{\mathrm{T}} \int_{\mathfrak{M}} \theta_- \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\mathrm{T}}(\theta_* - \theta)\bigg|_{p=p_1} + \theta_- \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\mathrm{T}}(\theta_* - \theta)\bigg|_{p=p_0} \, d\mathfrak{M} \\
&= \nu_{\mathrm{T}} \int_{\mathfrak{M}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\mathrm{T}}(\theta_*\theta_- + \theta_-^2)\bigg|_{p=p_1} + \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\mathrm{T}}(\theta_*\theta_- + \theta_-^2)\bigg|_{p=p_0} \, d\mathfrak{M} \\
&\geq 0,
\end{aligned}
$$

since $\theta_* \geq 0$. Also,

$$
\begin{aligned}
J &= \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}}\right)^2 \left[\left(\frac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \frac{\partial \theta}{\partial p} + \theta \frac{R}{p_0 c_{\mathrm{p}}} \left(\frac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}-1}\right] \\
&\qquad \cdot \left[\left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}} \frac{\partial \theta_-}{\partial p} - \frac{p_0 R}{p^2 c_{\mathrm{p}}} \left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}-1} \theta_-\right] d\mathcal{M} \\
&= \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}}\right)^2 \left[\frac{\partial \theta}{\partial p} + \theta \frac{R}{p c_{\mathrm{p}}}\right] \cdot \left[\frac{\partial \theta_-}{\partial p} - \theta_- \frac{R}{p c_{\mathrm{p}}}\right] d\mathcal{M} \\
&= \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}}\right)^2 \left[\theta_-^2 \left(\frac{R}{p c_{\mathrm{p}}}\right)^2 - \left(\frac{\partial \theta_-}{\partial p}\right)^2\right] d\mathcal{M}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}} \frac{\partial}{\partial p}\left[\left(\frac{gp}{R\bar{T}}\right)^2 \frac{\partial}{\partial p}\left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} \theta\right)\right] \theta_- \, d\mathcal{M} \\
&\qquad \geq \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}}\right)^2 \left[\theta_-^2 \left(\frac{R}{p c_{\mathrm{p}}}\right)^2 - \left(\frac{\partial \theta_-}{\partial p}\right)^2\right] d\mathcal{M}.
\end{aligned}
$$

- The final term is simply

$$
-\int_{\mathcal{M}} Q_E \theta_- \, d\mathcal{M} \leq 0.
$$

Collecting all these terms, we find

$$
\begin{aligned}
&\frac{c_{\mathrm{p}}}{2} \frac{d}{dt} |\theta_-|^2 + \mu_{\mathrm{T}} \int_{\mathcal{M}} |\nabla \theta_-|^2 \, d\mathcal{M} + \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}}\right)^2 \left(\frac{\partial \theta_-}{\partial p}\right)^2 d\mathcal{M} \\
&\qquad \leq \nu_{\mathrm{T}} \int_{\mathcal{M}} \left(\frac{g}{c_{\mathrm{p}} \bar{T}}\right)^2 \theta_-^2 \, d\mathcal{M}.
\end{aligned}
\tag{3.2}
$$

Thus, $\dfrac{d}{dt}|\theta_-|^2 \leq c|\theta_-|^2$, where $c = \dfrac{2\nu_{\mathrm{T}} g^2}{{c_{\mathrm{p}}}^3 \bar{T}_*^2}$. Therefore, $|\theta_-(t)|^2 \leq e^{ct}|\theta_-(0)|^2$, and since $\theta_-(0) = 0$, we obtain $\theta_- \equiv 0$. □

3.2. **Upper bounds.** In this section we establish a uniform upper bound for θ using a classical maximum principle method. For this, we will need to make more explicit the θ-equation in (1.6) and calculate the derivatives in the ν_{T}-term. We find:

$$\nu_{\mathrm{T}}\left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}}\frac{\partial}{\partial p}\left[\left(\frac{gp}{R\bar{T}}\right)^2\frac{\partial}{\partial p}\left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}}\theta\right)\right]=\alpha\frac{\partial^2\theta}{\partial p^2}+\beta\frac{\partial\theta}{\partial p}+\gamma\theta, \tag{3.3}$$

where

$$\begin{cases}\alpha=\nu_{\mathrm{T}}\left(\frac{gp}{R\bar{T}}\right)^2\geq 0,\\ \beta=\alpha\left(\frac{2}{p}-\frac{2}{\bar{T}}\frac{\partial\bar{T}}{\partial p}+\frac{2R}{pc_{\mathrm{p}}}\right),\\ \gamma=\alpha\left(\left(\frac{R}{pc_{\mathrm{p}}}\right)^2+\frac{R}{p^2c_{\mathrm{p}}}-\frac{2}{\bar{T}}\frac{\partial\bar{T}}{\partial p}\frac{R}{pc_{\mathrm{p}}}\right).\end{cases} \tag{3.4}$$

