OPTIMAL PRODUCTION POLICY UNDER THE CARBON EMISSION MARKET

ARASH FAHIM AND NIZAR TOUZI

ABSTRACT. The aim of this paper is to address the effect of the carbon emission allowance market on the production policy of a polluter production firm. We investigate this effect in three cases; when the firm is not a large polluter, when it has a large contribution in emission of carbon but can not affect the risk premium of the allowance market, and when it is a large contributor in the emission of carbon and can change the risk premium by its production. In our simple model, we ignore any possible investment of the firm in pollution reducing technologies. We formulate the problem of optimal production by a stochastic optimization problem. Then, we show that, as expected, the market reduces the optimal production policy in the first two cases. However, when the large producer activities can change the market risk premium, the cut on the production and consequently pollution cannot be guaranteed. In fact, our numerical test shows that for some models in our framework, the increase in production, and thus pollution, can increase the profit of the firm.

Key words: EU ETS, Carbon emission allowance, Optimal production policy, HJB equations

AMS 2000 subject classifications: 91G80, 93E20, 91B70.

1. INTRODUCTION

The long term costs of global warming is believed to be significantly more than the cost of controlling it by reducing the pollution due to greenhouse gases (see [16]). One direct way to reduce the emission is to impose the taxation on the installations whose production increases the pollution. One can propose the standard taxation system which imposes a limitation level on the production of each installation over a time period and any amount of production above this level will be penalized. This taxation method has some significant disadvantages. First, there is no change in the production of the installations whose current optimal production policy does not reach

This Research is supported by the Chair Financial Risks of the Risk Foundation sponsored by Société Générale, the Chair Derivatives of the Future sponsored by the Fédération Bancaire Française, and the Chair Finance and Sustainable Development sponsored by EDF and CA-CIB.

Arash Fahim is partially supported by the NSF (DMS-1209519).
the level. Second, there is no benefit for those who are below their level to keep their position.

The Kyoto protocol in 1997 concerns with the reduction of the greenhouse gases including CO$_2$ and is accepted by several countries e.g. European Union members. In 2000, the European Commission launches European Climate Change Program (ECCP) to implement Kyoto protocol in Europe. As an alternative to standard taxation, ECCP proposed European Union Emission Trading Scheme (EU ETS) which provides a way to control the emission of CO$_2$ within carbon polluters through a cap-and-trade scheme.

More precisely, ECCP imposes a cap on the carbon emission. Within EU ETS, certain installations with intensive carbon pollution are given free carbon emission allowances (a.k.a allowance or carbon credit) and some other allowance by auctioning. If any installation wants to produce more than its initially given allowances, it should buy some more through EU ETS. However, the allowances will be needed only if the carbon emission per member state violates the imposed cap. On the other hand, if such installations, are below their imposed production limit, they could sell their allowances through the market.

First phase of the program was run from January 2005 to the end of 2007, in which a cap was imposed for each member state. All relevant installations within a member state should provide enough allowances, only if the member state cap is reached. In the second phase 2008–2012, the caps have been revised after the collapse in the first phase in April 2006 due to the release of the information about the unreachability of the emission caps. Moreover in the second phase, ECCP proposed to relevant installations to put off execution of the first phase allowances to the second phase by paying 40 euros per tonne. The same mechanism is set between second phase and third phase by the cost of 100 euros per tonne. This mechanism, which is referred to as banking, proposes an option for the allowance holder to execute the allowance to offset the excess production or to keep it for the next phase after paying a predetermined charge. In the third phase 2013–2020, there are two main changes comparing two the previous phases. First, the allocation of 40% of free allowance is replaced by auctioning to make the process more transparent. In addition, instead of imposing a nation-wide cap, the third phase has a single EU-wide cap. For more details of the specifics of each phase, see [11].

The first cap-and-trade emission reduction program, which was the result of 1990 Clean Air Act, dates back to 1995 aiming to reduce sulfur dioxide and nitrogen dioxide is US and is one of the most successful cap-and-trade schemes around the world (See [3]). Nowadays, there are other regional markets implementing similar schemes as EU ETS, e.g. the US REgenial CLean
In this paper, we analyze the effect of emission market in reducing the carbon emission through the change on production policy of the relevant production firms. Our setting is similar to the third phase of EU EST where a single EU-wide cap on total emission is imposed. The firm’s objective is to maximize its utility on wealth which is made of both the profit gained from production and the value of its carbon allowance portfolio over its production policy and its portfolio strategy. We solve the utility maximization problem on portfolio strategy by the duality argument and then on the production by the use of Hamilton–Jacobi–Bellman (HJB) equations. The terminal condition of our HJB equation is discontinuous and therefore, we face a challenge similar to [1]. However, we will partially avoid the issue by assuming that the forward equation modeling the pollution index is non-degenerate and therefore has no atom. In this manner, we only have to deal with a minor issue of discontinuity of terminal condition in our uniqueness problem.

We further categorize the relevant firms by their impact on the risk premium of the price of carbon allowance and public perception of the pollution. A small producer is a price taker and can neither change the allowance price nor the total emission significantly; while a large producer can affect at least one of the two through large amount of production. We observe that the market always reduces the optimal production policy of the small producers and large producers who cannot affect the risk premium of the allowance price. However, our study shows that in certain cases, a large producer with impact on the risk premium can take a manipulating role and its optimal production behavior does not necessarily reduce the emission. The key result to establish this comparison is that the price of the carbon allowance is equal to negative sensitivity of the value function of the firm with respect to its emission.

To better address the manipulative nature of a large producer, we make certain simplification in our model. We assume that the price of allowance is given by a risk-neutral measure and the emission market is complete. In addition, we ignore the affect of abatement and leave it for the future research. As for the profit function of the firm, it usually depends on the price of raw material and the product of the producer which are all stochastic. Here, we eventually ignore the stochastic nature of the profit of the firm in establishing our main results and the numerical experiment. This can be justified for a period of time in which the supply and demand remain stationary and non-volatile. We also assume that the emission dynamics is governed by an Itô process. This process usually represents the perception of the firm on
the total emission. While in the standard cap–and–trade each inclusive firm has to adjust its position yearly, here we assumed that to avoid the penalty, the firm needs to provide enough allowance only at the end of the period. Finally for the sake of simplicity, we do not extend our study to multi period with banking, where the holder of the allowance is given an option to keep the allowance for the next period by paying a predetermined fee.

Although the idea behind all these markets is cap–and–trade, they are different in certain details. A standard cap–and–trade system allocates allowances at the beginning of the period for no charge based on regulator’s discretion and penalizes the excess emission units of each installation at the end of the period or even shorter sub-periods. Alternatively, the regulator can distribute the allowances by auctioning at the beginning or even over the duration of the period. Banking also offers to transfer the allowance over the future periods by paying a certain charge. Also, it can impose the penalties only if the total emission exceeds a global cap, as it is so in the third phase of EU ETS. About the efficiency of the design of the market, [7] comprehensively studies the standard cap–and–trade system and verifies the presence of windfall profit based on the real market data and proposes a more efficient allocation scheme. In [6], the authors study some alternatives to the standard cap–and–trade system which can potentially lead to less windfall profit for a dominant player in the market and less cost for the consumers of the product of the polluter firms. More precisely, in addition to auctioning, they propose that the distribution of allowance over time rather than the beginning of the period will make it more flexible for the regulator to achieve its pollution reduction target with less cost on the economy.

Several studies target the dynamics of the price of the allowance. In the presence of banking, the price of allowance in the current period can be viewed as an option on the price of allowance for the next period (See for example [8]). This approach is obviously not capable of explaining the price of the most farthest period, and therefore, it is important to take a different approach to explain the dynamics for the price in the last period. In [6], by adopting a stochastic game setting in discrete-time, the authors show the existence of a Nash equilibrium for the price of emission allowance, which appears to be a martingale. In addition, they show that this equilibrium price is equal to the marginal cost of total abatement which can be obtained by solving the problem of minimizing the total abatement cost in the market. In [5], they study the formation of equilibrium price in a continuous–time setting through a system of forward–backward SDEs (FBSDE) with singularity at terminal time. In their study, the singularity of the terminal condition is two-folded; one caused by the discontinuity of terminal condition of backward equation and the other by the degeneracy of the forward
equation at terminal time. Then for some specific pathological examples, they showed the existence and the uniqueness of the solution to the system of FBSDEs, by approximating the terminal condition with smooth functions in a certain manner. Their main contribution, besides modeling allowance price by FBSDEs, is to bold the difficulties caused by discontinuous terminal condition of backward component and the demand for a more inclusive theory of FBSDEs with discontinuous terminal condition.

The study of a large producer manipulating the emission allowance market is back to [17] where they consider a monopoly (or monopsony) firm whose production impacts both allowance price and product price. They show, in a static context, that the monopoly firm can manipulate the market by transferring the abatement costs to its rivals and as a consequence increasing the cost of production for fringe firms. In their study, the monopoly firm strives to maximize its profit subject to demand and price impact constraints. In [13] and [15], the authors study the market power of a large producer in analogy to the market power in the context of exhaustible resource market in a dynamic setting where the firm decides how to buy/sell the allowance permits and how to use them over time. They show that the large producer firm will cover its total emission in a competitive manner unless the initial allocation is not sufficient for its optimal production plan. In all the above mentioned literature, the stochastic nature of the allowance price in the market is ignored.

