Foliations, topology and geometry of 3-manifolds: $\mathbb{R}$-covered foliations and transverse pseudo-Anosov flows

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1 Introduction

In this article we analyse the topological and geometrical consequences that foliations induce on 3-manifolds. More specifically we study the transverse structure of an $\mathbb{R}$-covered foliation in a 3-manifold, where $\mathbb{R}$-covered means that in the universal cover the leaf space of the foliation is Hausdorff. When the manifold is aspherical we prove that either there is a region in the leaves where the geometry does not change much transversely, yielding an incompressible torus in the manifold; or there is a transverse pseudo-Anosov flow which captures the directions of maximal stretch/contraction transverse to the foliation. Hence if the manifold is atoroidal and aspherical there is always a transverse pseudo-Anosov flow. As a consequence manifolds with $\mathbb{R}$-covered foliations satisfy the weak hyperbolization conjecture.

The goal of this article is to understand the geometrical/topological implications of the existence of a foliation in a 3-manifold. First we review some basic facts of foliation theory. The main villain in 3-manifold foliation theory is the Reeb component: a foliation of the solid torus where the boundary is a leaf and the interior leaves are topological planes spiralling towards the boundary leaf. Reebless foliations, that is those without Reeb components, are extremely useful in understanding the topology of 3-manifolds: fundamental work of Novikov, and later Rosenberg, Palmeira showed that leaves inject in the fundamental group level (incompressible leaves) [No], the manifold is irreducible (that is every embedded sphere bounds a ball) [Ro] and the universal cover is homeomorphic to $\mathbb{R}^3$ [Pa]. Such foliations have excellent properties and they reflect the topology of the manifold. On the other hand Gabai constructed Reebless foliations in large classes of 3-manifolds which are not Haken [Rob] and jointly with Delman used this to prove property P for alternating knots [De-Ro]. Notice that the Reebless property is crucial here, since any closed 3-manifold admits a codimension one foliation [Li, Wo], most of which are not useful for topology - for instance $S^3$ has many foliations (with Reeb components).

Our focus will be on the transverse geometric structure of a Reebless foliation. Thursto [Th8, Th9, Th10] recently showed that foliations are much better behaved in the transverse direction than was previously expected: nearby leaves stay nearby forever in many directions of the leaf. This gives a tremendous boost in understanding the global structure of foliations and it aids the understanding of the geometry of the foliation and the manifold in connection with the geometrization conjecture [Th4].

There is a natural breakup into two cases here: the lifted foliation in the universal cover is a foliation by planes (or spheres) and the leaf space of this lifted foliation is a 1-manifold which may be Hausdorff or not. In a lot of situations the question of Hausdorff/non Hausdorff turns out to play an important role and have strong consequences [Ve, Ba1, Ba2, Ba3, Fe2, Fe3, Fe4]. If the leaf space is Hausdorff then it is homeomorphic to the real numbers $\mathbb{R}$ and the foliation is said to be $\mathbb{R}$-covered [Pl2, Fe2].

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§1. Introduction

In this article we analyse $\mathbb{R}$-covered foliations in 3-manifolds - the simplest case in studying the global structure of foliations in the universal cover. Examples of this large class of foliations are: 1) fibrations over the circle; 2) foliations defined by non singular closed 1-forms; 3) stable/unstable foliations of large classes of Anosov flows in 3-manifolds [Fe2]; 4) slitherings over the circle as defined by Thurston [Th7] - roughly a slithering is a map from the universal cover of the manifold to the circle $S^1$ which is a fibration equivariant under covering translations, inducing a foliation in the manifold; 5) Uniform foliations: a foliation is uniform if any two leaves in the universal cover are a bounded distance from each other (the bound depends on the pair of leaves) - they are closely related to slitherings [Th7]; 6) Many examples $\mathbb{R}$-covered foliations not induced by slitherings [Cal2]. On the other hand, Reebless finite depth foliations [Ga1, Ga3] are not $\mathbb{R}$-covered unless the compact leaf is a fiber of $M$ over the circle [Go-Sh].

The case of fibrations is very illuminating and is a precursor of the whole idea of analysing the transverse geometry of foliations. In a seminal work Thurston proved that in the aspherical case either there is an incompressible torus transverse to the foliation or there is a suspension flow which is a pseudo-Anosov flow producing singular stable/unstable foliations [Th2, Bl-Ca]. He went on to prove that the pseudo-Anosov case yields hyperbolic manifolds, establishing a deep relationship with geometry [Th3, Th4, Th5]. We concentrate on the first step. Thurston’s result can be summarized as follows from the foliations point of view: any transversal flow to the foliation produces homeomorphisms between leaves. There may be a region in the fiber whose geometry stays bounded under the transversal flow – this produces an invariant curve and a transverse incompressible torus. The second option is that transversely there will be unbounded distortion of the geometry everywhere and this produces a transverse flow which is (pseudo) hyperbolic - a pseudo-Anosov flow.

The goal of this article is to extend this result to general $\mathbb{R}$-covered foliations:

**Main theorem** - Let $\mathcal{F}$ be a transversely oriented, $\mathbb{R}$-covered foliation in $M^3$ closed, aspherical. Then either there is a $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$ or there is a (singular) pseudo-Anosov $\Phi$ transverse to $\mathcal{F}$.

Calegari [Cal1] has independently also proved the main theorem. Many of the tools used by Calegari are similar to those used in this article and the strategy for the proof of a preliminary result follows general ideas of Thurston [Th9, Th10], which have never been written up. On the other hand this article is more complete than [Cal1] and contains full details. This work was done independently of [Cal1].

The aspherical condition is only used to rule out manifolds finitely covered by $S^2 \times S^1$, see below. As in the fibering case this shows that either there is a region where the geometry varies boundedly in the transverse direction or there are directions of maximal stretch/contraction everywhere.

Thurston produced a transverse pseudo-Anosov flow in the case that the foliation is associated to a slithering, which implies that the foliation is uniform [Th7]. General $\mathbb{R}$-covered foliations need not be uniform: an easy example is the stable foliation of an Anosov flow which is the suspension of an Anosov diffeomorphism of the torus. Thurston asked whether any $\mathbb{R}$-covered foliation in atoroidal manifolds had to be uniform. This is not true in general: recently Calegari [Cal2] has produced many examples of $\mathbb{R}$-covered, non uniform foliations in hyperbolic 3-manifolds. In the uniform situation Thurston used the existence of projectively invariant measures in the appropriate setting to produce transversal laminations to the foliation. The proof in the general case is completely different and is more topological.

We now explain the basic ideas in the proof. First of all the intrinsic geometry of the leaves plays a fundamental role in our analysis. Two manifolds can be uniformized to be spherical, euclidean or hyperbolic and to a great extent the same is true for 2-dimensional foliations. This study started with Reeb’s result [Re] on stability of compact leaves. Then there was the seminal work of Plante [Pl1] on holonomy invariant transverse measures, which was extended by Sullivan [Sul] and put in the context of spaces which are negatively curved in the large by Gromov [Gr]. As a result there is a fundamental trichotomy for general foliations of 3-manifolds:

1) There is a sphere or projective plane leaf,

2) There is a holonomy invariant transverse measure of 0 Euler characteristic, approximated by a torus (either transverse or in a leaf);

3) Leaves are uniformly Gromov negatively curved in the large.
§1. Introduction

In case 1) Reeb showed that $M$ is finitely covered by $S^2 \times S^1$ with the product foliation [Re]. In case 2) if the foliation is Reebless, then the torus in question is incompressible and the manifold is toroidal. As spherical and toroidal manifolds are in some sense rare, this implies that 3) is the generic case if $\mathcal{F}$ is Reebless. In addition if $\mathcal{F}$ is Reebless then in cases 1) and 2) the manifold $M$ can be decomposed into geometric pieces [Th3, Th4, Th5] and is well understood. More recently Candel [Can] showed that in case 3) there is a metric in the manifold which makes all leaves hyperbolic (constant Gaussian curvature equal to $-1$). Therefore case 3) is the remaining case to be analysed in the proof of the main result.