As in (1.12) of [8], we assume that

$$R\left(\frac{R\bar{T}}{c_{\mathrm{p}}}-p\frac{\partial\bar{T}}{\partial p}\right)=c^2=\text{constant}. \tag{3.5}$$

Using this, we may rewrite β as

$$\beta=\alpha\frac{2}{p}\left(1+\frac{c^2}{R\bar{T}}\right)\geq 0, \tag{3.6}$$

and γ as

$$\gamma=\alpha\left(\frac{R}{p^2c_{\mathrm{p}}}\left(1+\frac{2c^2}{R\bar{T}}\right)-\left(\frac{R}{pc_{\mathrm{p}}}\right)^2\right). \tag{3.7}$$

Thus, we have $\gamma_*\leq\gamma\leq\gamma^*$, where we have set $\gamma_*=-\nu_{\mathrm{T}}\left(\frac{gp}{R\bar{T}_*}\right)^2\left(\frac{R}{p_0c_{\mathrm{p}}}\right)^2$ and $\gamma^*=\nu_{\mathrm{T}}\left(\frac{gp_1}{R\bar{T}_*}\right)\frac{R}{p_0^2c_{\mathrm{p}}}\left(1+\frac{2c^2}{R\bar{T}_*}\right)$.

We may therefore rewrite the θ-equation of (1.6) as

$$c_{\mathrm{p}}\frac{d\theta}{dt}-\mu_{\mathrm{T}}\Delta\theta-\alpha\frac{\partial^2\theta}{\partial p^2}-\beta\frac{\partial\theta}{\partial p}-\gamma\theta=Q_E. \tag{3.8}$$

If we set $\tilde{\theta}=e^{-\lambda t}\theta$, where $\lambda>0$ will be specified later, then (3.8) yields

$$c_{\mathrm{p}}\frac{d\tilde{\theta}}{dt}-\mu_{\mathrm{T}}\Delta\tilde{\theta}-\alpha\frac{\partial^2\tilde{\theta}}{\partial p^2}-\beta\frac{\partial\tilde{\theta}}{\partial p}+(\lambda-\gamma)\tilde{\theta}=Q_Ee^{-\lambda t}. \tag{3.9}$$

Consider a solution to (3.9) from time $t=0$ to time $t=t_1$, with global maximum $\tilde{\theta}_{\max}$ at some point $x\in\mathcal{M}\times[0,t_1]$. We consider the following cases:

- $x\notin\partial(\mathcal{M}\times[0,t_1])$, that is, x is in the interior of $\mathcal{M}\times[0,t_1]$.

  In this case, since $\tilde{\theta}$ has a maximum at x, we see that $\frac{d\tilde{\theta}}{dt}=0$, $\Delta\tilde{\theta}\leq 0$, $\frac{\partial^2\tilde{\theta}}{\partial p^2}\leq 0$, and $\frac{\partial\tilde{\theta}}{\partial p}=0$ at x. Thus, at x, (3.9) becomes

  $$(\lambda-\gamma)\tilde{\theta}_{\max}\leq\|Q_E\|_{L^\infty}e^{-\lambda t},$$

  Therefore, provided we choose $\lambda\geq 2\gamma^*$, we may write

  $$\tilde{\theta}\leq\tilde{\theta}_{\max}\leq\frac{\|Q_E\|_{L^\infty}e^{-\lambda t}}{\gamma^*}. \tag{3.10}$$

- x is at time $t = 0$.

  In this case, we simply bound $\tilde{\theta}$ by the initial condition,

$$\tilde{\theta} \leq \|\tilde{\theta}_0\|_{L^\infty} = \|\tilde{\theta}|_{t=0}\|_{L^\infty}. \tag{3.11}$$

- x is at time $t = t_1$, but in not on $\partial\mathcal{M}$.

  In this case, we have $\dfrac{d\tilde{\theta}}{dt} \geq 0$, $\Delta\tilde{\theta} \leq 0$, $\dfrac{\partial^2\tilde{\theta}}{\partial p^2} \leq 0$, and $\dfrac{\partial\tilde{\theta}}{\partial p} = 0$ at x, and (3.9) becomes

$$(\lambda - \gamma)\tilde{\theta}_{\max} \leq \|Q_E\|_{L^\infty} e^{-\lambda t},$$

  and thus, using again $\lambda \geq 2\gamma^*$, we see that

$$\tilde{\theta} \leq \tilde{\theta}_{\max} \leq \frac{\|Q_E\|_{L^\infty} e^{-\lambda t}}{\gamma^*}. \tag{3.12}$$

- x is in $\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}$, for some time $t \in (0, t_1]$.