This paper is organized as follows. In Section 2, we present the general non-Markovian model and derive a characterization for the optimal production policy of a small production firm. The tool we use in this section is convex duality for utility maximization which helps us separate the trading activity of the producer and its profit from the production. In Section 3, we repeat the analysis of Section 2 for a large producer in two cases based on the power of impact the large producer on the risk premium of the market. We start Section 3 in a general framework by using convex duality in a similar fashion to Section 2. Then later in section, we narrow our study to Markovian setting and derive HJB equation for the profit function of the firm. We use this HJB equation to study the impact of the large producer both in the analytical and numerical results. In Section 4, we present out numerical results. Appendix sections will cover the existence and uniqueness for the HJB equation, the existence of optimal production policy, and the implementation details of the numerical results.
2. Small producer with one-period carbon emission market

Let \((\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})\) be a filtered probability space satisfying the usual assumptions which hosts one-dimensional Brownian motion \(W\), and by \(E_t := \mathbb{E}[\cdot|\mathcal{F}_t]\) the conditional expectation operator given \(\mathcal{F}_t\).

We consider a production firm with risk preference described by the utility function \(U : \mathbb{R} \to \mathbb{R} \cup \{\infty\}\) assumed to be strictly increasing, strictly concave and \(C^1\) over \(\{U < \infty\}\). We denote by \(\pi_t(\omega, q)\) the (random) rate of profit of the firm for a production rate \(q\) at time \(t\). Here \(\pi : [0, T] \times \Omega \times \mathbb{R}_+ \to \mathbb{R}\) is an \(\mathbb{F}\)-progressively measurable map\(^\text{1}\). We shall omit \(\omega\) from the notations wherever appropriate. For fixed \((t, \omega)\), we assume that the function \(\pi_t(\cdot) := \pi(t, \cdot)\) is \(C^1\) and strictly concave in \(q\), and satisfies

\[ 0 < \pi_t'(0^+) \quad \text{and} \quad \pi_t'(\infty) < 0. \]

The above assumption on \(\pi\) can be justified as follows. If the production of the firm increases above a certain amount determined by the current demand, the price of the product of the firm decreases. Consequently, the profit drops.

Let us denote by \(\eta_t(q)\) the rate of carbon emission generated by a production rate \(q\). Here, \(\eta : [0, T] \times \Omega \times \mathbb{R}_+ \to \mathbb{R}\) is an \(\mathbb{F}\)-progressively measurable map such that for each \((t, \omega)\), \(\eta_t(q) := \eta(t, \cdot)\) is \(C^1\) and increasing in \(q \in \mathbb{R}_+\). Then the total amount of carbon emission induced by a production policy \(\{q_t, t \in [0, T]\}\) is given by

\[
(2.1) \quad E^d_T := \int_0^T \eta_t(q_t) dt.
\]

The aim of the carbon emission market is to incur some cost to the producer so as to obtain an overall reduction on the carbon emission.

From now on, we analyze the effect of the presence of the carbon emission market within the cap-and-trade scheme.

In order to model the allowance price, we introduce a state variable \(Y\) given by the dynamics:

\[
(2.2) \quad dY_t = \mu_t dt + \gamma_t dW_t,
\]

where \(\mu\) and \(\gamma\) are two bounded \(\mathbb{F}\)-adapted processes and \(\gamma > 0\).

---

\(^1\) An \(\mathbb{F}\)-progressively measurable map is usually defined for a mapping \(\pi\) from \([0, T] \times \Omega\) to \(\mathbb{R}\). However, we can simply extend it by calling a mappings \(\pi : [0, T] \times \Omega \times \mathbb{R}_+ \to \mathbb{R}\) \(\mathbb{F}\)-progressively measurable if and only if \(\pi : [0, t] \times \Omega \times \mathbb{R}_+ \to \mathbb{R}\) is \(B([0, t]) \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+\))-measurable for all \(t \in [0, T]\). In this manner, if \(q(\omega)\) is a \(\mathbb{F}\)-progressively measurable process in the usual sense and \(\pi\) is \(\mathbb{F}\)-progressively measurable in the extended sense, then \(\pi(\cdot, \omega, q(\omega))\) is \(\mathbb{F}\)-progressively measurable in the usual sense.
Remark 2.1. The state variable $Y$ should be interpreted as the perception of the firm on the total carbon emission. Since the total emission will only be revealed at the end of the period, the process $Y$ involves uncertainty and is considered stochastic because of the following reasons. First, it depends on the total production in the environment which makes it impossible for a single firm to precisely apperceive it. Second, the amount of production and thus the emission depends on several factors such as fuel price, weather changes, economic growth, etc; each of which are believed to be stochastic. Later on, we will assume that a large producer firm can benefit from knowing its contribution in the trend of state variable.

We assume that there is one single period $[0, T]$ during which the carbon emission market is in place. At each time $t \geq 0$, the random variable $Y_t$ indicates the aggregated market opinion on the cumulated carbon emission. At time $T$, $Y_T \geq \kappa$ (resp. $Y_T < \kappa$) means that the cumulated total emission have (resp. not) exceeded the cap $\kappa$, fixed by the trading scheme. We simply take $\kappa = 0$. Let $\alpha$ be the penalty per unit (tonne) of carbon emission. Then, the value of the carbon emission contract at time $T$ is:

$$S_T := \alpha 1_{\{Y_T \geq 0\}}.$$

The carbon emission allowance can be viewed as a derivative security on $Y$ defined by the above payoff. (See [18] and [9].) The carbon emission market allows for trading this contract in continuous-time throughout the time period $[0, T]$. Assuming that the market is frictionless, it follows from the classical no-arbitrage valuation theory that the price of the carbon emission contract at each time $t$ is given by

$$S_t := \mathbb{E}^Q_t [S_T] = \alpha \mathbb{Q}_t [Y_T \geq 0],$$

where $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$, the so-called equivalent martingale measure, $\mathbb{E}^Q_t$ and $\mathbb{Q}_t$ denote the conditional expectation and probability given $\mathcal{F}_t$. Given market prices of the carbon allowances, the risk-neutral measure may be inferred from the market prices. Since the market is frictionless, the value of the initial holdings in (free) allowances, $E^{\text{max}}$, can be expressed equivalently in terms of their value in cash $S_0 E^{\text{max}}$.

In the present context, and in contrast to a standard taxation (Remark 2.2), production firms have more incentive to reduce emission as they have the possibility to sell their allowances on the emission market.

We now formulate the objective function of the firm in the presence of the emission market. The primary activity of the firm is the production modeled by the rate $q_t$ at time $t$. This generates a gain $\pi_t(q_t)$. The resulting carbon emission are given by $\eta_t(q_t)$. Given that the price of the allowance
is available on the market, the profit on the time interval \([0, T]\) is given by:

\[
\int_0^T \pi_t(q_t) dt - S_T \int_0^T \eta_t(q_t) dt.
\]  

(2.4)

In addition to the production activity, the company trades continuously on the carbon emission market. Let \(\{\theta_t, t \geq 0\}\) be an \(\mathbb{F}\)-adapted process such that \(\int_0^T \theta_t^2 d\langle S \rangle_t < \infty\) \(\mathbb{P}\)-a.s.. For every \(t \geq 0\), \(\theta_t\) indicates the number of allowances held by the company at time \(t\). Under the self-financing condition, the wealth accumulated by trading in the emission market is:

\[
x + \int_0^T \theta_t dS_t,
\]

(2.5)

where \(x\) is the initial capital of the company, including the market value of its free allowances, i.e. \(S_0 E_{\text{max}}\). By (2.4) and (2.5), together with an integration by parts, the total wealth of the firm at time \(T\) is \(X_T^\theta + B_T^q\) where

\[
X_T^\theta := x + \int_0^T \theta_t dS_t, \quad B_T^q := \int_0^T (\pi_t(q_t) - S_t \eta_t(q_t)) dt - \int_0^T E_t^q dS_t,
\]

and

\[
E_t^q := \int_0^t \eta_u(q_u) du, \quad \text{for all } t \in [0, T].
\]

We assume that the firm is allowed to trade with no constraint. Then, the objective function of the manager is:

\[
V^{(1)} := \sup \left\{ E \left[ U \left( X_T^\theta + B_T^q \right) \right] : \theta \in \mathcal{A}, q \in \mathcal{Q} \right\},
\]

(2.6)

where \(\mathcal{A}\) is the collection of all \(\mathbb{F}\)-progressively measurable processes \(\{\theta_t\}_{t \geq 0}\) such that \(\int_0^T \theta_t^2 d\langle S \rangle_t < \infty\) \(\mathbb{P}\)-a.s. and \(X_t := x + \int_0^t \theta_s dS_s\) is bounded from below by a martingale, and \(\mathcal{Q}\) is the collection of all non-negative \(\mathbb{F}\)-progressively measurable processes such that \(E_T^q < \infty\).

Notice that the stochastic integrals with respect to \(S\) can be collected together in the expression of \(X_T^\theta + B_T^q\). Since \(\mathcal{A}\) is a linear subspace, it follows that the maximization with respect to \(q\) and \(\theta\) are completely decoupled; this problem is easily solved by optimizing successively with respect to \(q\) and \(\theta\).

The partial maximization with respect to \(q\) provides an optimal production level \(q_t^{(1)}\) defined by the first order condition:

\[
\frac{\partial \pi_t}{\partial q_t^{(1)}} (q_t^{(1)}) = S_t \frac{\partial \eta_t}{\partial q_t^{(1)}} (q_t^{(1)}).
\]

(2.7)

Because of the assumptions on \(\pi_t(.)\) and \(\eta_t(.)\), we deduce immediately that \(q_t^{(1)}\) is less than the business-as-usual optimal production \(q_t^{\text{bau}}\) of the firm in the absence of any restriction on the emission, which is determined by the first order condition \((\partial \pi_t/\partial q)(q_t^{\text{bau}}) = 0\). In other words, the emission
market leads to a reduction of the production, and therefore a reduction of the carbon emission.