A fibration over the circle is very nice because any transverse flow which induces homeomorphisms between leaves (in $M$ or in the universal cover $\tilde{M}$). This homeomorphism was used to analyse the transversal distortion of the geometry of leaves. General foliations have holonomy, so it only makes sense to look for homeomorphisms between leaves in the universal cover. This is not possible for non $\mathbb{R}$-covered foliations, so the $\mathbb{R}$-covered property is necessary here. One of the biggest difficulties in general is the lack of a transversal flow which gives homeomorphisms between leaves in the universal cover. Any transversal flow gives local homeomorphism between subsets of leaves in $\tilde{M}$ but it is far from clear they should give global homeomorphisms. In fact there are many natural counterexamples: for instance let $\Psi$ be a geodesic flow on the unit tangent bundle of a closed hyperbolic surface $R$, so $\Psi$ is an Anosov flow [An, An-Si]. Let $\mathcal{F}$ be the (weak) stable foliation of $\Psi$ which is $\mathbb{R}$-covered and choose the transversal flow to be generated by the strong unstable foliation of $\Psi$. This transversal flow produces local homeomorphisms between leaves in $\tilde{M}$ which definitely are not global homeomorphisms [Ba1, Fe2].

But all is not lost. For foliations with hyperbolic leaves one useful strategy is to first analyse the variation of distance between leaves of $\tilde{\mathcal{F}}$ in $\tilde{M}$ to obtain relationships between ideal boundaries of leaves. Each leaf of $\tilde{\mathcal{F}}$ has a circle at infinity. Thurston [Th9, Th10] explained how to use contracting directions between different leaves to locally and then globally collate these individual circles at infinity into a single universal circle which encodes all circles at infinity. For $\mathbb{R}$-covered foliations it turns out the local stitching between different circles at infinity is in fact a homeomorphism of ideal boundaries. There are two cases: if leaves are a bounded distance from each other in $\tilde{M}$ (uniform case), this yields a quasi-isometry between leaves and hence a homeomorphism between ideal boundaries. If leaves are not a bounded distance from each other, this forces an arbitrary pair of leaves of $\tilde{\mathcal{F}}$ to contract together in a dense set of directions also producing a homeomorphism between circles at infinity of leaves of $\tilde{\mathcal{F}}$. These boundary identifications are group equivariant. The common identified circle is called the universal circle of the foliation in this setting [Th9, Th10]. Universal circles for foliations with hyperbolic leaves were introduced by Thurston recently [Th9, Th10]. In general these ideal maps between circles at infinity come from maps defined only between strict subsets of leaves of $\tilde{\mathcal{F}}$ even for $\mathbb{R}$-covered foliations.

This identification of circles at infinity can be used to produce natural maps between the entire leaves in the universal cover. Given any two leaves $F, E$ of $\tilde{\mathcal{F}}$ there is a homeomorphism $\beta$ between the circles at infinity of $F$ and $E$. If $\beta$ can be continuously extended to an isometry between $F$ and $E$, one calls $\beta$ a Möbius map. In general one can quantify how far $\beta$ is from a Möbius map: one way is to look for the best possible extension of $\beta$ to a map from $F$ to $E$ - one such tightest extension was called an earthquake map by Thurston [Th6]. Another way is to use the universal circle and check the distortion on the geometry of various ideal quadrilaterals in leaves of the foliation. In either case one possibility is that the distortion in geometry (measured via earthquakes or ideal quadrilaterals) is in some sense globally bounded. This corresponds to the notion that the geometry does not change very much in the transversal direction and yields an incompressible torus in the manifold. The other option is that the analysis of the distortion produces either a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of the fundamental group $\pi_1(M)$ of $M$ or a transverse lamination to $\mathcal{F}$ - this result was announced by Thurston in 1997 [Th9, Th10].

We analyse the second option in much more detail here. First we show that if the homeomorphisms between circles at infinity are not uniformly bounded then there is always a transversal lamination to $\mathcal{F}$, that is, in this case even if there is $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(M)$ there will be one transverse lamination which encodes regions of maximal distortion. Thurston had obtained either one or the other conclusion. We then analyse the atoroidal case in much more detail: we show there are in fact two distinct transverse
laminations which have the behavior of stable and unstable laminations. These laminations are transverse to each other and fill $M$. They intersect in an orientable 1-dimensional foliation producing a flow in the intersection of the laminations. Collapse the complementary regions of the union of the laminations to produce a flow $\Phi$ in $M$. The transverse laminations blow down to singular foliations $F^s, F^u$, which are shown to have “hyperbolic” behavior, so $\Phi$ is pseudo-Anosov. In the first option of bounded distortion of geometry we prove a rigidity result: up to topological conjugation the foliation $\mathcal{F}$ admits a transverse foliation which is a local isometry between leaves - a transversely hyperbolic foliation [Ep, Th3].

The laminations constructed here are genuine essential laminations [Ga-Ka]. Using results of Gabai and Kazez [Ga-Ka] it immediately implies the following result, also proved by Calegari [Ca1]:

**Corollary** — Suppose that $M$ aspherical supports an $R$-covered foliation $\mathcal{F}$. Then $M$ satisfies the weak hyperbolization conjecture: either there is $\mathbb{Z} \oplus \mathbb{Z} < \pi_1(M)$ or $\pi_1(M)$ is Gromov negatively curved.

We mention a potential but extremely important possible use of the results here. In the atoroidal case above the geometrization conjecture predicts that the manifold is hyperbolic [Th4]. The pseudo-Anosov flow can be used to compare the geometries of leaves of $\widetilde{\mathcal{F}}$ and this can possibly be used as a starting point to geometrize $M$. A similar approach was successful in the case of fibrations [Th5].

The flow $\Phi$ constructed here is regulating for $\mathcal{F}$. This means that every orbit of $\Phi$ intersects every leaf of $\widetilde{\mathcal{F}}$ and vice versa: there is a (topological) product picture in the universal cover. Hence $\Phi$ produces global homeomorphisms between leaves of $\widetilde{\mathcal{F}}$ - as was desired in the initial analysis of the $R$-covered case. In [Fe5] we analyse when a transverse pseudo-Anosov flow to an $R$-covered foliation can fail to be regulating. It turns out that this can only occur if $\Phi$ itself has an $R$-covered stable foliation of a flow.

In [Fe7] we use the results of this article and of [Fe5, Fe6] to prove that except in the case of $R$-covered Anosov foliations, then up to topological conjugacy there is only one transverse pseudo-Anosov flow transverse to the $R$-covered foliation. Hence our construction is in fact canonical.

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## 2 Uniform foliations, compact leaves and minimality

Throughout the article $\mathcal{F}$ will denote a 2-dimensional foliation of a closed 3-manifold $M$. The universal cover of $M$ will be denoted by $\widetilde{M}$ and $\widetilde{\mathcal{F}}$ is the lifted foliation to $\widetilde{M}$. The map

$$\pi : \widetilde{M} \to M$$

will always denote the universal covering map. The fundamental group $\pi_1(M)$ is identified with the group of covering translations of $\widetilde{M}$. For any subset $B$ of $\widetilde{M}$, let $\mathcal{I}(B)$ denote the isotropy group of $B$, which is the subgroup of $\pi_1(M)$ leaving $B$ invariant. Finally $\mathcal{H}$ is the leaf space of $\mathcal{F}$, which is a topological space.

**Definition 2.1.** ([Fe2]) $\mathcal{F}$ is $R$-covered if $\mathcal{H}$ is homeomorphic to the real numbers $R$.

A weaker property is equivalent to $R$-covered:

**Lemma 2.2.** $\mathcal{F}$ is $R$-covered if and only if $\mathcal{H}$ is Hausdorff.

**Proof.** Assume first that $\mathcal{H}$ is Hausdorff. Suppose there is a closed curve $\gamma$ in $\widetilde{M}$ which is transverse to $\widetilde{\mathcal{F}}$. Then $\gamma$ bounds an immersed disk $D$ which can be put into general position with respect to $\widetilde{\mathcal{F}}$ [Ha, No, So]. An analysis of the induced (singular) foliation in $D$ shows there are leaves $\alpha_1, \alpha_2$ so that $\alpha_1$ is a closed curve and $\alpha_2$ spirals towards $\alpha_1$. Let $F_1$ be the leaf of $\mathcal{F}$ containing $\alpha_1$. Then $F_2$ limits on $F_1$ so any neighborhood of $F_1$ contains $F_2$ and hence $\mathcal{H}$ is not Hausdorff contradiction. Hence any
transversal curve to \( \tilde{F} \) projects injectively to \( \mathcal{H} \), so \( \mathcal{H} \) is a 1-manifold with a countable base. It is also Hausdorff so it can only be the circle or the real line. In the circle case, construct a closed transversal to \( \tilde{F} \), contradiction. Hence \( F \) is \( \mathbb{R} \)-covered. The converse implication is immediate.