  Then, at x, $\dfrac{\partial\tilde{\theta}}{\partial n} \geq 0$, and we have the boundary condition

$$\frac{\partial\theta}{\partial n} = \alpha_{\mathrm{T}}(\theta_* - \theta) - \frac{R}{pc_{\mathrm{p}}}\theta,$$

  which in terms of $\tilde{\theta}$ at the point x is

$$\frac{\partial\tilde{\theta}}{\partial n} = \alpha_{\mathrm{T}}\tilde{\theta}_* - \left(\alpha_{\mathrm{T}} + \frac{R}{pc_{\mathrm{p}}}\right)\tilde{\theta}_{\max}.$$

  Thus,

$$\tilde{\theta} \leq \tilde{\theta}_{\max} \leq \frac{\alpha_{\mathrm{T}}\|\tilde{\theta}_*\|_{L^\infty}}{\alpha_{\mathrm{T}} + \frac{R}{p_1 c_{\mathrm{p}}}}. \tag{3.13}$$

- x is in Γ_ℓ, for some time $t \in (0, t_1]$.

  Then, we have simply

$$\tilde{\theta} \leq \tilde{\theta}_{\max} \leq \|g_{\tilde{\theta}}\|_{L^\infty}, \tag{3.14}$$

  where $g_{\tilde{\theta}} = e^{-\lambda t} g_\theta$.

So, in any case, we can bound $\tilde{\theta}_{\max}$ by a certain function of the data, namely,

$$\tilde{\theta}_{\max} \leq K = \max\left\{\frac{\|Q_E\|_{L^\infty}}{\gamma^*}, \|\tilde{\theta}_0\|_{L^\infty}, \frac{\alpha_{\mathrm{T}}\|\tilde{\theta}_*\|_{L^\infty}}{\alpha_{\mathrm{T}} + \frac{R}{p_1 c_{\mathrm{p}}}}, \|g_{\tilde{\theta}}\|_{L^\infty}\right\}, \tag{3.15}$$

or, as well,

$$\tilde{\theta}_{\max} \leq K = \max\left\{\frac{\|Q_E\|_{L^\infty}}{\gamma^*}, \|\theta_0\|_{L^\infty}, \frac{\alpha_{\mathrm{T}}\|\theta_*\|_{L^\infty}}{\alpha_{\mathrm{T}} + \frac{R}{p_1 c_{\mathrm{p}}}}, \|g_\theta\|_{L^\infty}\right\}, \tag{3.16}$$

since the L^∞-norms of $\tilde{\theta}_0$, $\tilde{\theta}_*$, and $g_{\tilde{\theta}}$ are bounded by those of θ_0, θ_*, and g_θ.

Therefore, we have the following theorem, where we have used $\lambda = 2\gamma^*$:

Theorem 3.2. *If $u = (v, \theta)$ is a solution to* (2.9) *on the interval $[0, t_1]$ and all of the data Q_E, θ_0, θ_*, and g_θ are uniformly bounded, then we can bound $\theta_{\max}$, the maximum value of θ, by*

$$\theta_{\max} \leq K e^{2\gamma^* t_1}. \tag{3.17}$$

That is, we may bound the maximum value of T by

$$T_{\max} \leq \left(\frac{p_1}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} K e^{2\gamma^* t_1}.$$

Remark 3.1. There is some flexibility in the modelling of the dissipation term D in (1.1), and the choice $D = L_{\mathrm{T}}T$ is not the only one which is used (see a discussion about this in [12]). Hence, assume that we replace the fourth equation of (1.6) by

$$c_{\mathrm{p}}\frac{\partial\theta}{\partial t} + c_{\mathrm{p}}(v\cdot\nabla)\theta + c_{\mathrm{p}}\omega\frac{\partial\theta}{\partial p} - \mu_{\mathrm{T}}\Delta\theta - \frac{\partial}{\partial p}\left(\alpha(\theta,p)\frac{\partial\theta}{\partial p}\right) + \gamma\theta = Q_E, \tag{3.18}$$

with $\alpha, \gamma \geq 0$; more hypotheses on α and γ would be needed to guarantee existence, but the emphasis here is not on existence of solutions. If we also assume that (3.1) holds and furthermore that $Q_E = 0$, then we can prove a stronger form of (3.17) with no growth in time, namely, for all $t \geq 0$,

$$0 \leq \theta \leq K' = \max\{\|\theta_0\|_{L^\infty}, \|\theta_*\|_{L^\infty}, \|g_\theta\|_{L^\infty}\}. \tag{3.19}$$

This estimate can be obtained by using the Stampacchia version of the maximum principle using truncations: we multiply (3.18) by $(\theta - K')_+$, integrate over $\mathcal{M}$, and integrate by parts using the Stokes formula. We arrive at a differential inequality of the form

$$c_{\mathrm{p}}\frac{d}{dt}|(\theta-K')_+|^2 \leq \gamma_*|(\theta-K')_+|^2,$$

with the same γ_* as before; this implies, at all times t,

$$|(\theta-K')_+(t)|^2 \leq |(\theta_0-K')_+|^2 e^{-\frac{\gamma_* t}{c_{\mathrm{p}}}},$$

and the right-hand side of this last inequality vanishes since $\theta_0 \leq K'$ a.e.