We next turn to the optimal trading strategy by solving:

$$\sup_{\theta} \mathbb{E} \left[ U \left( X_{T,\theta} - E^{(1)} + B^{q (1)} \right) \right]$$

where $$B^{q (1)}_T := \int_0^T (\pi_t(q_t) - S_t \eta_t(q_t)) \, dt$$.

In the present context of a complete market, the solution is given by:

$$x + \int_0^T \left( \theta^{(1)}_t - E^q_t \right) \, dS_t + B^{q (1)}_T = (U')^{-1} \left( y^{(1)} \frac{dQ}{d\mathbb{P}} \right)$$

where the Lagrange multiplier $$y^{(1)}$$ is defined by:

$$\mathbb{E}^Q \left[ (U')^{-1} \left( y^{(1)} \frac{dQ}{d\mathbb{P}} \right) \right] = x + \mathbb{E}^Q \left[ B^{q (1)}_T \right]$$.

Let us sum up the present context of a small firm:

- the trading activity of the company has no impact on its optimal production policy,
- the firm’s optimal production $$q^{(1)}$$ is smaller than that of the business-as-usual situation, so that the emission market is indeed a good tool for the reduction of carbon emission,
- the emission market assigns a price to the externality that the firm manager can use in order to optimize his production scheme.

**Remark 2.2.** Let us examine the case where there is no possibility to trade the carbon emission allowances. This is the standard taxation system where $$\alpha$$ is the amount of tax to be paid at the end of period per unit of carbon emission. Assuming again that the firm’s horizon coincides with this end of period, its objective is:

$$V_0 := \sup_{q \in \mathbb{Q}} \mathbb{E} \left[ U \left( \int_0^T \pi_t(q_t) \, dt - \alpha \left( E^q_T - E^{max} \right)^+ \right) \right]$$

where $$E^{max}$$ is the free allowances of the firm. Direct calculation leads to the following characterization of the optimal production level:

$$\frac{\partial \pi_t \left( q^{(0)}_t \right)}{\partial q} = \alpha \frac{\partial \eta_t \left( q^{(0)}_t \right)}{\partial q} \mathbb{E}_t \left[ 1_{\mathbb{R}^+} \left( E^{q (0)}_t - E^{max} \right) \right]$$

where

$$\frac{dQ^{(0)}}{d\mathbb{P}} = \frac{U' \left( \int_0^T \pi_t(q^{(0)}_t) \, dt - \alpha \left( E^{q (0)}_t - E^{max} \right)^+ \right)}{\mathbb{E} \left[ U' \left( \int_0^T \pi_t(q^{(0)}_t) \, dt - \alpha \left( E^{q (0)}_t - E^{max} \right)^+ \right) \right]}.$$
The natural interpretation of (2.8) and (2.9) is that the production firm assigns an individual price to its emission:

\begin{equation}
S_t := \alpha \mathbb{E}_{t}^{Q^{(0)}} \left[ 1_{\mathbb{R}^+} \left( E_t^{(0)} - E^{\text{max}} \right) \right],
\end{equation}

i.e. the expected value of the amount of tax to be paid under the measure $Q^{(0)}$ defined by its marginal utility as a density. The probability measure $Q^{(0)}$ is the so-called risk-neutral measure in financial mathematics, or the stochastic discount factor of the firm. Given this evaluation, the firm optimizes its adjusted profit function, $\pi_t(q) - \eta(q) S_t$:

\begin{equation}
\frac{\partial \pi_t(q^{(0)})}{\partial q} = \frac{\partial \eta_t(q^{(0)})}{\partial q} S_t.
\end{equation}

We continue by commenting on the optimal production policy defined by (2.8)-(2.9):

- assuming that the firms know the nature of their utility functions, the system of equations (2.8)-(2.9) is still a nontrivial nonlinear fixed point problem.
- This problem would be considerably simplified if the manager were to know the market price for carbon emission (2.10). But of course, in the present context, this is an individual subjective price which is not quoted on any financial market.
- The present situation, based on a classical taxation policy, offers no incentive to reduce emission beyond $E^{\text{max}}$. Indeed, if the optimal production in the absence of taxes produces carbon emission below the level $E^{\text{max}}$, then it is indeed the same as the business-as-usual situation. So, the taxation does not contribute to reduce the carbon emission. As a consequence, the only way to benefit from having carbon emission below the level $E^{\text{max}}$ is to merge with another firm whose emission are above its given free emission allowances. Hence, such a policy puts a clear incentive to mergers.

The emission market provides an evaluation of the externality of carbon emission by firms. Given this information there is no more need to know precisely the utility function of the firm in order to solve the nonlinear system (2.8)-(2.9). The quoted price of the externality is then very valuable for the managers as it allows them to better optimize their production scheme.

3. LARGE PRODUCER WITH ONE-PERIOD CARBON EMISSION MARKET

In this section, we consider the case of a large carbon emitting production firm. We shall see that this leads to different considerations as the trading activity will have an impact on the production policy of the company.
We model this situation by assuming that the state variable \( Y \) is affected by the production policy of the firm:

\[
\frac{dY_t^q}{Y_t^q} = (\mu_t + \beta \eta_t(q_t)) \, dt + \gamma_t \, dW_t
\]

where \( \beta > 0 \) is a given impact coefficient. The price process \( S \) of the carbon emission allowances is, as in the previous section, given by the no-arbitrage valuation principle:

\[
S_t^q = \alpha Q_t^q \left[ Y_T^q \geq 0 \right]
\]

and is also affected by the production policy \( q \). The equivalent martingale measure \( Q_t^q \) is characterized by its Radon-Nykodim density which can be represented as a Doléans-Dade exponential martingale generated by some risk premium process \( \lambda \). In general, the risk premium process \( \lambda \) may depend on the path of the control process \( q \). For technical reasons, we shall restrict our analysis to those risk-neutral probability measures with risk premium process depending on the current value of the control process:

\[
\frac{dQ_t^q}{dP} \bigg|_{\mathcal{F}_t} = \exp \left( -\int_0^T \lambda_t(q_t) \, dW_t - \frac{1}{2} \int_0^T \lambda_t(q_t)^2 \, dt \right)
\]

where \( \lambda : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R} \) is an \( \mathcal{F} \)-progressively measurable map. Under \( P \), the dynamics of the price process \( S \) is given by

\[
\frac{dS_t^q}{S_t^q} = \sigma_t^q \left( dW_t + \lambda_t(q_t) dt \right), \ t < T,
\]

where the volatility function \( \sigma_t^q \) is progressively measurable and depends on the control process \( \{q_s, 0 \leq s \leq T\} \). As in the previous section, the wealth process of the company is given by:

\[
X_T^{x, \theta} := x + \int_0^T \theta_t \, dS_t^q \quad \text{and} \quad B_T^q := \int_0^T \pi_t(q_t) \, dt - S_T^q \int_0^T \eta_t(q_t) \, dt
\]

**Remark 3.1.** The study performed in [18] supports the assumption of existence of a martingale measure. In fact, by using empirical data, they showed that the discounted price of the allowance is martingale. As a consequence, there is no seasonal effect in the price and we can simply assume that \( \sigma_t^q \) is independent of time \( t \).

3.1. **Large carbon emission with no impact on risk premium.** In this subsection, we restrict our attention to the case of large emitting firm with no impact on the risk premium, i.e.

\[
\lambda_t(q) \quad \text{is independent of} \ q \quad \text{for any} \ t \geq 0.
\]

The objective of the large emitting firm is:

\[
V_0^{(2)} := \sup_{q \in \mathcal{Q}, \ \theta \in \mathcal{A}} \mathbb{E} \left[ U \left( X_T^{x, \theta} + B_T^q \right) \right].
\]
Proposition 3.2. Assume (3.5), and that the market is complete with unique risk-neutral measure $Q$. Then, the optimal production policy is independent of the utility function of the producer $U$, and obtained by solving:

$$\sup_{q \in Q} \mathbb{E}^Q \left[ B_T^q \right].$$

Moreover, if $q^{(2)}$ is an optimal production scheme, then the optimal investment strategy $\theta^{(2)}$ is characterized by

$$X_T^{x,\theta^{(2)}} + B_T^{q^{(2)}} = (U')^{-1} \left( y^{(2)} \frac{dQ}{dP} \right),$$

where the Lagrange multiplier $y^{\theta}$ is defined by

$$\mathbb{E}^Q \left[ (U')^{-1} \left( y^{\theta} \frac{dQ}{dP} \right) \right] = x + \mathbb{E}^Q \left[ B_T^q \right].$$

Proof. We first fix some production strategy $q$. Since the market is complete, the partial maximization with respect to $\theta$ can be performed by the duality method in [10, Theorem 3.1] to obtain

$$X_T^{x,\theta} + B_T^q = (U')^{-1} \left( y \frac{dQ}{dP} \right),$$

where the Lagrange multiplier $y$ is defined by

$$\mathbb{E}^Q \left[ (U')^{-1} \left( y \frac{dQ}{dP} \right) \right] = x + \mathbb{E}^Q \left[ B_T^q \right].$$

This reduces the problem to:

$$\sup_{q \geq 0} \mathbb{E} \left[ U \circ (U')^{-1} \left( y^{\theta} \frac{dQ}{dP} \right) \right].$$

Notice that $U \circ (U')^{-1}$ is decreasing and the density $\frac{dQ}{dP} > 0$. Then (3.8) reduces to

$$\inf \{ y^{\theta} : q \geq 0 \}.$$ 

Since $(U')^{-1}$ is also decreasing, (3.9) converts the problem into

$$\sup \left\{ \mathbb{E}^Q \left[ B_T^q \right] : q \in Q \right\}.$$ 

Finally, given the optimal strategy $q^{(2)}$, the optimal investment policy is characterized by (3.7). □

In order to push further the characterization of the optimal production policy $q^{(2)}$, we specialize the discussion to the Markov case by assuming the following for the triple $(\pi, \eta, \lambda)$.