Here is how Reeb components interact with the \( \mathbb{R} \)-covered property: if \( F \) is \( \mathbb{R} \)-covered then \( F \) is Reebless unless \( M \) is finitely covered by \( S^2 \times S^1 \) [Go-Sh]. The restriction is necessary: glue two Reeb components along their boundaries to produce \( S^2 \times S^1 \) with an \( \mathbb{R} \)-covered foliation.

Most of the time we assume that \( F \) is transversely orientable. The non orientable case usually follows from some additional considerations.

**Definition 2.3.** (foliated I-bundle) A foliated I-bundle (in dimension 3) is a pair \((N, \mathcal{G})\), with \( N \) homeomorphic to a product \( R \times I \), where \( R \) is a surface, which may be compact or not, and may have boundary or not; and \( I \) is the unit interval. In addition \( \mathcal{G} \) is a foliation in \( N \) so that:

- \( R \times \{0\} \) and \( R \times \{1\} \) are leaves of \( \mathcal{G} \);
- \( \mathcal{G} \) is transverse to the \( I \)-fibers in \( N \) (including \( \partial R \times [0,1]\)).

Up to topological conjugacy, the foliation \( \mathcal{G} \) is completely determined by the monodromy, which is a map from \( \pi_1(R) \) to the group of homeomorphisms of \( I \). Sometimes we abuse the notation and say that \( N \) or \( \mathcal{G} \) is a foliated \( I \)-bundle.

A fundamental concept here is that of a uniform foliation:

**Definition 2.4.** ([Th7]) A foliation \( F \) is uniform if given any two leaves \( E, F \) of \( \tilde{F} \), there is a positive constant \( b \), so that the Hausdorff distance between \( E, F \) is smaller than \( b \). Explicitly, for any point \( x \) of \( E \) there is \( y \) in \( F \) so that \( d(x, y) \) is less than \( b \) and conversely. The bound depends on the pair \( E, F \).

Obviously it is not at all important that leaves be hyperbolic to define uniform foliations. In the case of \( \mathbb{R} \)-covered foliations, the existence of compact leaves implies that the foliation is uniform.

**Lemma 2.5.** (compact leaves) Let \( F \) be an \( \mathbb{R} \)-covered foliation in \( M^3 \), closed and not finitely covered by \( S^2 \times S^1 \). Then \( F \) is taut. In addition if \( F \) has a compact leaf \( R \) then \( F \) is uniform.

**Proof.** If necessary lift to a finite cover and assume that \( F \) is transversely orientable and \( M \) is orientable. If the lifted foliation is taut then so it the original one. The uniform condition concerns objects in the universal cover so the same holds concerning this property. Since \( F \) is \( \mathbb{R} \)-covered and \( M \) is not finitely covered by \( S^2 \times S^1 \) then \( F \) is Reebless as shown by Goodman and Shields [Go-Sh].

With the orientation conditions, if \( F \) is not taut then there is a codimension 0 submanifold \( V \) bounded by a union \( T_1, \ldots, T_n \) of tori so the transversal flow is say outgoing along the boundary [Go]. Hence there are no transversals connecting distinct lifts of the \( T_i \) to \( \tilde{M} \). As \( F \) is \( \mathbb{R} \)-covered any two leaves of \( \tilde{F} \) are connected by a transversal. The only possibility is that \( n = 1 \) and there is only one lift of \( T_1 \) to \( \tilde{M} \). Hence \( \pi_1(T_1) \) surjects in \( \pi_1(V) \). But if \( T_1 \) is incompressible this is impossible by theorem 10.5 of [He]. But \( F \) is Reebless so \( T_1 \) in incompressible. This contradiction shows that \( F \) is taut.

If in addition \( F \) has a compact leaf \( R \), then Goodman and Shields proved that \( R \) is the fiber of a fibration of \( M \) over the circle [Go-Sh] (this uses \( F \) transversely orientable). Cut \( M \) along \( R \) to produce a manifold \( M_1 \) homeomorphic to \( R \times I \) with an induced foliation \( F_1 \). We want to show that \( F_1 \) is a foliated \( I \)-bundle. Let \( A \) be an annulus in \( M_1 \) of the form \( \gamma \times I \) where \( \gamma \) is a simple closed curve in \( R \). Isotope \( A \) to be in general position with respect to \( F_1 \). By Euler characteristic arguments \( A \) has no singularities. Since \( F \) is taut, Gabai [Ga5] showed that \( A \) can be made transverse to \( F_1 \). Cut along \( A \) to produce a new manifold with a new foliation transverse to the vertical boundary. Continue cutting along transverse annuli and disks until obtaining a union of manifolds homeomorphic to \( D^2 \times I \), where \( D^2 \) is the closed disk. The foliation in the boundary of these balls has a tangential part \( D^2 \times \partial I \) and a transverse part \( \partial D^2 \times I \). The transverse part has no holonomy because this is a ball. Therefore (up to topological conjugacy) this is a foliation by horizontal disks. Conclusion: we can isotope \( F_1 \) so that it is transverse to the \( I \)-fibers in \( M_1 \) and hence \( F_1 \) is a foliated \( I \)-bundle. Glue back along \( R \) and lift to \( M \). As \( M_1 \) is a foliated \( I \)-bundle, it now follows that \( F \) is uniform. \( \square \)
§2. Uniform foliations, compact leaves and minimality

When there are no compact leaves we can reduce to the minimal case:

**Proposition 2.6.** (minimal case) Suppose that $\mathcal{F}$ is $\mathbf{R}$-covered, does not have compact leaves and it is not minimal. Then $\mathcal{F}$ can be collapsed to a minimal foliation: there is a foliation $\mathcal{F}'$ which is minimal and which is obtained from $\mathcal{F}$ by collapsing at most countably many foliated $I$-bundles of $\mathcal{F}$ to single leaves.

**Proof.** Suppose first that $\mathcal{F}$ is transversely orientable.

Let $Z$ be a minimal set of $\mathcal{F}$. By hypothesis $Z$ is not a compact leaf and not all of $M$. This implies that $Z$ intersects any transversal curve to $\mathcal{F}$ in a Cantor set.

Let $U$ be a component of the complement of $Z$ in $M$ and $\bar{U}$ the metric completion of $U$. The interior of $\bar{U}$ embeds in $M$ and there is an induced map in the boundary, but the boundary may be a double cover of a leaf $B_0$ of $\mathcal{F}$. In the last case the leaf $B_0$ will be isolated on both sides. Since $B_0$ is a minimal set, it follows that $B_0$ is a compact leaf. This contradicts the hypothesis. Hence $\bar{U}$ embeds in $M$.

We claim that $\bar{U}$ is a foliated $I$-bundle. By the general theory of codimension one foliations $[D]$ the set $\bar{U}$ is equal to a union of two subsets $K$ and $A$, intersecting only in their boundaries and satisfying:

- $K$ is a compact, connected, codimension 0 submanifold of $\bar{U}$, which is called the core of $\bar{U}$,
- $A$ is a union of manifolds $Q_i$ which are homeomorphic to products $R_i \times I$, where each $R_i$ is a non compact, connected surface with compact, connected boundary and $\mathcal{F}$ restricted to $Q_i$ is a foliated $I$-bundle (see also $[C-C0]$).

In addition in the induced metric the $I$-fibers in $Q_i$ are very short, so we can choose them to be the transversal flow segments of $\Phi$.

Let $\bar{U}$ be a component of $\pi^{-1}(\bar{U})$. Since $\bar{U}$ embeds in $M$ and is not all of $M$, then $\bar{U}$ is not all of $\bar{M}$. Therefore it has boundary components. Then $\bar{U}$ is closed, connected, has non empty interior and is not all of $\bar{M}$. Since the leaf space of $\bar{M}$ is $\mathbf{R}$, then $\bar{U}$ has leaf space which is a closed interval $J$. Here it is fundamental that $\mathcal{F}$ is $\mathbf{R}$-covered. Let $L_1, L_2$ be the boundary leaves of $\bar{U}$. If an element of the isotropy group of $\bar{U}$ switched $L_1$ and $L_2$ then $\mathcal{F}$ would not be transversely oriented, contradiction. It follows that the isotropy group of $\bar{U}$ is the same as the isotropy group of $L_1$ or $L_2$.

Let $\bar{K}$ be a component of $\pi^{-1}(K)$ contained in $\bar{U}$;
- $E$ be the intersection of $\bar{K}$ and $L_1$;
- $B_1$ be a component of $E$. Let $C_1 = \pi(B_1)$ which is a subset of $\partial K$.