Reinterpreting (3.18) in terms of T, we find a somewhat complicated expression for D:

$$\begin{aligned} c_{\mathrm{p}}\frac{\partial T}{\partial t} &+ c_{\mathrm{p}}(v\cdot\nabla)T + c_{\mathrm{p}}\omega\frac{\partial T}{\partial p} - \frac{R\omega}{p}T - \mu_{\mathrm{T}}\Delta T \\ &- \left(\frac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}}\frac{\partial}{\partial p}\left[\alpha(\theta,p)\left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}}\frac{\partial T}{\partial p}\right] \\ &+ \left(\frac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}}\frac{\partial}{\partial p}\left[\beta(\theta,p)\left(\frac{p_0}{p}\right)^{\frac{R}{c_{\mathrm{p}}}}T\right] + \gamma T = \left(\frac{p}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}} Q_E \ (=0), \end{aligned}$$

where $\beta(\theta,p) = \dfrac{R}{pc_{\mathrm{p}}}\alpha(\theta,p)$.

4. The Primitive Equations on the Sphere.

In this section, we will consider the primitive equations on the whole sphere of the earth. Once we overcome the difficulties related to the use of the geometrical tools, the results and the proofs are very similar. Therefore, we will only emphasize the main differences with the preceding results.

4.1. Mathematical setting.

We consider first the mathematical and geometrical setting necessary for the primitive equations on the sphere. We define $S = S^a$ to be the sphere with radius a in $\mathbb{R}^3$. We will use the spherical coordinates λ, φ, r in $\mathbb{R}^3$, where λ is the latitude, φ is the longitude, and r is the radius. Thus, $S^a = \{r = a\}$.

We set $\mathcal{M} = S \times (p_0, p_1)$, and $T_{(q,r)}\mathcal{M} = T_qS \times T_r(p_0,p_1) = T_qS \times \mathbb{R}$ is its tangent space at the point (q, r). We have $\partial\mathcal{M} = \Gamma_{\mathrm{i}} \cup \Gamma_{\mathrm{u}}$, where $\Gamma_{\mathrm{i}} = S \times \{p_1\}$ and $\Gamma_{\mathrm{u}} = S \times \{p_0\}$.

The Riemannian metric for $\mathcal{M}$ defined for $q \in S$ and $r \in (p_0, p_1)$ reads

$$g_{\mathcal{M}}((q,r);(v,\omega),(\hat{v},\hat{\omega})) = g_S(q;v,\hat{v}) + \omega\cdot\hat{\omega},$$

where g_S is the Riemannian metric for the sphere S. That is, we set

$$(g_{ij}) = a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} = g_{\mathcal{M}}\left((\lambda,\varphi,r); \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

The unit vectors in $T\mathcal{M}$ are

$$e_\lambda = \frac{1}{a}\frac{\partial}{\partial\lambda}, \quad e_\varphi = \frac{1}{a\cos\lambda}\frac{\partial}{\partial\varphi}, \quad e_r = \frac{1}{a}\frac{\partial}{\partial r}.$$

We now define some function spaces on $\mathcal{M}$. Let $C^\infty(\bar{\mathcal{M}})$ be the space of all smooth functions from $\bar{\mathcal{M}}$ into $\mathbb{R}$, $C^\infty(T\bar{\mathcal{M}})$ be the space of all smooth vector fields on $\bar{\mathcal{M}}$, $C^\infty(S)$ be the space of smooth functions on S, $C^\infty(TS)$ be the space of smooth vector fields on S, and $C^\infty(T\bar{\mathcal{M}}|TS)$ be the space of smooth functions $v : \bar{\mathcal{M}} \to TS$ with each $v(\lambda,\varphi,r) \in T_{(\lambda,\varphi)}S$.

For $v \in TS$, we write $v = v^\lambda e_\lambda + v^\varphi e_\varphi$. Consider $v, \hat{v} \in C^\infty(TS)$ and $\theta \in C^\infty(S)$. Then we define the following operations on $v, \hat{v}$, and θ:

$$\begin{aligned}
\nabla_v \hat{v} &= \frac{1}{a}\left(v^\lambda \frac{\partial \hat{v}^\lambda}{\partial\lambda} + \frac{v^\varphi}{\cos\lambda}\frac{\partial\hat{v}^\lambda}{\partial\varphi} + v^\varphi \hat{v}^\varphi \tan\lambda\right) e_\lambda \\
&\quad + \frac{1}{a}\left(v^\lambda \frac{\partial \hat{v}^\varphi}{\partial\lambda} + \frac{v^\varphi}{\cos\lambda}\frac{\partial\hat{v}^\varphi}{\partial\varphi} - v^\varphi \hat{v}^\lambda \tan\lambda\right) e_\varphi, \\
\nabla_v \theta &= \frac{v^\lambda}{a}\frac{\partial\theta}{\partial\lambda} + \frac{v^\varphi}{a\cos\lambda}\frac{\partial\theta}{\partial\varphi}, \\
\operatorname{grad}\theta &= \frac{1}{a}\frac{\partial\theta}{\partial\lambda} e_\lambda + \frac{1}{a\cos\lambda}\frac{\partial\theta}{\partial\varphi} e_\varphi, \\
\operatorname{div} v &= \frac{1}{a\cos\lambda}\left(\frac{\partial v^\lambda}{\partial\lambda} + \frac{\partial v^\varphi}{\partial\varphi}\right), \\
\Delta\theta &= \frac{1}{a^2\cos\lambda}\left(\frac{\partial}{\partial\lambda}\left(\cos\lambda\frac{\partial\theta}{\partial\lambda}\right) + \frac{1}{\cos\lambda}\frac{\partial^2\theta}{\partial\varphi^2}\right), \\
\Delta v &= \frac{1}{a^2}\left(\Delta v^\lambda + \frac{2\sin\lambda}{\cos^2\lambda}\frac{\partial v^\varphi}{\partial\varphi} - \frac{v^\lambda}{\cos^2\lambda}\right) e_\lambda \\
&\quad + \frac{1}{a^2}\left(\Delta v^\varphi - \frac{2\sin\lambda}{\cos^2\lambda}\frac{\partial v^\lambda}{\partial\varphi} - \frac{v^\varphi}{\cos^2\lambda}\right) e_\varphi.
\end{aligned}$$

Here, $\nabla_v \hat{v}$ and $\nabla_v \theta$ are the covariant derivatives of $\hat{v}$ and θ in the direction of v, and Δ is the Laplace-Beltrami operator on the sphere for the scalar function θ; for the vector function v, Δv denotes the Laplacian of v as defined by A. Lichnerowicz [5], namely, for the sphere of radius a,

$$(\Delta v)^i = -\nabla^k \nabla_k v^i - \frac{1}{a^2} v^i,$$

where ∇^k and ∇_k are the covariant and contravariant derivatives, respectively, in the kth direction.[1]

Note that we have the following Stokes formula on the sphere:

$$\int_{\mathcal{M}} (-\Delta v)\cdot\hat{v}\, d\mathcal{M} = \int_{\mathcal{M}} \left(\nabla_{e_\lambda} v \cdot \nabla_{e_\lambda}\hat{v} + \nabla_{e_\varphi} v \cdot \nabla_{e_\varphi}\hat{v} + \frac{1}{a^2} v\cdot\hat{v}\right) d\mathcal{M}.$$

[1]We recall that, for vector functions, and contrarily to the Cartesian case, there are several nonequivalent definitions of the Laplacian of a vector function tangent to a manifold, and they differ in the lower order terms.

We also consider the following inner products:

$$((v,\hat{v})) = \int_{\mathcal{M}} \left(\nabla_{e_\lambda} v \cdot \nabla_{e_\lambda} \hat{v} + \nabla_{e_\varphi} v \cdot \nabla_{e_\varphi} \hat{v} + \frac{\partial v}{\partial r} \cdot \frac{\partial \hat{v}}{\partial r} + v \cdot \hat{v} \right) d\mathcal{M}$$
$$((\theta,\hat{\theta})) = \int_{\mathcal{M}} \left(\operatorname{grad}\theta \cdot \operatorname{grad}\hat{\theta} + \frac{\partial \theta}{\partial r} \cdot \frac{\partial \hat{\theta}}{\partial r} + \theta \cdot \hat{\theta} \right) d\mathcal{M}.$$

Here we define

$$V_v = \left\{ v \in H^1(T\bar{\mathcal{M}}|TS) \;\middle|\; \int_{p_0}^{p_1} \operatorname{div} v \, dp = 0 \right\},$$