Assumption A. $\pi_t(q) = \pi(t, q)$, $\eta_t(q) = \eta(t, q)$, and $\lambda_t(q) = \lambda(t)$ are in $C^{0,1}([0, T] \times \mathbb{R}_+)$ and satisfy

(i) $\pi$ is strictly concave in $q$, $\pi_t(0) = 0$ and $\frac{\partial \pi}{\partial q}(\infty) < 0$,

(ii) $\eta$ is convex and strictly increasing in $q$. 


(iii) \( \lambda \) is concave and nondecreasing in \( q \) and \( \lambda(t,0) \geq 0 \),
(iv) \( \lim \inf_{q \to \infty} \sup_{t \in [0,T]} \frac{\eta(t,q)}{\lambda(t,q)} > 1 \).

We also enforce a Markovian dynamics for process \( Y^q \) under measure \( P \); i.e.,
\[
dY^q_t = (\mu(t,Y^q_t) + \beta \eta(t,q_t)) \, dt + \gamma(t,Y^q_t) \, dW_t,
\]
for some deterministic functions \( \mu, \gamma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) and a nonnegative constant \( \beta \).

The state variable \( E \) is now defined by the dynamics
\[
dE^q_t = \pi(t,q_t) \, dt,
\]
which records the cumulated carbon emission of the company. The dynamic version of the producer planning problem (3.6) is given by:
\[
V^{(2)}(t,e,y) := \sup_{q \in \mathcal{Q}_t} \mathbb{E}_{t,e,y}^Q \left[ \int_t^T \pi(t,q_s) \, ds - \alpha E^q_t \mathbf{1}_{\{y > 0\}} \right],
\]
where \( \mathcal{Q}_t \) is the collection of all non-negative \( \mathbb{F} \)-progressively measurable processes such that \( \int_t^T \eta_s(q_s) \, ds < \infty \), and \( \mathbb{E}_{t,e,y}^Q \) is the expectation with respect to \( \mathbb{Q} \) conditional on \( E^q_t = e, Y^q_t = y \). Then, \( V^{(2)} \) is a viscosity solution of the dynamic programming equation with a terminal condition:
\[
0 = \frac{\partial V^{(2)}}{\partial t} + (\mu - \lambda \gamma)V^{(2)}_y + \frac{1}{2} \gamma^2 V^{(2)}_{yy} + \max_{q \geq 0} \left\{ \pi(t,q) + \eta(t,q)V^{(2)}_{y(t,q)} + \beta \eta(t,q) V^{(2)}_y \right\},
\]
(3.12)
\[
V^{(2)}(T,e,y) = -\alpha e \mathbf{1}_{\{y > 0\}}.
\]
For the moment assume that the value function \( V^{(2)} \) is smooth. Then, the optimal production \( q^{(2)} \) is given by the maximum
\[
q^{(2)}(t,e,y) \in \arg\max_{q \geq 0} \left\{ \pi(t,q) + \eta(t,q)(V^{(2)}_{e(t,q)} + \beta V^{(2)}_{y(t,q)})(t,e,y) \right\}.
\]
(3.13)
By Lemma (A.14), we have
\[
-V^{(2)}_{e(t,q)}(t,\eta_t,Y_t) = S_t.
\]
(3.14)
If the maximum in (3.13) is attained in an interior point, then it satisfies
\[
\frac{\partial \pi}{\partial q} \left( t,q^{(2)}_t \right) = \frac{\partial \eta}{\partial q} \left( t,q^{(2)}_t \right) \left( S_t - \beta V^{(2)}_y(t,E^q_{t(2)},Y^{(2)}_t) \right).
\]
(3.15)
Otherwise in case the maximum in (3.13) is not attained in an interior point, we have \( q^{(2)} = 0 \). Since supremum of nonincreasing functions is nonincreasing, we have \( V^{(2)}_y \leq 0 \). Thus, it follows from comparing (3.13) with (2.7) that \( q^{(2)} \leq q^{(1)} \). In fact, a larger \(-\beta V^{(2)}_y(t,E^q_{t(2)},Y^{(2)}_t)\) implies a larger the difference \( q^{(1)} - q^{(2)} \). For instance by choosing a large penalty term \( \alpha \), the optimal production and consequently the emission can be controlled to meet the target.
In other words, the impact of the production of the firm on the prices of carbon emission allowances increases the cost of the externality for the firm. This immediately affects the profit function of the firm and leads to a decrease of the level of optimal production. Hence, the presence of the emission market is playing a positive role in terms of reducing the carbon emission.

The following result shows that under certain assumptions, the above formal calculation is valid in our model.

**Theorem 3.3.** Let for any \((t,e,y)\) there exists an optimal control \(q^* = q_{t,e,y}^* \in \mathcal{Q}_t\) for the problem (A.1) and triple \((\pi, \eta, \lambda)\) satisfies Assumption A. Then \(V_{e}^{(2)}\) exists and (3.12) holds true. In addition, if problem (3.12) has a (bounded) solution in \(C^{1,1,2}([0,T) \times \mathbb{R}_+ \times \mathbb{R})\), then there exists a unique optimal production strategy satisfying (3.13).

**Proof.** Since \(V\) is convex in \(e\) (supremum of linear functions is convex), it has left and right partial derivatives with respect to \(e\) everywhere, and the partial gradient \(V_e\) exists almost everywhere. Under our conditions, the additional differentiability and the remaining characterizations of the theorem follow from Proposition A.13 and Lemma A.14.

For the last assertion of the Theorem, notice that by Theorems ?? and ??, \(V\) is the unique bounded viscosity solution of (3.12). Therefore, if \(V \in C^{1,1,2}([0,T) \times \mathbb{R}_+ \times \mathbb{R})\), one can use the dynamic programming principle to deduce \(q_{e}^{(2)}\) obtained from (3.13) is an optimal strategy. \(\square\)

### 3.2. Large carbon emission impacting the risk-neutral measure.

We now consider the general case where the risk premium process is impacted by the emission of the production firm:

\[
\frac{dQ^q}{d\mathbb{P}} \bigg|_{\mathcal{F}_T} = \exp \left( - \int_0^T \lambda(q_t) dW_t - \frac{1}{2} \int_0^T \lambda(q_t)^2 dt \right).
\]

The partial maximization with respect to \(\theta\), as in the proof of Proposition 3.2, is still valid in this context, and reduces the production firm’s problem to

\[
\sup_{q \in \mathcal{Q}} \mathbb{E} \left[ U \circ (U')^{-1} \left( y^q \frac{dQ^q}{d\mathbb{P}} \right) \right]
\]

where \(y^q\) is defined by

\[
\mathbb{E}^{Q^q} \left[ (U')^{-1} \left( y^q \frac{dQ^q}{d\mathbb{P}} \right) \right] = x + \mathbb{E}^{Q^q} \left[ B_T^q \right].
\]

In order to move further, we assume that the preferences of the production firm are defined by an exponential utility function

\[
U(x) := -e^{-ax}, \quad x \in \mathbb{R}, \quad a > 0.
\]
Then \( U \circ (U')^{-1}(y) = -y/a \), and (3.14) reduces to
\[
\inf_{q \geq 0} E \left[ y^q \frac{dQ^q}{dP} \right] = \inf_{q \geq 0} y^q.
\]

Finally, the budget constraint (3.17) is in the present case:
\[
x + E^Q \left[ B_t^q \right] = \frac{-1}{a} E^Q \left[ \ln \left( \frac{y^q}{a} \frac{dQ^q}{dP} \right) \right]
= \frac{-1}{a} \left\{ \ln \left( \frac{y^q}{a} \right) + E^Q \left[ \ln \left( \frac{dQ^q}{dP} \right) \right] \right\},
\]
so that the optimization problem (3.18) is equivalent to:
\[
\sup_{q \in Q} E^Q \left[ \int_0^T \left( \pi + \frac{\lambda^2}{2a} \right) (t, q_t)dt - S_T^q \int_0^T \eta_t(q_t)dt \right].
\]

Notice the difference between the above optimization problem, which determines the optimal production policy of the production firm, and the problem (3.6). In the present situation where the risk premium process is impacted by the carbon emission of the firm, the firm’s optimization criterion is penalized by the entropy of the risk-neutral measure with respect to the statistical measure.

The firm’s optimal production problem (3.19) is a standard stochastic control problem. We continue our discussion by considering the Markov case, and introducing the dynamic version of (3.19):
\[
V^{(3)}(t, e, y) := \sup_{q \in Q_t} E^Q_t \left[ \int_t^T \left( \pi + \frac{\lambda^2}{2a} \right) (s, q_s)ds - a E^Q_t 1_{Y^q_t \geq 0} \right],
\]
where the controlled state dynamics is given by:
\[
dY^q_t = (\mu(t, Y^q_t) + \beta \eta(t, q_t) - \gamma(t, Y^q_t) \lambda(t, q_t)) dt + \gamma(t, Y^q_t) dW^q_t;
\]
\[
dE^q_t = e(t, q_t)dt,
\]
\( W^q \) is a Brownian motion under \( Q^q \), and \( \mu \) and \( \gamma \) are \( C^{1,2} \) functions in \((t, y)\), and \( \mu, \eta \) and \( \lambda \) are \( C^{1,2} \) functions in \((t, q)\).