If there is another component $B_2$ of $E$, then there is a curve $\alpha$ contained in $\partial B_1$ separating $B_1$ from $B_2$ in $L_1$. But $\alpha$ projects to a closed curve $\beta$ in $\partial K$. Recall that $\partial K$ is equal to $\partial A$ (as subsets of $\bar{U}$) and then $\beta$ is contained in an annulus $\beta \times I$ in $\partial K$. The annulus $\beta \times I$ separates $K$ from a component of $A$ in $\bar{U}$. The component $C_2$ of $\pi^{-1}(\beta \times I)$ containing $\alpha$ separates $\bar{K}$ from the component of $\bar{U} - C_2$ containing $B_2$. Therefore the lift $\bar{K}$ of $K$ does not intersect $B_2$. This is a contradiction. We conclude that $E$ is connected and therefore equal to $B_1$. As any covering translation preserving $\bar{K}$, also preserves $E$ then it also preserves $B_1$. It follows that the isotropy groups $\mathcal{I}(B_1)$ and $\mathcal{I}(K)$ are equal. Therefore $\pi_1(C_1)$ surjects in $\pi_1(K)$.

But $K$ is compact and irreducible, so theorem 10.5 of Hempel $[H]$ implies that $K$ is homeomorphic to $C_1 \times I$. As in the previous lemma isotope the foliation to be transverse to the $I$-fibers in $K$. This proves the claim that $\bar{U}$ is a foliated $I$-bundle.

Notice that this discussion shows that there is a unique minimal set $Z$. This is because we just proved that the complement of $Z$ is a union of foliated $I$-bundles with non compact bases. Any leaf in the interior of the $I$-bundles will limit in points that the boundary leaves also limit on, that is they will have limit points in $Z$. But $Z$ is a minimal set so the additional leaf is not part of a minimal set. It is fundamental in all of this discussion that $\mathcal{F}$ is $\mathbf{R}$-covered - clearly these results do not work in more generality.

So we can consider the at most countably many components of $M - Z$. There is a positive number $\epsilon$ so that any two points in $M$ which are less than $\epsilon$ apart, then their local leaves are connected by a very
small transversal arc. At most finitely components of $M - Z$ many have thickness bigger than $\epsilon$, hence in the other ones the transversal flow already produces an $I$-bundle structure. For the finitely many other ones change the original transversal flow in the core part (which is an $I$-bundle) to consist of the $I$-fibers in the particular component. This is done only in finitely many compact pieces - so we may assume the flow is smooth. Finally blow down all the complementary regions of $Z$ using the new transversal flow to produce a foliation $\mathcal{F}'$. Because the minimal set $Z$ intersects transversals to the original $\mathcal{F}$ in a Cantor set, the collapsed object is still $M$ with a foliation $\mathcal{F}'$. Also $\mathcal{F}'$ is minimal - if there is a non trivial minimal set of $\mathcal{F}'$, it would generate a complementary component of $Z$ which was not collapsed. This finishes the proof in the transversely orientable case.

Suppose now that $\mathcal{F}$ is not transversely orientable. In any case choose a transverse line field to $\mathcal{F}$. There is a double cover $M_2$ and a lift $\mathcal{F}_2$ of $\mathcal{F}$ which is transversely orientable. The cover is normal and there is an involution $f$ of $M_2$ so that $M$ is the quotient of $M_2$ by $f$. Certainly $\mathcal{F}_2$ is $\mathbb{R}$-covered and the results above work for $\mathcal{F}_2$.

Let $Z$ be a minimal set of $\mathcal{F}$ and $Z'$ be its inverse image in $M_2$. Let $Z_2$ be a minimal set contained in $Z'$. Then $Z_2$ projects to a set in $M$ which is closed and contained in $Z$, hence the projection is $Z$. Furthermore $f(Z_2)$ is also a minimal set of $\mathcal{F}_2$, hence by the above discussion $f(Z_2)$ equals $Z_2$. This now implies that $Z_2$ equals $Z'$ and then $Z$ is the unique minimal set of $\mathcal{F}$.

Let $U$ be a complementary region of $Z$ which is an $I$-bundle. One option is that it projects homeomorphically to $M$ and as above we can use $I$-fibers to collapse this to a single leaf. The other option is that it double covers a set in $M$, which may have one or two boundary components. Isotope the $I$-fibration in $\bar{U}$ so that it is invariant under $f$ — only one needs to do this in the compact pieces of the $I$-bundle. Then one can collapse the resulting $I$-fibration in $M$. If the region has only one boundary component, then it collapses to a leaf which is not transversely orientable. The resulting foliation is minimal as above. 

Notice $\mathcal{F}$ is a blow up of at most countably many leaves of $\mathcal{F}'$. To prove the main theorem, if we find a subgroup $Z \oplus Z$ in $\pi_1(M)$ we are done. If we find a pseudo-Anosov flow transverse to $\mathcal{F}'$, then it pulls back by the blow up operation to a pseudo-Anosov flow transverse to $\mathcal{F}$. Hence from now on assume that $\mathcal{F}$ is minimal if it does not have compact leaves.

3 Ideal geometry and the universal circle

In order to prove the main theorem, the remaining case is when the leaves are Gromov hyperbolic. Using Candel's theorem [Can] we assume each leaf of $\mathcal{F}$ is hyperbolic — notice the metric may vary only continuously in the transversal direction [Can]. Thurston explained how to locally stitch the circles at infinity of different leaves and then to globalize the local stitching to produce a universal circle which encodes all circles at infinity [Th9, Th10]. In the case of $\mathbb{R}$-covered foliations we show how to do the identifications so that the local maps are homeomorphisms between the circles at infinity.

Remark. In this section and the next there are no orientability conditions.

If $E,F$ are leaves of $\overline{\mathcal{F}}$, let $(E,F)$ denote the set of leaves of $\overline{\mathcal{F}}$ separating $E$ from $F$. As $\mathcal{H}$ is homeomorphic to $\mathbb{R}$, then if $E,F$ are distinct the set $(E,F)$ is homeomorphic to an interval. Let $[E,F]$ be the union of $(E,F)$ and the two leaves $E,F$. Each leaf of $\overline{\mathcal{F}}$ is isometric to the hyperbolic plane $\mathbb{H}^2$ and has an ideal circle at infinity $S^1_\infty(F)$. We now come to a key object of our study:

**Definition 3.1.** (cylinder at infinity) Let $\mathcal{F}$ be an $\mathbb{R}$-covered foliation with hyperbolic leaves. Let

$$\mathcal{A} = \bigcup_{F \in \overline{\mathcal{F}}} S^1_\infty(F),$$

which is the cylinder at infinity of $\overline{\mathcal{F}}$ — the union of all ideal circles of leaves of $\overline{\mathcal{F}}$. 

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Since $\mathcal{F}$ is $\mathbf{R}$-covered, then set wise $\mathcal{A}$ is an infinite cylinder $S^1 \times \mathbf{R}$. First of all we put a topology in $\mathcal{A}$ so that it is also homeomorphic to a cylinder.

**Notation:** To be used throughout the article: given $x \in \widehat{M}$ let $F(x)$ denote the leaf of $\widehat{\mathcal{F}}$ containing $x$. The same holds for $x$ in $\mathcal{M}$.

Each geodesic ray in $F(x)$ starting at $x$ defines a unique ideal point in $S^1_{\infty}(F(x))$ giving a homeomorphism between the unit tangent bundle of $F(x)$ at $x$ and $S^1_{\infty}(F(x))$. Let $T_1\widehat{\mathcal{F}}$ be the unit tangent bundle of $\widehat{\mathcal{F}}$. Given any $B \subset \widehat{M}$, let $\widehat{\mathcal{F}}_B$ be the union of leaves of $\widehat{\mathcal{F}}$ which intersect $B$ and

$$\mathcal{A}_B = \bigcup_{F \in \widehat{\mathcal{F}}_B} S^1_{\infty}(F).$$

This is particularly useful if $B = \mu$ is a transversal arc to $\widehat{\mathcal{F}}$. In addition if $F$ is a leaf of $\widehat{\mathcal{F}}$, then $d_F$ denotes the path distance in $F$. The term “open transversal” will be used for a transversal to $\mathcal{F}$ or $\widehat{\mathcal{F}}$ which is homeomorphic to an open interval (0, 1).