$$V_\theta = H^1(\mathcal{M}),$$

$$V = V_v \times V_\theta.$$

On the sphere, instead of (1.6), the primitive equations read

$$\begin{cases} \dfrac{\partial v}{\partial t} + \nabla_v v + \omega \dfrac{\partial v}{\partial p} + f(k \times v) + \nabla\Phi \\ \qquad -\mu_{\mathrm{V}} \Delta v - \nu_{\mathrm{V}} \dfrac{\partial}{\partial p} \left[\left(\dfrac{gp}{R\bar{T}} \right)^2 \dfrac{\partial v}{\partial p} \right] = 0 \\ \dfrac{\partial \Phi}{\partial p} + \dfrac{R}{p} \left(\dfrac{p}{p_0} \right)^{\frac{R}{c_{\mathrm{p}}}} \theta = 0 \\ \operatorname{div} v + \dfrac{\partial \omega}{\partial p} = 0 \\ c_{\mathrm{p}} \dfrac{\partial \theta}{\partial t} + c_{\mathrm{p}} \nabla_v \theta + c_{\mathrm{p}} \omega \dfrac{\partial \theta}{\partial p} - \mu_{\mathrm{T}} \Delta\theta \\ \qquad -\nu_{\mathrm{T}} \left(\dfrac{p_0}{p} \right)^{\frac{R}{c_{\mathrm{p}}}} \dfrac{\partial}{\partial p} \left[\left(\dfrac{gp}{R\bar{T}} \right)^2 \dfrac{\partial}{\partial p} \left(\left(\dfrac{p}{p_0} \right) \theta \right) \right] = Q_E. \end{cases} \tag{4.1}$$

Here we have furthermore replaced the vertical variable r by the pressure variable p as before.

The appropriate boundary conditions become:

$$\begin{cases} \omega = 0 & \text{on } \Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_{\mathrm{u}}, \\ \dfrac{\partial v}{\partial n} = \alpha_{\mathrm{V}}(v_* - v) & \text{on } \Gamma_{\mathrm{i}}, \\ \dfrac{\partial \theta}{\partial n} = \alpha_{\mathrm{T}}(\theta_* - \theta) - \dfrac{R}{pc_{\mathrm{p}}}\theta & \text{on } \Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}. \end{cases} \tag{4.2}$$

We redefine a_v, a_θ, b_v, and b_θ from (2.5)-(2.8) as follows:

$$\begin{cases} a_v(v,\hat{v}) = \mu_{\rm V} \int_M \nabla_{e_\lambda} v \cdot \nabla_{e_\lambda} \hat{v} + \nabla_{e_\varphi} v \cdot \nabla_{e_\varphi} \hat{v} + v \cdot \hat{v}\, dM \\ \qquad\qquad + \nu_{\rm V} \int_M \left(\frac{gp}{R\bar{T}}\right)^2 \frac{\partial v}{\partial p} \cdot \frac{\partial \hat{v}}{\partial p}\, dM, \\ a_\theta(\theta,\hat{\theta}) = \mu_{\rm T} \int_M \operatorname{grad}\theta \cdot \operatorname{grad}\hat{\theta}\, dM + \nu_{\rm T} \int_M \left(\frac{gp}{R\bar{T}}\right)^2 \frac{\partial \theta}{\partial p}\frac{\partial \hat{\theta}}{\partial p}\, dM \\ \qquad\qquad - \nu_{\rm T} \int_M \left(\frac{g}{c_{\rm p}\bar{T}}\right)^2 \theta\hat{\theta}\, dM, \\ b_v(v,\hat{v},\tilde{v}) = \int_M \left(\nabla_v \hat{v} + w\frac{\partial \hat{v}}{\partial p}\right) \cdot \tilde{v}\, dM, \\ b_\theta(v,\hat{\theta},\tilde{\theta}) = c_{\rm p} \int_M \left(\nabla_v \hat{\theta} + w\frac{\partial \hat{\theta}}{\partial p}\right) \tilde{\theta}\, dM, \end{cases} \tag{4.3}$$

and with these new definitions, the weak formulation of (4.1) supplemented with boundary conditions (4.2) is:

$$\begin{cases} \text{To find } u(t) \in V, \text{ such that} \\ \left(\frac{\partial v}{\partial t}, \hat{v}\right) + c_{\rm p}\left(\frac{\partial \theta}{\partial t}, \hat{\theta}\right) + a(u,\hat{u}) + b(u,u,\hat{u}) + e(u,\hat{u}) \\ \qquad\qquad - d(u,\hat{u}) = (Q_E, \hat{\theta}), \qquad \text{for all } \hat{u} \in V. \end{cases} \tag{4.4}$$

We have the following lemma corresponding to Lemma 2.1:

Lemma 4.1. *For $v, \hat{v}, \tilde{v} \in H^1(\mathcal{M})^2$, $\theta, \tilde{\theta} \in H^1(\mathcal{M})$, we have*