By classical arguments, we then see that \( V^{(3)} \) solves a dynamic programming equation with a terminal condition:
\[
0 = \frac{\partial V^{(3)}}{\partial t} + \mu V^{(3)}_y + \frac{1}{2} \gamma^2 V^{(3)}_{yy} + \max_{q \in R_+} \theta(t, y, q, V^{(3)}_y, V^{(3)}_y)
= -\alpha e 1_{y > 0},
\]
where
\[
\theta(t, y, q, p_e, p_y) = \pi(t, q) + \frac{1}{2a} \lambda(t, q)^2 + \eta(t, q)(p_e + \beta p_y) - \gamma(t, y) \lambda(t, q)p_y
\]
In terms of the value function \( V(3) \), the optimal production policy is obtained as the maximizer in the above equation, i.e.

\[
q^{(3)}(t, e, y) \in \arg\max_{q \geq 0} \left\{ \pi(t, q) + \frac{1}{2a} \lambda(t, q)^2 + q(t, q)(V_e^{(3)} + \beta V_y^{(3)})(t, e, y) - \gamma(t, y)\lambda(t, q)\lambda_y^{(3)}(t, e, y) \right\}.
\]

Observe that if we assume \( V^{(3)} \) is regular enough, then Assumption \( A \) implies that \( \arg\max \) is a singleton and \( q^{(3)} \) is unique. In addition if an interior maximum occurs, then the first order condition is:

\[
\frac{\partial \pi}{\partial q}(q^{(3)}) + \frac{1}{a} (\lambda \frac{\partial \lambda}{\partial q})(q^{(3)}) + \frac{\partial \eta}{\partial q}(q^{(3)})(V_e^{(3)} + \beta V_y^{(3)}) - \gamma \frac{\partial \lambda}{\partial q}(q^{(3)})\lambda^{(3)} = 0.
\]

Moreover, we shall show in Lemma \( A.14 \) that the price of the carbon emission allowance, as observed on the emission market, is given by:

\[
S_t = -V_e^{(3)}(t, E_t^{q^{(3)}}, Y_t^{q^{(3)})}.
\]

Then, it follows that the optimal production policy of the firm is defined by:

\[
\frac{\partial \pi}{\partial q}(t, q^{(3)}) = \frac{\partial \eta}{\partial q}(t, q^{(3)}) \left( S_t - \beta V_y^{(3)}(t, E_t^{q^{(3)}}, Y_t^{q^{(3)})} \right) \\
+ \frac{\partial \lambda}{\partial q}(t, q^{(3)}) \left( \gamma V_y^{(3)}(t, E_t^{q^{(3)}}, Y_t^{q^{(3)})} - \frac{1}{a} \lambda(t, q^{(3)}) \right).
\]

Contrary to the previous case where the risk premium process was not impacted by the carbon emission of the large firm, we can not always conclude from the above formula that \( q^{(3)} \) is smaller than \( q^{(1)} \); the optimal production policy in the absence of a financial market given by (2.7). More precisely, if \( \tau \) defined below is non-negative, then we can conclude that \( q^{(3)} \leq q^{(1)} \).

\[
\tau := \left( \beta \frac{\partial \eta}{\partial q}(t, q^{(3)}) - \gamma \frac{\partial \lambda}{\partial q}(t, q^{(3)}) \right) V_y^{(3)}(t, E_t^{q^{(3)}}, Y_t^{q^{(3)})} + \frac{1}{a} \frac{\partial \lambda}{\partial q}(t, q^{(3)}) \lambda(t, q^{(3)})
\]

However, \( \tau \) has no known sign, and there is no economic argument supporting that it should have some specific sign. By means of assumption Assumption \( A \) we can only be sure that \( \frac{1}{a} \frac{\partial \lambda}{\partial q}(t, q^{(3)}) \lambda(t, q^{(3)}) \geq 0 \). However, while \( V_y^{(3)} \leq 0, \beta \frac{\partial \eta}{\partial q}(t, q^{(3)}) - \gamma \frac{\partial \lambda}{\partial q}(t, q^{(3)}) \) does not have a known sign. Therefore due to the impact on the emission market, the optimal production of the large producer can potentially be higher than the case when there is not emission market. Based on the discussion above, the case where we can make sure \( q^{(3)} \leq q^{(1)} \) is provided in the following proposition. The above discussion, lead to the following proposition.
Proposition 3.4. Let Assumption A holds true for triple \( \bar{\pi} := \pi + \frac{\lambda^2}{2\alpha}, \eta, \lambda \) and problem (3.22) has a solution in \( C^{1,1,2}([0, T) \times \mathbb{R}_+ \times \mathbb{R}) \). If for \( q^{(3)} \) given by (3.22) we have
\[
\beta \frac{\partial \eta}{\partial q}(t, q^{(3)}) - \gamma \frac{\partial \lambda}{\partial q}(t, q^{(3)}) \geq 0,
\]
then \( q^{(3)} \leq q^{(1)} \).

Proof. The proof is straightforward from Lemmas A.13 and A.14 in the appendix. \( \square \)

The above discussion is made rigorous in the following theorem.

Theorem 3.5. Let for any \((t, e, y)\) there exists an optimal control \( q^* = q^*_{t, e, y} \in Q \) for the problem (A.1) and triple \( \bar{\pi} := \pi + \frac{\lambda^2}{2\alpha}, \eta, \lambda \) satisfies Assumption A. Then \( V_e^{(3)} \) exists and (3.23) holds true. In addition, if problem (3.22) has a solution in \( C^{1,1,2}([0, T) \times \mathbb{R}_+ \times \mathbb{R}) \), then there exists an optimal production strategy satisfying (3.23).

Proof. The proof follows the same line of argument as the proof of Theorem 3.3 with the use of Lemma A.13 and Proposition A.13. \( \square \)

Proposition 3.6. Let for any \((t, e, y)\) there exists an optimal control \( q^* = q^*_{t, e, y} \in Q \) for the problem (A.1) and problem (3.22) has a solution in \( C^{1,1,2}([0, T) \times \mathbb{R}_+ \times \mathbb{R}) \). If, in addition, \( q^{(3)} \in \text{argmax}_{q \in \mathbb{R}_+} \theta(t, y, q, V_e^{(3)}, V_y^{(3)}) \) and
\[
(3.26) \quad \beta \frac{\partial \eta}{\partial q}(t, q^{(3)}) - \gamma(t, y) \frac{\partial \lambda}{\partial q}(t, q^{(3)}) \leq 0
\]
for all \((t, y)\), then we have \( q^{(3)} \leq q^{(1)} \).

In the next section, we discuss the cases where (3.26) does not hold through numerical implementation of HJB equation (3.22). The main question is when (3.26) fails to hold, what is the probability that the total emission does not meet the target set by the regulator.

4. Numerical results

The main goal of the numerical results is to understand the behavior of the optimal strategy \( q^{(3)} \) in (3.25) and more precisely to study the case where \( q^{(3)} > q^{(1)} \). If we consider \( \pi(q) = q(1 - q), \eta(q) = \lambda(q) = q, \beta = 1, \gamma = .65 \) and at this moment \( \alpha = 1 \), then (3.22) reduces to
\[
(4.1) \quad V_t + \mu V_y + \frac{1}{2} \gamma^2 V_{yy} + \frac{1}{4q} (1 + V_e + (1 - \gamma)V_y)^2_+ = 0.
\]
Figure 1. When $\gamma = 1.5$ and $\alpha = 1$

Figure 2. When $\gamma = 0.5$ and $\alpha = 1$

Note that this example satisfies Assumption A except (iv). However by remark A.12, we can still have sufficient regularity to conclude Lemmas A.13 and A.14. Therefore, $V_{e+} = -S_t$ and optimal control is given by

$$q^{(3)} = \frac{1}{2\varrho}(1 + V_e + (1 - \gamma)V_y)_+,$$

where $\varrho = (1 - \frac{1}{2})$, and we used direct calculations to obtain

$$\max_{q \geq 0} \theta(t, y, q, V_e, V_y) = \frac{1}{4\varrho}(1 + V_e + (1 - \gamma)V_y)_+^2.$$

In order to compare the optimal strategy $q^{(3)}$ with $q^{(1)}$, we momentarily assume that $q^{(3)}$ is obtained in an interior point and use (3.24) to write

$$\pi'(q^{(3)}) = \eta'(q^{(3)}) S_t + \tau(e, y),$$

where the correction term $\tau(e, y)$ is defined by

$$\tau(e, y) = (1 - \gamma)V_y + (1 - \varrho^{-1})(1 + V_{e+} + (1 - \gamma)V_y)_+.$$

Even though the above formula is assuming that $q^{(3)}$ is obtained in an interior point, if $\tau(e, y) > 0$, one can still conclude that $q^{(3)} > q^{(1)}$. For parameters $\mu = 0.1$, $\rho = .9$ ($a = 5$) and the final time is $T = 10$, we approximated the value function, correction term $\tau$, and optimal control by a finite-difference Trotter-Kato based scheme whose details is given in Appendix B.
If Figure 1, we set $\gamma = 0.5$ which by proposition 3.4 reduces the optimal production $q(3)$ to below $q(1)$. However, in Figure 2 where $\gamma = 1.5$, we have $q(3) > q(1)$ in the region where $\tau < 0$, i.e. region with orange-to-blue color.

As a matter of fact, the position of the large producer can be distanced from region $\{\tau < 0\}$ by having enough of allowance in the portfolio at time 0. On the other hand, too much uncertainty about the process $Y$ measure with the value of $\gamma$ can eventually put the producer in distress, since the size of this region grows as $\gamma$ becomes larger. Therefore, we can avoid large producer from producing more than $q(1)$ by giving sufficient free allowances or/and reducing the uncertainty of the pollution index by frequent announcements.

