Final exam (practice)  mac2313-08-F15

This practice exam is a fair approximation of the final exam which carries 30% of the total grade. I would like to draw your attention to the following important remarks.

- I will not provide the solution of this practice exam in a file. The only way you can learn about the solutions is to attend the class on Thursday Dec. 3 or during the office hours.

- There might still be some typos in the practice test. Please inform them as soon as you find one.

- The exam starts right at 10:00am (Tuesday Dec. 8) at the same room of the class and continues till 12pm.

- Calculators are not necessary but there is no restriction in using them. You don’t need to simplify your answers.

- You can bring a note sheet containing all formulas. No problem, example, or solution is allowed on the note sheet. All the note sheets must have names and turned in with your exam sheet at the end of the exam. Any missing note sheet will result in a score of zero in the test.

Start of the practice unit test

1. Complete the following steps to calculate the flux of the curl of vector field \( F = -yi + xj + zk \) through the surface \( S \) given by the triangle with vertices \( P(1,0,0), Q(0,1,0), \) and \( R(0,0,1) \) in two ways (See (d) and (e)).

   (a) Find \( \text{curl}(F) \).

   (b) Find the upward unit normal vector \( n \) of \( S \).

   (c) Compute \( \text{curl}(F) \cdot n \).

   (d) Evaluate \( \int_{S} \text{curl}(F) \cdot dS \).

   (e) Evaluate the line integral \( \int_{C} F \cdot dr \) where curve \( C \) is the positively oriented triangle viewed from above made of the edges of \( S \).

2. Determine which one of the following are vector, scalar or non-sense. \( F \) and \( G \) are vector fields, \( f \) is a scalar function, and \( u, v \) and \( w \) are vectors. Give a very short explanation in case it does not make sense.

   (a) \( \frac{\partial f}{\partial z} \)

   (b) \( \text{div} (\text{div} (F)) \)

   (c) \( v \times u \cdot w \)

   (d) \( v \times u \times w \)

   (e) \( \nabla f \)

   (f) \( \text{div} (\nabla (f)) \)

   (g) \( \text{curl}(F) \times \text{curl}(G) \)
(h) $v \times (u \cdot w)$
(i) $\text{curl} \left( \text{curl} \ (F) \right)$
(j) $\nabla \text{div} \ (F)$

3. Use Green’s theorem to find $\oint_C F \cdot dr$ where $F = -xy\mathbf{i} + 2xz\mathbf{j}$ and $C$ is the circle centered at the origin with radius 4 positively oriented.

4. Evaluate the flux of $F = 2xi + 2yj + 2zk$ over the surface $S$ given by $r(u, v) = ui + vj + (u^2 - v)k$ with $0 \leq u, v \leq 1$ oriented upward by following the steps below.
   (a) Find $r_u \times r_v$.
   (b) Find the upward normal vector of the surface.
   (c) Write and evaluate a surface integral over the parametrized surface.

5. Evaluate $\int_C F \cdot dr$ by following the steps below where $C$ is a positively oriented triangle with vertices $P(2, 0, 0)$, $Q(0, 1, 0)$, and $R(0, 0, 1)$ and $F = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + 2x^2)\mathbf{k}$.
   (a) Find $\text{curl} \ (F)$. (1 point)
   (b) Find the normal vector of the triangle. (1 point)
   (c) Find $\text{curl} \ (F) \cdot n$. (1 point)
   (d) Use Stoke’s theorem to complete the answer. (1 point)

6. Change the order of integration to evaluate $\int_0^1 \int_y^1 e^{-x^2} dx dy$.

7. True or false?
   (a) The spherical coordinates of the point $(x, y, z) = (0, 0, 1)$ is $(\rho, \theta, \phi) = (1, 0, 0)$.
   (b) If $u$ and $v$ are both unit vectors and $u \perp v$, then so is $u \times v$.
   (c) The equation of the plane $x = 1$ in cylindrical coordinates is $r = \frac{1}{\sin(\theta)}$.
   (d) If $u$ and $v$ are both unit vectors, then $|u \times v| = \sin(\theta)$ where $\theta$ is the angle between the vectors.
   (e) Plane $x + y - z = 0$ and line $\frac{x - 1}{2} = \frac{y - 2}{1} = z + 2$ are parallel.
   (f) A parametric equation of a sphere centered at $(1, 0, 0)$ with radius 1 is given by $r(\theta, \phi) = (1 + \sin(\phi)\cos(\theta))\mathbf{i} + \sin(\phi)\sin(\theta)\mathbf{j} + \cos(\phi)\mathbf{k}$ for $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$.
   (g) Two orientation of the surface $z = xy$ are given by vector fields $\hat{n} = \pm(-yi - xj + k)$.
   (h) Two lines given by parametric equation $r_1(t) = (2t, 1 - t, 3t)$ and $r_1(s) = (1 + 2s, -s, 2s)$ are skew.

8. Use the divergence theorem to compute the flux of the vector field $F = (-2x + z)i + xj + yk$ out of the ellipsoid $x^2 + 9y^2 + 4z^2 = 1$. (Hint: the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4}{3} \pi abc$.)
9. Match the equations with the level curve. (Equations are more than the plots.)

(1) \( z = xy \)
(2) \( z = x^2 + y^2 \)
(3) \( z^2 = x^2 + y^2 \)
(4) \( z = x^2 + 4y^2 \)
(5) \( z = 4x^2 + y^2 \)
(6) \( z = x^2 \)
(7) \( z = y^2 \)
(8) \( z = x + y \)
(9) \( z = x - y \)
(10) \( z = x^2 - y^2 \).

Figure 1: Level sets for problem 9
10. Match the equations with the surface. (Equations are more than the plots.)

(1) \( r^2 + z^2 = 16 \) and \( z \geq 0 \)
(2) \( x^2 + \frac{z^2}{4} = \frac{y^2}{4} \)
(3) \( x^2 - y^2 + 4y - 5 + z^2 = 0 \)
(4) \( x^2 + y^2 - z^2 = -1 \)
(5) \( r(u, v) = u^2i + vu\mathbf{j} + v^2k, \ -2 \leq u \leq 2 \) and \( v \leq 2 \).
(6) \( r(u, v) = \sqrt{v} \cos(u)i + v^2j + \sqrt{v} \sin(u)k, \ 0 \leq u \leq 2\pi \) and \( v \geq 0 \).
(7) \( r(u, v) = 4 \cos(\theta) \sin(\phi)i + 4 \sin(\theta) \sin(\phi)j + 4 \cos(\phi)k, \ 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq \pi \).
(8) \( \rho = 4 \).

Figure 2: Plots for problem 10