**Lemma 3.2. (topology of $\mathcal{A}$)** Let $\mu$ be an open transversal to $\widehat{\mathcal{F}}$. Then $T_1\widehat{\mathcal{F}}$ restricted to $\mu$ is homeomorphic to an open cylinder $S^1 \times (0, 1)$. This provides an identification of $A_\mu$ with an open cylinder $S^1 \times (0, 1)$ and defines a topology in $\mathcal{A}$ making it homeomorphic to a cylinder. The union $\mathcal{M} \cup \mathcal{A}$ has a natural topology making it homeomorphic to $D^2 \times \mathbf{R}$, where $D^2$ is the closed disk and $D^2 \times \{t\}$ correspond to the union $F \cup S^1_{\infty}(F)$ for $F$ a leaf of $\widehat{\mathcal{F}}$.

**Proof.** If $\mu, \mu'$ are two transversals to $\widehat{\mathcal{F}}$ so that $\widehat{\mathcal{F}}_\mu, \widehat{\mathcal{F}}_{\mu'}$ intersect, we need to show that the topology $\mathcal{T}_\mu$ induced by $\mu$ in the intersection of $\mathcal{A}_\mu$ and $\mathcal{A}_{\mu'}$ is the same as the topology $\mathcal{T}_{\mu'}$ induced by $\mu'$ in this intersection. By restricting to their intersection we can assume that the sets $\mathcal{A}_\mu, \mathcal{A}_{\mu'}$ are equal.

Since both topologies induced in $\mathcal{A}_\mu$ are first countable it suffices to consider the behavior of sequences. Consider a sequence $y_i, i \in \mathbb{N}$ converging to $y_0$ in $\mathcal{T}_\mu$. Then $y_i$ are in $S^1_{\infty}(F_i)$, for uniquely defined leaves $F_i$ which are in $\widehat{\mathcal{F}}_\mu$ (equal to $\widehat{\mathcal{F}}_{\mu'}$). Let

$$x_i = F_i \cap \mu, \quad z_i = F_i \cap \mu'.$$

Then the sequence $F_i$ converges to $F_0$ in $\mathcal{H}$ with $x_0$ in $F_0$ and $F_0$ a leaf of $\widehat{\mathcal{F}}_\mu$. Also

$$x_i \to x_0 \quad \text{in } \mu \quad \text{and} \quad z_i \to z_0 \quad \text{in } \mu'.$$

For each $i$ let $l_i, s_i$ geodesic rays in $F_i$ from $x_i, z_i$ respectively with ideal point $y_i$ in $S^1_{\infty}(F_i)$. These come from identifications of the unit tangent bundle to $\widehat{\mathcal{F}}$ at $x_i, z_i$ with $S^1_{\infty}(F_i)$ respectively. Since the sequence $y_i$ converges to $y_0$ in $\mathcal{T}_\mu$, then the directions of $l_i$ at $x_i$ converge to the direction of $l_0$ at $x_0$.

Notice $y_i$ converges to $y_0$ in $\mathcal{H}$ if and only if the directions of $s_i$ in $F_i$ converge to the direction of $s_0$ in $F_0$. We use a couple properties of the hyperbolic metric. Since $x_i$ converges to $x_0$ and $z_i$ converges to $z_0$ then $d_{F_i}(x_i, z_i)$ is bounded above for all $i$. In addition the rays $l_i, s_i$ for $i \geq 1$ define the same ideal point $y_i$ in $S^1_{\infty}(F_i)$. Hence $l_i$ and $s_i$ are asymptotic in $F_i$. These two facts imply that given any positive $\epsilon$ there is a positive $a(\epsilon)$ so that except for initial length $a(\epsilon)$, the remainder of the rays $l_i, s_i$ are within $\epsilon$ of each other in $F_i$ for each $i \geq 1$. Notice that the constants are independent of $i$ — this only uses the fact that all leaves are hyperbolic and the distance between $x_i$ and $z_i$ in $F_i$ is bounded above.

Consider any subsequence $s_{i(k)}$ so that $s_{i(k)}$ converges to a ray $v_0$ in $F_0$ or equivalently that the directions of $s_{i(k)}$ at $z_{i(k)}$ converge to the direction of $v_0$ at $z_0$. For notational simplicity assume this is the original sequence $s_i$. As $l_i$ converges to $l_0$ and $s_i$ converges to $v_0$, the above property implies that except for initial segments of length smaller than $a(\epsilon)$, the remainder of the rays $l_0, v_0$ are within $\epsilon$ of each other in $F_0$. Explicitly, if $w$ is a point in $l_0$ which is more than $a(\epsilon)$ away from $x_0$ in $F_0$, then

$$w = \lim_{i \to \infty} w_i \quad \text{with} \quad w_i \in l_i \quad \text{and} \quad d_{F_i}(x_i, w_i) > a(\epsilon).$$
Also $d_{F_i}(x_i, w_i)$ is bounded above. By the property above there are $u_i$ in $s_i$ with $d_{F_i}(w_i, u_i)$ smaller than $\epsilon$ and up to subsequence again we may assume $u_i$ converges. As $d_{F_i}(x_i, w_i)$ is bounded above, then so is $d_{F_i}(u_i, v_i)$. Therefore $w_i$ has to converge to a point $u$ in $v_0$. Then $d_{F_0}(w, u)$ is bounded above by $\epsilon$ and conversely. This implies that $l_0, v_0$ have subrays which are at most $\epsilon$ distant from each other, so it again follows from hyperbolic geometry that they are asymptotic in $F_0$. That means that $v_0$ defines the ideal point $y_0$ in $S^{1}_\infty(F_0)$. Therefore the rays $v_0, s_0$ are equal. This is equivalent to the sequence $s_i$ converging to $s_0$: the directions of $s_i$ converge to that of $s_0$. But notice that this is in fact a subsequence of the original sequence! This proves that any sequence $y_i$ converging to $y_0$ in $T_\mu$ has a subsequence which converges to $y_0$ in $T_\mu'$. This then implies that the original sequence converges to $y_0$ in $T_\mu'$ as we wanted to prove. This shows that the topology in $\mathcal{A}$ is well defined. Clearly

$$\mathcal{A} = \bigcup_{i \in \mathbb{N}} \mathcal{A}_{\mu_i},$$

with $\mu_i$ transversals intersecting more and more of the leaf space of $\mathcal{F}$. Each $\mathcal{A}_{\mu_i}$ is homeomorphic to a cylinder hence $\mathcal{A}$ is homeomorphic to $S^1 \times \mathbb{R}$.

Similar arguments show that there is a natural topology on

$$\bigcup_{x \in \mathcal{F}} (F(x) \cup S^{1}_\infty(F(x))),$$

making it homeomorphic to $D^2 \times (0, 1)$, where each leaf with its ideal circle corresponds to $D^2 \times \{t\}$. It follows that $\mathcal{M} \cup \mathcal{A}$ is naturally homeomorphic to $D^2 \times \mathbb{R}$. \hfill \Box

If $g$ is a covering translation of $\mathcal{M}$ and $L$ a leaf of $\mathcal{F}$, then $g$ maps $L$ to $g(L)$ by an isometry which extends to a homeomorphism $g^L_\infty$ between their circles at infinity. This produces a bijection $g_\infty$ from $\mathcal{A}$ to itself. Similar arguments as in the lemma above show that $g_\infty$ is a homeomorphism of $\mathcal{A}$, which will be called a covering homeomorphism. Many times we will abuse notation and write $g$ instead of $g_\infty$ for this “ideal” map. In this way $\pi_1(M)$ acts in the cylinder at infinity.

Clearly $\mathcal{A} \cong S^1 \times \mathbb{R}$ has a natural foliation by circles, which comes from the circles at infinity of leaves. This is what we call the “horizontal” foliation of $\mathcal{A}$. Natural means that this foliation is left invariant by the action of $\pi_1(M)$. In general the action of $\pi_1(M)$ on $\mathcal{A} \cong S^1 \times \mathbb{R}$ does not respect the vertical foliation by $\{x\} \times \mathbb{R}$. The main goal of this section is to produce a natural “vertical” foliation of $\mathcal{A}$ which also is associated to the geometry of the foliation. This will creat the universal circle of the foliation $\mathcal{F}$ (or $\mathcal{F}$). First recall the definition of quasi-isometries:

**Definition 3.3.** (quasi-isometry)[Th3] A quasi-isometry is a map $\varphi : (M_1, d_1) \rightarrow (M_2, d_2)$ between metric spaces so that there is positive $k$ satisfying: for any $x, y$ in $M_1$ then

$$\frac{1}{k} d_1(x, y) - k < d_2(\varphi(x), \varphi(y)) < kd_1(x, y) + k$$

and in addition there is a positive $k'$ so that for any point $z$ of $M_2$ there is $x$ of $M_1$ with $d_2(z, \varphi(x))$ smaller than $k'$. If the constant is important we say that $\varphi$ is a $k$-quasi-isometry.