**Appendix A. Uniqueness, verification and existence of optimal control**

Throughout the appendix, we assume that $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions which hosts a Brownian motion $\{W_t\}_{t \geq 0}$ and let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$. Let

$$V(t, e, y) = \sup_{q \in \mathbb{Q}_+} J_q(t, e, y)$$

(A.1)

$$J_q(t, e, y) = \mathbb{E}_{t,e,y} \left[ \int_t^T \tilde{\pi}(s, q_s) \, ds - \alpha E_{t}^{\pi} \mathbf{1}_{\{Y_{t}^{\pi} > y\}} \right],$$

where

$$dY_t^q = \left( \mu_t(Y_t^q) + \beta \eta_t(q_t) - \gamma_t(Y_t^q) \lambda_t(q_t) \right) \, dt + \gamma_t(Y_t^q) \, dW_t, \quad dE_t^q = \eta_t(q_t) \, dt$$

where $\mu, \gamma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous in $t$ and Lipschitz in $y$ with $\gamma \geq 0$.

**Remark A.1.** Notice that in the current Appendix, the reference probability measure $\mathbb{P}$ is different from the physical probability measure introduced at the beginning of Section 2. This setting helps us extend the results in this appendix to both value functions $V^{(2)}$ and $V^{(3)}$. More precisely, if we set $\tilde{\pi} = \pi$ and $\mathbb{P} = \mathbb{Q}$, then $V = V^{(2)}$. Else if $\tilde{\pi} = \pi + \frac{\lambda^2}{2\alpha}$ and $\mathbb{P} = \mathbb{Q}^q$, then $V = V^{(3)}$; here the dependency of martingale measure $\mathbb{Q}^q$ with respect to $q$ in the definition of $V^{(3)}$ is absorbed in the dynamic of $Y_t^q$.

We would like to show that $V$ can be characterized by the HJB equation

$$0 = -\partial_t V - H(t, y, \partial_y V, \partial_e V, \partial_{yy} V)$$

(A.2)

$$V(T, e, y) = -\alpha e \mathbf{1}_{\{y > 0\}},$$
where

\[ H(t, y, v_1, v_2, v_{11}) := \mu(t, y) v_1 + \frac{1}{2} \gamma^2(t, y) v_{11} + \sup_{0 \leq q} \theta(t, y, v_1, v_2, v_{11}) \]

\[ \theta(t, y, v_1, v_2, v_{11}) := \tilde{\pi}(t, q) + \eta(t, q) v_2 + (\beta \eta(t, q) - \gamma(t, y) \lambda(t, q)) v_1. \]

Because of discontinuity in terminal condition, we adopt the definition of discontinuous viscosity solutions from [19, Section 6.2] or [1, Section 4.2]. For a locally bounded measurable function \( u \), we denote by \( u^* \) and \( u_* \) the upper semicontinuous and lower semicontinuous envelopes of \( u \), respectively.

**Definition A.2.** Let \( h(e, y) \) be a locally bounded measurable function.

(a) A locally bounded measurable function \( u \), upper semicontinuous on \([0, T) \times \mathbb{R}_+ \times \mathbb{R} \), is called a viscosity subsolution of (A.2) if

(i) \( u^*(T, e, y) \leq h^*(e, y) \)

(ii) for any smooth function \( \phi \) such that \( \max(u^* - \phi) = (u^* - \phi)(t_0, e_0, y_0) \) with \( t_0 < T \), at \((t_0, e_0, y_0)\) we have

\[ -\frac{\partial \phi}{\partial t} - H_*(t, y, \phi_y, \phi_e, \phi_{yy}) \leq 0 \]

(b) A locally bounded measurable function \( u \), lower semicontinuous on \([0, T) \times \mathbb{R}_+ \times \mathbb{R} \), is called a viscosity supersolution of (A.2) if

(i) \( u_*(T', e', y') \geq h_*(e, y) \)

(ii) for any smooth function \( \phi \) such that \( \min(u_* - \phi) = (u_* - \phi)(t_0, e_0, y_0) \) with \( t_0 < T \), at \((t_0, e_0, y_0)\) we have

\[ -\frac{\partial \phi}{\partial t} - H^*(t, y, \phi_y, \phi_e, \phi_{yy}) \geq 0 \]

(c) A locally bounded measurable function \( u \), continuous on \([0, T) \times \mathbb{R}_+ \times \mathbb{R} \), is called a viscosity solution of (A.2) if it is a viscosity sub- and supersolution.

**Theorem A.3.** Let triple \((\tilde{\pi}, \eta, \lambda)\) satisfy Assumption [A.1]. Then \( V \) is the unique\(^2\) viscosity solution of (A.2) on \([0, T] \times \mathbb{R}_+ \times \mathbb{R} \).

**Proof.** By Assumption [A.1], \( H \) is locally bounded, and can be approximated by a net of continuous functions \( \{H_M\}_{M>0} \)

\[ H_M(t, y, v_1, v_2, v_{11}) := \mu(t, y) v_1 + \frac{1}{2} \gamma^2(t, y) v_{11} + \sup_{0 \leq q \leq M} \theta(t, y, v_1, v_2, v_{11}). \]

Thus, one can apply [19, Theorems 7.4 and 6.8] to obtain (a.ii) and (b.ii) in Definition A.2. To show (a.i) and (b.i), we approximate the terminal condition \( h(e, y) = -\alpha e \chi_{\{y \geq 0\}} \) by two smooth functions \(-\alpha e \rho_{e}^-(y)\) and \(-\alpha e \rho_{e}^+(y)\) from below and above respectively, i.e. \( \rho_{e}^-(y) = 1 \) on \( y \geq 0 \), \( \rho_{e}^+(y) = 0 \)

\(^2\)The specific sense of uniqueness here is discussed in Remark A.4.
on \( y \leq -\varepsilon \) and \( 0 \leq \rho_{\varepsilon}(y) \leq 1 \), and \( \overline{\rho}_{\varepsilon}(y) = 1 - \rho_{\varepsilon}(-y) \). Then by [14, Theorems 7.4 and 7.6], the value functions \( V_{\varepsilon} \) and \( \overline{V}_{\varepsilon} \) defined below are the unique continuous viscosity solutions\(^3\) of problem (A.2) with terminal condition \( V_{\varepsilon}(T, e, y) = -ae\rho_{\varepsilon}(y) \) and \( \overline{V}_{\varepsilon}(T, e, y) = -ae\overline{\rho}_{\varepsilon}(y) \), respectively.

\[
V_{\varepsilon}(t, e, y) = \sup_{q \in \mathbb{Q}} \mathbb{E}_t E_y \left[ \int_t^T \pi(t, q_t) dt - E_{Y_t}^2 \alpha \overline{\rho}_{\varepsilon}(Y_t^2) \right]
\]

On the other hand, it follows from the the optimal control problems of \( V_{\varepsilon} \), \( \overline{V}_{\varepsilon} \), and \( V \) that \( V_{\varepsilon} \leq V \leq \overline{V}_{\varepsilon} \). Therefore, by taking upper semicontinuous and lower semicontinuous envelopes from both sides and then sending \( \varepsilon \to 0 \), we obtain the desired result.

For uniqueness, first notice that \( V_{\varepsilon} - \overline{V}_{\varepsilon} \to 0 \) as \( \varepsilon \to 0 \). By standard comparison, e.g. [14, Theorem 6.21], Since any upper semicontinuous sub-solution \( u \) (lower semicontinuous supersolution \( v \)) of (A.2), is also an upper semicontinuous subsolution (a lower semicontinuous supersolution) of HJB problem for \( V_{\varepsilon} \), we have \( u \leq V_{\varepsilon} \geq \overline{V}_{\varepsilon} \). Thus, \( u - v \leq V_{\varepsilon} - \overline{V}_{\varepsilon} \) and by sending \( \varepsilon \to 0 \), we obtain uniqueness for \( t < T \). \( \square \)

**Remark A.4.** The above continuity result does not imply that \( u(T, e, 0) \leq v(T, e, 0) \). In fact, it only implies that \( u \leq v + \alpha \varepsilon \delta(y) \delta(t) \). In this case, the uniqueness may be violated along the half-line \( \{(T, e, 0) : e > 0\} \). But, we can see that it does not affect the main results of this study.

Theorem [A.3] requires minimal regularity of the value function \( V \). However to achieve the results of Section 3.1 for the large production firm, we need to show that an optimal control exists and can be expressed in terms of derivatives of \( V \). To do so, we need to impose the following assumption.

**Assumption Y.** \( \mu, \gamma : [0, T] \times \mathbb{R} \to \mathbb{R} \) are \( C^{\infty, \infty} \) and there is some positive constant \( c \), such that for all \( (t, y) \) \( \gamma(t, y) \geq c > 0 \).

**Remark A.5.** The above assumption implies that semigroup \( \{P_t\}_{t \geq 0} \) generated by \( \mathcal{L} := \frac{\partial^2}{2} \partial_{yy} + \mu \partial_y \) is \( C^{\infty, \infty}([0, T] \times \mathbb{R}) \) in the sense that for any bounded measurable function \( f \), \( P_t f \in C^{\infty, \infty}([0, T] \times \mathbb{R}) \) for all \( t > 0 \); see proof of [12, Theorem 10.1].