i) $|a_v(v,\hat{v})| \le c_1^v (\!(v,\hat{v})\!)$,
ii) $a_v(v,v) \ge c_2^v \|v\|^2$,
iii) $|a_\theta(\theta,\hat{\theta})| \le c_1^T (\!(\theta,\hat{\theta})\!)$,
iv) $a_\theta(\theta,\theta) \ge c_2^T \|\theta\|^2 - \nu_{\rm T} c_3 |\theta|^2$,
v) $b_v(v,\hat{v},\hat{v}) = 0$, *if* $v \in V_v$,
vi) $|b_v(v,\hat{v},\tilde{v})| \le |v| \cdot \|\nabla\theta\|_{L^\infty(\mathcal{M})} \cdot |\tilde{v}|$,
vii) $b_\theta(v,\theta,\theta) = 0$, *if* $v \in V_v$,
viii) $|b_\theta(v,\theta,\hat{\theta})| \le |v| \cdot \|\nabla\theta\|_{L^\infty(\mathcal{M})} \cdot |\hat{\theta}|$,
ix) $|d_v(v,\hat{v})| \le \nu_{\rm V}\alpha_{\rm V} c_1 |v_* - v|_{L^2(\partial\mathcal{M})} \cdot \|\hat{v}\|$,
x) $d_v(v,v) = \nu_{\rm V} \int_{\Gamma_{\rm i}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\rm V} v_* \cdot v\, d\Gamma_{\rm i} - \nu_{\rm V} \int_{\Gamma_{\rm i}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\rm V} |v|^2\, d\Gamma_{\rm i}$,
where $\nu_{\rm V} \left| \int_{\Gamma_{\rm i}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\rm V} v_* \cdot v\, d\Gamma_{\rm i} \right| \le \nu_{\rm V}\alpha_{\rm V} c_1 |v_*|_{L^2(\partial\mathcal{M})} \cdot \|v\|$,
and $\nu_{\rm V} \int_{\Gamma_{\rm i}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\rm V} |v|^2\, d\Gamma_{\rm i} \ge \nu_{\rm V}\alpha_{\rm V} c_2 \|v\|^2$,
xi) $|d_\theta(\theta,\hat{\theta})| \le \nu_{\rm T}\alpha_{\rm T} c_1 |\theta_* - \theta|_{L^2(\partial\mathcal{M})} \cdot \|\hat{\theta}\|$,
xii) $d_\theta(\theta,\theta) = \nu_{\rm T} \int_{\Gamma_{\rm u}\cup\Gamma_{\rm i}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\rm T} \theta_* \theta\, d\Gamma_{\rm u}\cup\Gamma_{\rm i} - \nu_{\rm T} \int_{\Gamma_{\rm u}\cup\Gamma_{\rm i}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\rm T} \theta^2\, d\Gamma_{\rm u}\cup\Gamma_{\rm i}$,
where $\nu_{\rm T} \left| \int_{\Gamma_{\rm u}\cup\Gamma_{\rm i}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\rm T} \theta_* \theta\, d\Gamma_{\rm u}\cup\Gamma_{\rm i} \right| \le \nu_{\rm T}\alpha_{\rm T} c_1 |\theta_*|_{L^2(\partial\mathcal{M})} \cdot \|\theta\|$,
and $\nu_{\rm T} \int_{\Gamma_{\rm u}\cup\Gamma_{\rm i}} \left(\frac{gp}{R\bar{T}}\right)^2 \alpha_{\rm T} \theta^2\, d\Gamma_{\rm u}\cup\Gamma_{\rm i} \ge \nu_{\rm T}\alpha_{\rm T} c_2 \|\theta\|^2$,
xiii) $|e_c(v,\hat{v})| \le c_4 |v| \cdot |\hat{v}|$,
xiv) $e_c(v,v) = 0$,

xv) $|e_\Phi(\theta, v)| \le c_5|\theta| \cdot |\nabla v|$, *if* $v \in V_v$,

where $c_1 = \left(\dfrac{gp_1}{R\bar{T}_*}\right)^2$, $c_2 = \left(\dfrac{gp_0}{R\bar{T}^*}\right)^2$, $c_3 = \left(\dfrac{gp_1}{p_0 c_{\mathrm{p}} \bar{T}_*}\right)^2$, $c_4 = \|f\|_{L^\infty}$, $c_5 = 2(p_1 - p_0)\dfrac{R}{p_0}\left(\dfrac{p_1}{p_0}\right)^{\frac{R}{c_{\mathrm{p}}}}$, $c_1^v = \max\{\mu_{\mathrm{V}}, \nu_{\mathrm{V}} c_1\}$, $c_1^T = \max\{\mu_{\mathrm{T}}, \nu_{\mathrm{T}} c_1, \nu_{\mathrm{T}} c_3\}$, $c_2^v = \min\{\mu_{\mathrm{V}}, \nu_{\mathrm{V}} c_2\}$, *and* $c_2^T = \min\{\mu_{\mathrm{T}}, \nu_{\mathrm{T}} c_2\}$.