First we produce the natural vertical foliation in the uniform case:

**Proposition 3.4.** (vertical foliation - uniform case) Let $\mathcal{F}$ be an uniform $\mathbb{R}$-covered foliation with hyperbolic leaves. Then given any two leaves $E, F$ of $\mathcal{F}$, there is a canonical homeomorphism between $S^{1}_\infty(E)$ and $S^{1}_\infty(F)$. This yields a universal circle which is naturally homeomorphic to any circle at infinity. There is a “vertical” foliation in $\mathcal{A}$ which is transverse to the horizontal foliation and is group invariant. The homeomorphisms between $S^{1}_\infty(E)$ and $S^{1}_\infty(F)$ are given by the holonomy of this vertical foliation.
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Proof. There is a brief proof of this result in [Th7] — for completeness we provide the details here.

Fix $E, F$ in $\overline{\mathcal{F}}$ and positive $b_1$ so that their Hausdorff distance is less than $b_1$. Define a map $\varphi : E \to F$:

$$\varphi(x) = y \text{ for some } y \text{ in } F \text{ with } d(x, y) < b_1.$$ 

The map $\varphi$ is not well defined, but it is coarsely defined. This follows from a fundamental property of $\mathbf{R}$-covered foliations: If $\mathcal{F}$ is $\mathbf{R}$-covered, then for any positive $b_2$, there is positive $b_3 = f(b_2)$ satisfying:

$$\forall \ z, w \in \overline{\mathcal{M}} \text{ with } w \in F(z), \text{ then } d(z, w) < b_2 \Rightarrow d_{F(z)}(z, w) < b_3,$$

see [Fe1]. The important thing is that $b_3$ depends only on $b_2$ and not on individual leaves or points. This property is in fact equivalent to the $\mathbf{R}$-covered property (for Reebless foliations) and does not hold in general. Hence there is positive $b_1$ so that if $x \in E$ and $y, z \in F$ with

$$d(x, y) < b_1, \ d(x, z) < b_1, \text{ then } d_{F}(y, z) < b_1 = f(2b_1).$$

We conclude that $\varphi(x)$ is well defined up to a set of diameter $b_1$ in $F$. This is what we mean by coarsely defined. We want to show that $\varphi$ is a quasi-isometry from $E$ to $F$.

For any $x, y \in E$ choose a geodesic arc from $x$ to $y$ in $E$, having length $a_1$ and let $n$ be the integer $[a_1]$ where $[\ ]$ is the greatest integer function. Then $d_E(x, y)$ is a number in the interval $[n, n + 1]$. Split $\gamma$ to produce points $x_0 = x, x_1, \dots, x_n, x_{n+1} = y$ with $d_E(x_{i-1}, x_i)$ equal to $1$ for any $i$ smaller than $n$ and $d_E(x_n, x_{n+1})$ less than $1$. Then

$$d(\varphi(x_{i-1}, \varphi(x_i))) \leq d(\varphi(x_{i-1}), x_i) + d(x_{i-1}, x_i) + d(x_i, \varphi(x_i))$$

$$\leq b_1 + 1 + b_1 = (2b_1 + 1).$$

Let $b_5 = f(2b_1 + 1)$ so if $w, z$ are in the same leaf of $\overline{\mathcal{F}}$ and $d(w, z)$ is smaller than $(2b_1 + 1)$ then $d_{F(z)}(z, w)$ is smaller than $b_5$. It follows that

$$d_F(\varphi(z), \varphi(y)) \leq (n + 1)b_5 < (d_E(x, y) + 1)b_5 = b_5d_E(x, y) + b_5.$$

This shows one side of the required inequalities for quasi-isometries. In the same way there is a map $\xi$ from $F$ to $E$ with $d(w, \xi(w))$ smaller than $b_1$ for all $w$ in $F$. Hence

$$d(w, \varphi\xi(w)) \leq 2b_1 \text{ and so } d_F(w, \varphi\xi(w)) \leq b_1 = f(2b_1)$$

for all $w$ in $F$. This shows that $\varphi$ is almost onto as required in the definition of quasi-isometry. Similarly $d_F(x, \xi\varphi(x))$ is smaller than $b_1$ for all $x$ in $E$. Given $x, y$ in $E$, let $z = \varphi(x)$, $w = \varphi(y)$. An argument as above implies that

$$d_E(\xi(z), \xi(w)) \leq b_5d_E(z, w) + b_5.$$

So

$$d_E(x, y) \leq d_E(x, \xi\varphi(x)) + d_E(\xi\varphi(x), \xi\varphi(y)) + d_E(\xi\varphi(y), y)$$

$$\leq 2b_1 + b_5d_F(\varphi(x), \varphi(y)) + b_5,$$

or

$$\frac{1}{b_5}d_E(x, y) - \left(\frac{2b_1}{b_5} + 1\right) \leq d_F(\varphi(x), \varphi(y)).$$
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We conclude that \( \varphi : E \rightarrow F \) is a quasi-isometry. Therefore it extends to a homeomorphism \( \phi \) between \( S^1_\infty(E) \) and \( S^1_\infty(F) \) [Gr, Th2]. This works for any pair of leaves \( L, G \) of \( \tilde{\mathcal{F}} \), producing corresponding maps: \( \varphi_L^G \) from \( L \) to \( G \) — a quasi-isometry; and \( \phi_L^G \) homeomorphism between \( S^1_\infty(L) \) and \( S^1_\infty(G) \).

We now produce a natural “vertical” foliation in \( \mathcal{A} \). Fix \( E \) in \( \tilde{\mathcal{F}} \). For any \( y \) in \( S^1_\infty(E) \) and any \( F \) in \( \tilde{\mathcal{F}} \), then \( \phi_E^F(y) \) is a point in \( S^1_\infty(F) \). Let

\[
\alpha_y = \bigcup_{F \in \tilde{\mathcal{F}}} \phi_E^F(y).
\]

By the above \( \alpha_y \) intersects every circle \( S^1_\infty(F) \) in a single point. We claim that \( \alpha_y \) is a continuous curve in \( \mathcal{A} \). Let \( \mu \) be a transversal to \( \tilde{\mathcal{F}} \) and \( x_i \) a sequence in \( \mu \) converging to \( x_0 \). Let \( F_i = F(x_i) \). We want to show that \( y_i = \varphi_{F_i}^E(y) \) produces a sequence converging to \( y_0 = \varphi_{F_0}^E(y) \).

Consider \( l_i \) geodesic rays in \( F_i \) starting in \( x_i \) and with ideal point \( y_i \). For simplicity assume that all \( F_i \), with \( i \) bigger than 1 are in the interval \((F_0, F_1)\). Such \( F_i \) separate \( F_0 \) from \( F_1 \) in \( \tilde{M} \). The Hausdorff distance is monotone increasing: if \([F, G] \) is a subset of \([L, H] \) in \( \mathcal{H} \), then \( d_H(F, G) \) is smaller than \( d_H(L, H) \). Therefore

\[
d_H(F_0, F_1) \text{ is bounded above by } d_H(F_0, F_1)
\]

for all \( i \). Using the arguments above and this uniform bound on \( d_H(F_0, F_1) \) this implies that that all \( \varphi_{F_0}^{F_i} \) are uniform quasi-isometries — they are all \( k \)-quasi-isometries for some fixed \( k \). The images \( \varphi_{F_0}^{F_i} (l_0) \) are uniform quasigeodesics in \( F_i \) with ideal point \( y_i \). Hence they are a bounded distance from a geodesic ray in \( F_i \) starting in \( z_i \) and with ideal point \( y_i \). Since they are uniform quasigeodesics starting in \( z_i \) which is a uniformly bounded distance from \( x_i \), then the images \( \varphi_{F_0}^{F_i} \) are a uniform bounded distance from \( l_i \) in \( F_i \).