**Lemma A.6.** Let Assumption [A.1] holds. Then, \( \mathbb{P}(Y_T = q, y) = 0 \) for all \( q \in \mathbb{Q} \).

Proof. By Assumption [A.1](iv), there exists a \( \bar{q} > 0 \) such that for all \( (t, y) \), \( \beta \eta(t, q) - \gamma(t, y) \lambda(t, q) > 0 \) and \( \tilde{\pi}_t(q) < 0 \) for \( q \geq \bar{q} \). Therefore in problem

---

\(^3\)in class of functions with linear growth
one can restrict to all strategies \( q \in \mathcal{Q} \) bounded by \( \bar{q} \). This, in particular, implies that
\[
\frac{d\mathbb{P}}{d\mathbb{P}} = \exp \left( -\int_0^T \zeta_t dW_t - \frac{1}{2} \int_0^T \zeta_t^2 dt \right) \in L^1(\mathbb{P})
\]
where \( \zeta_t := \beta \eta_t(q_t) - \gamma_t(Y^\theta_t) \lambda_t(q_t) \). Then, one can write \( \mathbb{P}(Y^{q,y}_T = 0) = \mathbb{P}^E d\bar{\mathbb{P}}(\bar{Y}^y_{T} = 0) \] where \( \bar{Y} \) under \( \bar{\mathbb{P}} \) satisfies \( d\bar{Y} = \mu_t(\bar{Y}_t)dt + \gamma_t(\bar{Y}_t)d\bar{W}_t \), where \( \bar{W} \) is a Brownian motion under \( \bar{\mathbb{P}} \). Assumption \( \mathbb{I} \) implies that Aronson inequality holds for the density of \( \bar{Y}_T \); in particular, \( \bar{Y}_T \) has no atoms. Thus, \( \mathbb{P}(Y^{q,y}_T = 0) = \mathbb{E}^\mathbb{P}[d\bar{\mathbb{P}}(\bar{Y}^y_{T} = 0)] = 0 \). \( \square \)

**Lemma A.7.** Let Assumptions \( \mathbb{A} \) and \( \mathbb{I} \) hold. Suppose that \( v \in C^{1,0,2}([0,T) \times \mathbb{R}_+ \times \mathbb{R}) \) be such that \( v_{e+} \) exists for all \( (t,e,y) \). If \( v \) is a supersolution of (A.3), then \( v \geq V \) for all \( t < T \). In addition, if there exists \( q^* := q^*(t,e,y) \) such that
\[
0 = -\partial_t v - \frac{\gamma^2}{2} v_{yy} - \mu v_y - \pi(t,q^*) - \eta(t,q^*)v_{e+} - (\beta \eta(t,q^*) - \gamma \lambda(t,q^*))v_y
\]
\( v(T,t,e,y) = -\alpha \epsilon 1_{\{y > 0\}} \),

Then, \( V = v \).

**Proof.** For the moment, let \( v \in C^{1,1,2}([0,T) \times \mathbb{R}_+ \times \mathbb{R}) \). Then, for any \( q \in \mathcal{Q} \), Itô’s formula implies
\[
v(\theta, E^q_\theta, Y^q_\theta) = v(t,e,y) + \int_t^\theta \left( \partial_t v + \frac{\gamma^2}{2} v_{yy} + \mu v_y + \eta(t,q)v_e \\
+ (\beta \eta(t,q) - \gamma \lambda(t,q))v_y \right)(s, E^q_s, Y^q_s)ds + M_\theta - M_t
\]
where \( M \) is a continuous local martingale. Then, supersolution property of \( v \) implies that
\[
v(\theta, E^q_\theta, Y^q_\theta) \leq v(t,e,y) - \int_t^\theta \tilde{\pi}(t,q)(s, E^q_s, Y^q_s)ds + M_\theta - M_t.
\]
Let \( \{\tau_n\} \) be a sequence for \( \mathcal{M} \) in the definition of local martingale such that \( \tau_n \to \infty \). By choosing \( \theta = \tau_n \land T \), taking expectation \( \mathbb{E}_{t,e,y} \), and sending \( n \to \infty \), we obtain that
\[
v(t,e,y) \geq \mathbb{E}_{t,e,y} \left[ \int_t^\theta \tilde{\pi}(t,q)(s, E^q_s, Y^q_s)ds + v(T, E^q_T, Y^q_T) \right]
\]
\[
= \mathbb{E}_{t,e,y} \left[ \int_t^\theta \tilde{\pi}(t,q)(s, E^q_s, Y^q_s)ds - \alpha E^{q,e}_{t} 1_{\{Y^q_{\tau_n} > 0\}} \right].
\]
The equality in the above holds from Lemma \( \mathbb{A} \). If \( v \in C^{1,0,2} \), then by Krylov method of shaking coefficients \( \mathbb{K} \), proof of Theorem 2.2]
find a supersolution \( v_\varepsilon(t, e, y) \in C^{1,1,2} \) such that \( |v - v_\varepsilon| = o(\varepsilon) \). Thus, \( v_\varepsilon \geq V \) and the proof of the first part is complete after sending \( \varepsilon \to 0 \).

For the second part, one can see that all the above holds with equality, if one can show (A.3) holds with \( v_\varepsilon + \) in place of \( v_\varepsilon \). One can use a net of mollifiers \( \rho_\varepsilon \) with \( \rho_\varepsilon \) supported on \([0, \varepsilon]\); i.e. \( v_\varepsilon := v(t, \cdot, y) * \rho_\varepsilon(e) \). Then one can write (A.3) for \( v_\varepsilon \). Since \( \rho_\varepsilon \) is supported on \([0, \varepsilon]\], by sending \( \varepsilon \to 0, \partial_\varepsilon v_\varepsilon \to v_{e+} \), which completes the proof. \( \square \)

The first regularity result is covered by the following two lemmas.

**Lemma A.8.** \( V \) is convex and Lipschitz continuous in \( e \) uniformly on \((t, y) \in [0, T] \times \mathbb{R}\)

**Proof.** For \( q \in \overline{Q} \), we can write
\[
J_q(t, e, y) - J_q(t, e', y) = -\alpha(e - e') E \left[ 1_{\{Y^q_t \geq 0\}} | Y^q_t = y \right].
\]
Thus, \( |J_q(t, e, y) - J_q(t, e', y)| \leq \alpha|e - e'| \) and the inequality is uniform on \( q \in \overline{Q} \), which completes the proof. Convexity follows from the fact that \( J_q(t, e, y) \) is linear on \( e \) and the supremum of linear functions are linear. \( \square \)

The following corollary follows from the properties of convex functions and the above Lemma.

**Corollary A.9.** Right (left) partial derivatives of \( V \), i.e. \( \partial_{e+} V \) (\( \partial_{e-} V \)) exists, is non-decreasing and is right(left)-continuous and bounded in \([0, T] \times \mathbb{R}\).

**Remark A.10.** By Corollary A.9 in Definition A.2 of viscosity supersolution solution, a test function \( \phi \) satisfies \( \partial_{e-} V \leq \partial_e \phi \leq \partial_{e+} V \). Therefore, supersolution property can be written as
\[
-\frac{\partial \phi}{\partial t} - H^*(t, y, \phi_y, V_{e+}, \phi_{yy}) \geq 0
\]
provided that \( V \) is continuous for \( t < T \). For the subsolution property, the set of the test function is empty unless \( V_{e+} = V_{e-} \), i.e. \( V_e \) exists.

To establish the regularity property in \( t \) and \( y \), we present the following Lemma.

**Lemma A.11.** Let Assumption 1 hold. Then, for all \( t < T \) and \( e \in \mathbb{R}_+ \), the partial derivatives \( \partial_t V(t, e, y), \partial_y V(t, e, y) \) and \( \partial_{yy} V(t, e, y) \) exist and are continuous.

**Proof.** The opening argument in the proof of Lemma A.6 implies that in (A.2), \( H \) can be replaced by \( H_q \), i.e.
\[
0 = -\partial_t V - \mu V_y - \frac{1}{2} \gamma^2 V_{yy} - \sup_{0 \leq q \leq \bar{q}} \theta(t, y, q, V_y, V_e)
\]
\[ V(T, e, y) = -\alpha e 1_{\{y > 0\}}, \]
Notice that \( \sup_0 \leq q \leq q \theta(t, y, q, v_1, v_2) \) is bounded when \( v_1 \) is non-positive and \( v_2 \) is bounded. Thus, one can use Duhamel’s principle to write the above terminal value problem as

\[
V(t, \cdot) = P_{T-t} V(T, \cdot) + a \int_t^T P_{s-t} f(s, \cdot) ds.
\]

where

\[
f := \sup_{0 \leq q \leq q} \{ \pi(t, q) + \eta(t, q)V_e + (\beta q(t, q) - \gamma(t, y)\lambda(t, q))V_y \}
\]

By Assumption \( \mathfrak{Y} \) and Remark \( \mathfrak{A.5} \), the right hand side in the above is \( C^{1,2} \) in \( (t, y) \) for all \( e \) and \( t < T \).

**Remark A.12.** Let \( \pi(t, q) = -\rho q^2 + q, \eta(t, q) = \beta q \) and \( \lambda(t, q) = \lambda_0 + \lambda_1 q \) for \( \rho, \lambda_0, \lambda_1 > 0 \) and assume that \( \gamma(t, y) = \gamma \) is a positive constant. Then, triple \((\pi, \eta, \lambda)\) does not satisfy Assumption \( \mathfrak{A.1(iv)} \) when \( \beta < \gamma \lambda_1 \). However, one can still show that the regularity result of Lemma \( \mathfrak{A.7} \) holds by applying the change of variable \( W(t, y) := \exp(aV(t, e, y)) \) for some \( a > 0 \). Here we consider variable \( e \) as a fixed parameter. Then, \( W \) satisfies \(-\partial_t W - (\mu + \gamma \lambda_0)W_y - \frac{\gamma^2}{2} W_{yy} - gW = 0\), where

\[
g := \frac{1}{4\rho} (1 + \mu + (\beta - \gamma \lambda_1)V_y) + aV_y^2
\]

Then, appropriate choice of \( a \) makes \( g \) bounded, and the argument of Lemma \( \mathfrak{A.7} \) applies to show \( W \) is \( C^{1,2} \) in \( (t, y) \).

**Lemma A.13.** Let Assumption \( \mathfrak{Y} \) holds and triple \((\pi, \eta, \lambda)\) satisfy Assumption \( \mathfrak{A} \). Then, a Markovian optimal control \( q^*(t, e, y) \) exists and is given by

\[
q^* \in \arg\max_{q \geq 0} \pi(t, q) - \eta(t, q)V_e - (\beta q(t, q) - \gamma(t, y)\lambda(t, q))V_y.
\]

In addition, \( q^*(t, e, y) \) is continuous in \( (t, y) \in [0, T) \times \mathbb{R} \) and right-continuous in \( e \in \mathbb{R}_+ \).