Proof. We explicitly prove only (v) here, as the proofs of the other parts remain substantially unchanged from before.

$$
\begin{aligned}
b_v(v, \hat{v}, \hat{v}) &= \int_{\mathcal{M}} \left(\nabla_v \hat{v} + \omega \frac{\partial \hat{v}}{\partial p}\right) \cdot \hat{v}\, d\mathcal{M} \\
&= \frac{1}{2} \int_{\mathcal{M}} \left(\nabla_v |\hat{v}|^2 + \omega \frac{\partial |\hat{v}|^2}{\partial p}\right) d\mathcal{M} \\
&= \frac{1}{2} \int_{\mathcal{M}} \left(\mathrm{div}\,(|\hat{v}|^2 v) - |\hat{v}|^2 \mathrm{div}\, v + \omega \frac{\partial |\hat{v}|^2}{\partial p}\right) d\mathcal{M} \\
&= -\frac{1}{2} \int_{\mathcal{M}} |\hat{v}|^2 \left(\mathrm{div}\, v + \frac{\partial \omega}{\partial p}\right) d\mathcal{M} + \int_{\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}}} |\hat{v}|^2 \omega \Big|_{p=p_0}^{p_1} d\Gamma_{\mathrm{u}} \cup \Gamma_{\mathrm{i}} \\
&= 0.
\end{aligned}
$$

□

From this lemma, it is easy to prove, as before:

Theorem 4.2. *For any* u_0 *sufficiently regular, there exists a solution* u *to* (4.4) *satisfying* $u(0) = u_0$.

4.2. **Maximum principles.** In the spherical case, the proofs of positivity and boundedness are easier than in the previous case, since the lack of the vertical boundary only simplifies matters.

Theorem 4.3. *Under the assumptions*

$$
\begin{cases}
Q_E \ge 0, \\
\theta_* \ge 0, \\
\theta_0 \ge 0,
\end{cases}
$$

any solution $u = (v, \theta)$ *to* (4.4) *satisfies* $\theta(t) \ge 0$, *for all* $t \in [0, t_1]$. *Consequently,* $T(t) \ge 0$, *for all* $t \in [0, t_1]$, *as well.*

The proof is the same as in the previous case, except that any term involving the vertical boundary Γ_ℓ is now absent, so we omit it here.

Theorem 4.4. *If* $u = (v, \theta)$ *is a solution to* (4.4) *on the interval* $[0, t_1]$, *and all of the data* Q_E, θ_0, θ_* *are uniformly bounded, then we can bound* $\theta_{\max}$, *the maximum value of* θ, *by*

$$\theta_{\max} \le K e^{2\gamma^* t_1},$$

where

$$K = \max\left\{\frac{\|Q_E\|_{L^\infty}}{\gamma^*}, \|\theta_0\|_{L^\infty}, \frac{\alpha_{\mathrm{T}} \|\theta_*\|_{L^\infty}}{\alpha_{\mathrm{T}} + \frac{R}{p_1 c_{\mathrm{p}}}}\right\}.$$

Again, the proof is the same as in the previous case, except that the case where x, the point at which θ achieves its maximum, is on the vertical boundary Γ_ℓ is absent, and so we again omit the proof.

REFERENCES

[1] T. Aubin, "Nonlinear Analysis on Manifolds. Monge-Ampere Equations," Springer, New York, 1982.

[2] J. G. Charney, R. F. Fjørtaft, and J. von Neumann, *Numerical integration of the barotropic vorticity equation*, Tellus 2 (1950), 237–254.

[3] J. G. Charney, *Integration of the primitive and balance equations*, Proc. Intern. Symp. Numerical Weather Prediction, Tokyo (1962).

[4] G. Haltiner and R. Williams, "Numerical Weather Prediction and Dynamic Meteorology," 2nd edn., Wiley, New York, 1980.

[5] A. Lichnerowicz, *Propagateurs et Commutateurs en relativité génerále*, Presses Universitaires de France, Institut des Hautes Etudes Scientifiques, Publication Mathématique, vol. 10 (1961).

[6] J. L. Lions, R. Témam, and S. Wang, *New formulations of the primitive equations of the atmosphere and applications*, Nonlinearity 5 (1992), 237–288.

[7] ———, *On the equations of the large-scale ocean*, Nonlinearity 5 (1992), 1007–1053.

[8] ———, *Models of the coupled atmosphere and ocean, and Numerical analysis of the coupled models of atmosphere and ocean*, in "Computational Mechanics Advances," **1** (ed. J. T. Oden) (1993), 5–119.

[9] ———, *Mathematical study of the coupled models of atmosphere and ocean*, J. Math. Pures Appl. 74 (1995), 105–163.

[10] Kevin E. Trenberth, ed., "Climate System Modeling," Cambridge University Press, 1992.

[11] W. M. Washington and C. L. Parkinson, "An Introduction to Three-Dimensional Climate Modeling," Oxford University Press, Oxford, 1986.

[12] Qingcun Zun (Q. C. Zeng), "Mathematical and Physical Foundations of Numerical Weather Prediction," Science Press, Beijing, 1979 (in Chinese).

Received November 2000.

E-mail address: bewald@indiana.edu
E-mail address: temam@indiana.edu