If the sequence \( l_i \) does not converge to \( l_0 \), up to subsequence suppose that \( l_i \) converges to \( v_0 \) not equal to \( l_0 \). But \( d_H(l_i, l_0) \) is bounded above by \( a_0 \) for some globally defined \( a_0 \), where this Hausdorff distance is computed in \( \tilde{M} \). Hence \( d_H(v_0, l_0) \) is bounded above by \( a_0 \) as well. The \( \mathbf{R} \)-covered property implies that \( d_{F_0}(v_0, l_0) \) is bounded, contradicting the fact that \( v_0 \) and \( l_0 \) diverge exponentially in \( F_0 \). Therefore \( l_i \) converges to \( l_0 \).

Hence \( \alpha_y \) is a continuous curve in \( \mathcal{A} \). Consider the collection \( \{ \alpha_y \} \) where \( y \) is arbitrary in \( S^1_\infty(E) \). For any point \( z \) of \( \mathcal{A} \), \( z \) is in \( S^1_\infty(F) \) for some \( F \) of \( \tilde{\mathcal{F}} \) and \( z = \phi_E^F(y) \) for a unique \( y \) in \( S^1_\infty(E) \). Equivalently

\[
z = \alpha_y \cap S^1_\infty(F(z)) \text{ and hence } \mathcal{A} = \bigcup_{y \in S^1_\infty(E)} \alpha_y.
\]

Furthermore the sets \( \{ \alpha_y \} \) with \( y \) in \( S^1_\infty(E) \) are disjoint for distinct \( y \). Since they are continuous curves, this collection produces a vertical trivialization of \( \mathcal{A} \cong S^1 \times \mathbf{R} \). Since covering translations preserve distances and relations between distances, it is very easy to check that this foliation of \( \mathcal{A} \) is invariant under covering translations, producing a “natural” vertical foliation in \( \mathcal{A} \). This will be used to analyse how the geometry changes transverse to \( \tilde{\mathcal{F}} \). This finishes the proof of proposition 3.4. 

Before we analyse the non uniform situation we introduce contracting directions and markers.

**Definition 3.5.** (contracting direction) Let \( x \) be a point in a leaf \( L \) of \( \tilde{\mathcal{F}} \) and let \( \{ \gamma(t), t \in [0, +\infty) \} \) be a geodesic ray in \( L \) starting in \( x \) and tangent vector \( v \) at \( x \). Let \( p \) in \( S^1_\infty(L) \) be the ideal point of \( \gamma \). Then \( \gamma \) (or \( v \)) is a contracting direction if the following happens: there is a transversal \( \mu \) to \( \tilde{\mathcal{F}} \) containing \( x \) (maybe as an endpoint or maybe in the interior) so that for any leaf \( E \) of \( \mathcal{F} \) which intersects \( \mu \) the distance \( d(E, \gamma(t)) \) converges to 0 as \( t \to \infty \). In other words holonomy along \( \gamma \) (or in the \( v \) direction) contracts a neighborhood of leaves towards \( L \). Similarly define contracting directions in \( \mathcal{F} \).
Remarks: 1) Contracting directions can be defined for any foliation: it just means that nearby leaves get contracted together in that direction. Using harmonic measures Thurston [Th8] showed that contracting directions are quite common in codimension one foliations in closed manifolds (any dimension).

2) In our setting the contracting direction is really a property of the ideal point p in \( S^1_\infty(L) \) and is independent of the initial point x or the geodesic ray defining p. This is because all such rays are asymptotic, so a packet of leaves gets contracted together irrespective of the initial point or ray.

Lemma 3.6. Let \( x \) in \( L \) with a contracting direction given by a geodesic ray \( \{ \gamma(t) \} \) and \( \mu \) a transversal to \( \mathcal{F} \) contracted in the \( \gamma \) direction. For any \( E \) of \( \mathcal{F} \) intersecting \( \mu \), the contracting direction \( \gamma \) defines an ideal point \( \zeta(E) \) of \( E \) and any geodesic ray of \( E \) with ideal point \( \zeta(E) \) is contracted to \( L \). In addition for any \( F \) in \( \mathcal{F} \) there is at most one direction in \( F \) which gets contracted towards \( \gamma \).

Proof. Fix a transverse line field to \( \mathcal{F} \) and lift to \( \hat{\mathcal{F}} \). Let \( E \) in \( \hat{\mathcal{F}} \) intersecting \( \mu \). For any positive \( a_0 \) there is positive \( t_0 \) so that \( d(\gamma(t), E) \) is smaller than \( a_0 \) for \( t \) bigger than \( t_0 \). If \( a_0 \) is small, the translate of \( \gamma(t) \) for \( t \) bigger than \( t_0 \) along the transverse foliation is defined for all time (as they are very close) and is a curve with arbitrarily small geodesic curvature in \( E \). Hence the translate is a quasigeodesic in \( E \) [Th3] and it defines an ideal point in \( S^1_\infty(E) \) which is denoted by \( \zeta(E) \). Also smaller and smaller \( a_0 \) the translates have smaller and smaller geodesic curvature and become more and more geodesic. Hence a geodesic ray in \( E \) with ideal point \( \zeta(E) \) is asymptotic with the initial ray \( \gamma \) in \( E_0 \).

Finally suppose there is \( F \) in \( \hat{\mathcal{F}} \) and there are geodesic rays \( r_1, r_2 \) in \( F \) which are asymptotic to \( \gamma \). Therefore they are asymptotic to each other. But \( \mathcal{F} \) is Reebless, so there are no closed transversals to \( \mathcal{F} \). This implies when \( r_1 \) and \( r_2 \) are close in \( \hat{M} \), they have to be in the same local sheet of \( \hat{\mathcal{F}} \). This in turn implies that \( r_1, r_2 \) are also asymptotic in \( F \). Therefore they define the same direction in \( F \).

Contracting directions in \( \hat{\mathcal{F}} \) in turn produce markers in the cylinder at infinity:

Definition 3.7. (marker) Let \( L \) in \( \hat{\mathcal{F}} \) with a contracting direction given by the geodesic ray \( \gamma \) which contracts a transversal segment \( \mu \). For any \( E \) intersecting \( \mu \) let \( \zeta(E) \) be the unique ideal point of \( E \) defined in the previous lemma. The set of \( \{ \zeta(E) \} \) with \( E \) intersecting \( \mu \) is a subset of \( A \) which defines a marker in \( A \) associated to the pair \( (\gamma, \mu) \). For any \( E \) intersecting \( \mu \) we say there is a marker between \( S^1_\infty(L) \) and \( S^1_\infty(E) \) or equivalently a contracting direction between \( L \) and \( E \). Sometimes we abuse notation and say that this produces a marker between \( L \) and \( E \). Let \( \zeta \) denote the marker.

Remark: If \( \mathcal{F} \) is a non-R-covered foliation with hyperbolic leaves there is not a global cylinder at infinity. However the union of the circles at infinity associated to a transversal to \( \hat{\mathcal{F}} \) still is a cylinder and one can define markers associated to intervals of leaves in the leaf space.

Some needed properties of markers are now established. If \( \zeta \) is a marker and \( E \) in \( \hat{\mathcal{F}} \), let \( \zeta(E) \) be the intersection of \( \zeta \) and \( S^1_\infty(E) \) which is at most one point.

Lemma 3.8. If \( \alpha, \beta \) are markers in \( A \) which intersect each other, then they do not intersect transversely, that is: For any \( E \) in \( \hat{\mathcal{F}} \) with \( \alpha(E), \beta(E) \) not empty, then \( \alpha(E) = \beta(E) \).

Proof. Let \( \alpha, \beta \) be markers which intersect in a point \( p \) and let and \( E \) in \( \hat{\mathcal{F}} \) with \( \alpha(E), \beta(E) \) both non empty. There is \( L \) in \( \hat{\mathcal{F}} \) with \( p \) equal to \( \alpha(L) \) and \( \beta(L) \). Let \( r \) be a geodesic ray in \( L \) with ideal point \( p \). Let

\[ r_1, r_2 \text{ geodesic rays in } E \text{ with ideal points } \alpha(E), \beta(E) \text{ respectively.} \]

As \( p, \alpha(E) \) are in \( \alpha \) then \( r \) and \( r_1 \) are asymptotic (in \( \hat{M} \)). Similarly \( r \) and \( r_2 \) are asymptotic so \( r_1 \) and \( r_2 \) are asymptotic. As in lemma 3.6 this implies that \( r_1 \) and \( r_2 \) are asymptotic in \( E \). In other words \( \alpha(E), \beta(E) \) are equal.