**Proof.** This is a direct consequence of Lemmas \( \mathfrak{A.8}, \mathfrak{A.11} \) and \( \mathfrak{A.7} \).

**Lemma A.14.** Let triple \((\pi, \eta, \lambda)\) satisfies Assumption \( \mathfrak{A} \) and Assumption \( \mathfrak{M} \) holds true. Then,

\[
-V_{e+}(t, e, y) = S_t := aP_{t,e,y}(Y_T^{q^*} \geq 0),
\]

where \( q^*(t, e, y) \) is given by Lemma \( \mathfrak{A.12} \).

**Proof.** Suppose that \( e > e' \) and let \( q^* \) be the optimal control for problem \( \mathfrak{A.1} \) starting at \( (t, e, y) \). Then, by direct calculations one can write

\[
V(t, e, y) - V(t, e', y) \leq -(e - e') aP_{t,e,y}(Y_T^{q^*} \geq 0)
\]
Dividing both sides by \( e - e' \) and sending \( e' \to e \) yields to \( V_{e^+}(t, e, y) \leq -\alpha P_{t,e,y}(Y^q_T \geq 0) \). One can obtain the other inequality by the fact that according to Lemma 4.13, \( q^* \) is right-continuous in \( e \) and \( Y^q_T \) has no atoms. If \( q^* \) is the optimal control for problem 4.1 starting at \( (t, e', y) \), then

\[
V(t, e, y) - V(t, e', y) \geq -(e - e')\alpha P_{t,e',y}(Y^q_T \geq 0).
\]

Sending \( e' \to e \) yields to \( V_{e^+}(t, e, y) \geq -\alpha P_{t,e,y}(Y^q_T \geq 0) \). \( \square \)

Appendix B. Numerical scheme

In this section, we present details of numerical approximation of the nonlinear problem (3.11) from Section 3. It is worth mentioning that the techniques discussed here can be adjusted to a wider class of degenerate semilinear HJB equation. The first step is to discretize in time and in \( (e, y) \)-space. Let \( \Delta t := \frac{T}{N} \) be the time step and \( t^{(k)} = k\Delta t \), for \( k = 0, \ldots, N \).

We set a computational bounded domain \([0, L_e] \times [-L_y, L_y] \) for the \( (e, y) \) space domain and discretize the computational domain by an appropriately fine grid \( \{(e_i, y_j) : i = 0, \ldots, N_e \text{ and } j = -N_y, \ldots, N_y \} \) with \( e_i = i\Delta e \), \( y_j = j\Delta y \), \( \Delta e = \frac{L_e}{N_e} \) and \( \Delta y = \frac{L_y}{N_y} \). We set the discrete terminal data \( V_N(e_i, y_j) = -e_i 1_{y_j \geq 0} \). To solve (3.1) numerically, we need to consider the following: appropriate artificial boundary conditions (a.k.a. ABC) for the computational domain, stable approximation of the semi-linear terms in (3.1), and treatment of discontinuity of the terminal condition.

To set a correct ABC for \( y = \pm L_y \), we return to the optimization problem (3.20). If \( L_y \) is sufficiently large so that \( Y^q \) defined by (3.21) satisfies \( \mathbb{Q}^q(Y^q_T \geq 0 | Y^q_t = L_y) \approx 1 \) uniformly on \( q \in \mathcal{Q} \), then we can approximately set

\[
V(t, e, L_y) \approx \sup_{q \in \mathcal{Q}} \mathbb{E}^q \left[ \int_t^T (-q^2 s + q_s) ds - \alpha \left( e + \int_t^T q_s \right) \right]
= \sup_{q \in \mathcal{Q}} \mathbb{E}^q \left[ \int_t^T (-q^2 s + (1 - \alpha)q_s) ds \right] - \alpha e.
\]

The last supremum is a deterministic optimization. For the choice of parameters in Section 3, we obtain \( V(t, e, L_y) \approx -\alpha e + (1 - \alpha)^2(T - t) \). Similarly at \( y = -L_y \), we have \( \mathbb{Q}^q(Y^q_T \geq 0 | Y^q_t = -L_y) \approx 0 \) and thus our approximate ABC becomes \( V(t, e, -L_y) \approx \frac{1}{2\alpha}(T - t) \). We postpone the discussion on appropriate ABC for the \( e \)-boundaries after presenting the algorithm in (3.3) below.

In order to handle the discontinuity of terminal condition in the algorithm, Step 3 for solving the heat equation come first in the splitting method to regularize the terminal condition. Step 4 in the algorithm is motivated by the relaxation scheme of [2] and is to linearize the nonlinear term locally in time.
The splitting scheme for problem (B.1)

1: \( \hat{V}^N(T, e_i, y_j) = -\alpha e_i 1_{\{y_j \geq 0\}}. \)

2: for each \( n = N - 1, \ldots, 0 \) do

3: \( \hat{V}^{n+\frac{1}{2}}(e, y) := V(t_n, e, y) \) where \( V \) is the solution of \( \hat{V}_t + \frac{1}{2} \gamma^2 V_{yy} = 0 \) on \( [t^n, t^{n+1}] \) with boundary condition \( V(t, e, L_y) = -\alpha e + \frac{(1-\gamma)^2}{2\rho}(T - t_{n+1}) \) and \( V(t, e, -L_y) = \frac{1}{2\rho}(T - t_n) \) and terminal condition \( V(t_{n+1}, e, y) = \hat{V}^{n+1}(e, y). \)

4: \( \varphi^n(e, y) := \frac{1}{2\rho}(1 + \hat{V}^{n+\frac{1}{2}} + (1 - \gamma)V^{n+\frac{1}{2}})_+. \)

5: \( \hat{V}^n(e, y) := V(t_n, e, y) \) where \( V \) is the solution of

(B.1) \( \hat{V}_t + \mu V_y + \frac{\varphi^n}{4\rho} (1 + V + (1 - \gamma)V_y) = 0 \)

on \( [t^n, t^{n+1}] \) with boundary condition \( V(t, e, L_y) = -\alpha e + \frac{(1-\gamma)^2}{2\rho}(T - t_n) \) and \( V(t, e, -L_y) = \frac{1}{2\rho}(T - t_n) \) and (B.3), and terminal condition \( V(t_{n+1}, e, y) = \hat{V}^{n+\frac{1}{2}}(e, y). \)

6: end for

In both steps 3 and 5 in the algorithm, we use an implicit finite-difference scheme to solve the PDEs. Since ABC is known on both boundaries \( y = L_y \) and \( y = -L_y \), we use centered finite-difference to approximate the derivatives with respect to \( y \). However, to avoid the hassle of setting ABC on both \( e = L_e \) and \( e = 0 \), we approximate \( V_t(t_n, e_i, y_j) \) from one side by \( \frac{V(t_{n+1}, e_i+1, y_j) - V(t_n, e_i, y_j)}{\Delta e} \). Thus, we only need to set the ABC on \( e = 0 \). To do so in a reasonable way, notice that first order linear PDE (B.1) can easily be solved by the method of characteristics, which yields

(B.2) \( \hat{V}^n(t_n, 0, y) = \hat{V}^{n+\frac{1}{2}}(e + \psi^n(0, y), y + (1 - \gamma)\psi^n(0, y) + \mu \Delta t) - \varphi(\psi^n(0, y))^2 + \psi^n(0, y). \)

Therefore, at each interval \( [t_n, t_{n+1}] \), step 5 in the algorithm is solved by setting ABC at \( e = 0 \) by

(B.3) \( V(t, 0, y) = \hat{V}^{n+\frac{1}{2}}(e + \psi^n(0, y), y + (1 - \gamma)\psi^n(0, y) + \mu \Delta t) - \varphi(\psi^n(0, y))^2 + \psi^n(0, y). \)

Remark B.1. Estimation (B.2), which is based on the method of characteristics, can be equivalently derived from approximate dynamic programing principle for the following deterministic optimal control problem which corresponds to (B.1).

(B.4) \( \sup_{q \in Q} \int_0^T (-q g^2 + q_t) dt - \alpha E_T^q \mathbf{1}_{\{Y^q \geq 0\}}. \)
where $dE^q_t = q_t dt$ and $dY^q_t = (\mu + (1 - \gamma)q_t)dt$. The dynamic programming principle of problem \((B.4)\) over the interval \([t^{(n)}, t^{(n+1)}]\) is

\[
V(t_n, 0, y) = \sup_{q \in \mathcal{Q}} \int_{t_n}^{t_{n+1}} (-q \rho q_t^2 + q_t) dt \\
+ V(t_{n+1}, e + \int_{t_n}^{t_{n+1}} q_t dt, y + (1 - \gamma) \int_{t_n}^{t_{n+1}} q_t dt + \mu \Delta t)
\]

Observe that $\psi^n(0, y)$ is an approximation of the optimal control on interval \([t^{(n)}, t^{(n+1)}]\). Thus replacing $q_t$ by $\psi^n(0, y)$ yields \((B.2)\).

References


Arash Fahim, 1017 Academic Rd., Tallahassee, FL32306, USA, Tel: +1-850-644-0617, Fax: +1-850-644-0617
E-mail address: fahim@math.fsu.edu

Nizar Touzi, Centre de Mathmatiques Appliques, Ecole Polytechnique, UMR CNRS 7641, 91128 Palaiseau Cedex, FRANCE, Tel: +33-1-69-33-46-12, Fax: +33-1-69-33-30-11
E-mail address: nizar.touzi@polytechnique.edu