Lemma 3.9. Markers are continuous curves in \( A \).
We will now consider the case of $\mathbb{R}$-covered non uniform foliations. Again the goal is to produce a natural vertical foliation in $\mathcal{A}$. Lemma 2.5 shows that $\mathcal{F}$ has no compact leaves and by proposition 2.6 we may assume that that $\mathcal{F}$ is minimal. Hence:

**Running hypothesis for the rest of the section:** $\mathcal{F}$ is a minimal, $\mathbb{R}$-covered, non uniform foliation with hyperbolic leaves.

In the uniform case if one of the leaves has a point sufficiently far from the other, then the leaves can never get too close to each other at all (for ideas on this see [Th7]). The non uniform case is completely different: any pair of leaves of $\mathcal{F}$ has many directions where they are arbitrarily close.

We learned some ideas in this section from Danny Calegari in 1998 — at that point he was studying foliations with “confined regions”. This means there are $F$ and $L$ in $\mathcal{F}$ and a half plane of $F$ which is asymptotic to $L$. His goal was to prove that if in addition $\mathcal{F}$ is minimal then it is conjugate to the stable foliation of a suspension Anosov flow. We realized that some ideas of the confined case can be used to treat the general case. The article [Cal1] has a similar treatment of the general non uniform situation.

Fix an orientation in $\mathcal{H}$. We first prove several needed properties of non uniform $\mathbb{R}$-covered foliations.

**Lemma 3.10.** No two leaves of $\mathcal{F}$ are a bounded distance from each other.

**Proof.** Suppose there are leaves $E, F$ of $\mathcal{F}$ which are a bounded distance from each other. Let $J$ be the interval $[E, F]$ of $\mathcal{H}$ and consider the union of $J$ with all its translates under covering translations. Take the component $C$ containing $J$. Assume first that $C$ is a bounded interval in $\mathcal{H}$. Then translates of $C$ are either $C$ itself or disjoint from $C$. It follows that the closure of $C$ in $\mathcal{H}$ is precisely invariant. But then the leaves of $\mathcal{F}$ corresponding to the translates of $\partial C$ project to a non trivial closed set of $\mathcal{F}$ in $M$. This is not possible by hypothesis. If $C$ is unbounded below in $\mathcal{H}$ then it has to be invariant under all of $\pi_1(M)$ and so it must be unbounded above, that is $C$ is equal to $\mathcal{H}$. But as the Hausdorff distance is monotone increasing, that implies that any two leaves in $\mathcal{F}$ are a bounded distance from each other or that $\mathcal{F}$ is uniform, contrary to assumption. This finishes the proof. □

The following proposition is crucial for our results. It states that anything bounded can be put in between two arbitrary leaves and then uses that to produce contracting directions between the leaves.

**Proposition 3.11.** (compression of the universal cover and contracting directions) Given arbitrary distinct leaves $E, F$ of $\mathcal{F}$ and $B$ a bounded set, there is a covering translate of $B$ contained between $E$ and $F$. As a consequence there is at least one contracting direction between $E$ and $F$.

**Proof.** By hypothesis $d_H(E, F)$ is infinite. For simplicity we may assume without loss of generality that $\mathcal{F}$ is transversely orientable, $F$ is in front of $E$ and choose

$$p_i \in E \text{ with } d(p_i, F) \text{ converging to infinity.}$$
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Let $B$ be a bounded set in $\overline{M}$. We are looking for a translate $h(B)$ of $B$ so that $h(B)$ is in the front of $E$ and in the back of $F$, that is, between $E$ and $F$. Choose covering translations $g_i$ with $g_i(p_i)$ converging to $p_0$ and so $g_i(E)$ converge to $E_0$ containing $p_0$. Let $L$ leaf of $\tilde{F}$ very near $p_0$ and in front of $E_0$.

We claim that we can choose a covering translate $h(B)$ contained in the front of $L$. This seemingly obvious fact is not true in general, even for Reebless foliations! For example, start with a Reeb foliated annulus $A$ and consider $A \times S^1$ with the product foliation. Then glue the two boundary tori to produce a non-taut, but Reebless foliation $\mathcal{F}$. A noncompact leaf in $A$ produces an annulus leaf of $\mathcal{F}$. Lifting to $Z$ in $\tilde{F}$, one of the complementary regions of $Z$ in $\overline{M}$ has every point a bounded distance from $Z$, and there are sets of big diameter which cannot be mapped into that component. This example also shows that for general $\mathcal{F}$, given arbitrary leaves $G, H$ of $\tilde{F}$, the fact that $G$ is in a bounded neighborhood of $H$ does not imply that $H$ is in a bounded neighborhood of $G$ - this relation is not symmetric.

To prove the claim we use that $\mathcal{F}$ is taut. Suppose there is a finite supremum $a_0$ of $d(z, L)$ for $z$ in front of $L$. Let $z$ in front of $L$ with $d(z, L)$ very near $a_0$. Any geodesic arc from $z$ to $L$ with length very close to $a_0$ is almost perpendicular to $L$. There is positive $\epsilon$ so we can choose foliated box neighborhoods of these points in $L$ with $d(z, y)$ bigger than $a_0 + \epsilon$ for any $y$ in the other side of $L$ from these foliated boxes. As $\mathcal{F}$ is taut there is a transversal from $L$ to a translate $f(L)$ in the back of $L$ and not intersecting those neighborhoods. Then $z$ is in front of $f(L)$ and $d(z, f(L))$ is greater than $a_0 + \epsilon$. Hence $f^{-1}(z)$ is in front of $L$ and $d(f^{-1}(z), L)$ is bigger than $a_0$ contradiction to assumption. This proves the claim.

As $g_i(E)$ converges to $E_0$ then for $i$ big enough $L$ is in front of $g_i(E)$ and so is $h(B)$. Then

$$d(g_i(p_i), g_i(F)) \rightarrow +\infty, \quad \text{but} \quad d(g_i(p_i), h(B)) \quad \text{is bounded.}$$

It follows that $g_i(F)$ does not intersect $h(B)$ and does not separate it from $g_i(E)$ for $i$ big enough. Since $\mathcal{F}$ is $\mathbb{R}$-covered this implies that $h(B)$ is in the front of $g_i(E)$ and in the back of $g_i(F)$, that is between $g_i(E)$ and $g_i(F)$. Hence $g_i^{-1}h(B)$ is between $E$ and $F$, see fig. ?? This proves the first statement, compression of the universal cover.

Now take $B$ intersecting $E$ and $F$ and choose $g(B)$ to be between $E$ and $F$, see fig. ??, a. This implies that $g(E), g(F)$ are between $E$ and $F$ and so

$$g([E, F]) \subset (E, F).$$

As $[E, F]$ is an interval in $\mathcal{H}$, there is $F_0$ in $(E, F)$ with $g(F_0)$ equal to $F_0$ and $g'(F)$ converging to $F_0$ when $i$ converges to infinity. Hence there is a geodesic $\gamma$ in $F_0$ with $g(\gamma) = \gamma$ and which has contracting holonomy in one side, that is $g$ contracts the interval $[F_0, F]$ to $F_0$ as $i$ converges to infinity, see fig. ??, b.

By the same arguments as above then for any two leaves $U, V$ of $\tilde{F}$ one can map the interval $[U, V]$ of $\mathcal{H}$ inside $[F_0, F]$. Since $F_0$ has a contracting direction with $F$ it produces a contracting direction between any two leaves in $[F_0, F]$. Pulling back by a covering translation, this produces a contracting direction between $U$ and $V$. This finishes the proof of the proposition.

The goal is to use these contracting directions to produce identifications of the circles at infinity.

**Lemma 3.12.** There are at least two contracting directions between any two leaves $E, F \in \tilde{F}$.

**Proof.** Otherwise there are $E_0, F_0$ in $\tilde{F}$ with only one contracting direction between them. For any $E, F$ in $\tilde{F}$ we can map $[E_0, F_0]$ inside $[E, F]$, so there is only one contracting direction between $[E, F]$ as well. Choosing leaves $E_i, F_i$ of $\tilde{F}$ with $E_i, F_i$ escaping to opposite ends of $\mathcal{H}$ and the packets $[E_i, F_i]$ increasing, it follows that there is a unique "vertical" curve $\alpha$ in $\mathcal{A}$, which contains all markers. For any leaf $F$ let $\alpha_F$ be the intersection of $\alpha$ and $F$. The action of $\pi_1(M)$ on $\mathcal{A}$ sends markers to markers, therefore any covering translation acts in $\mathcal{A}$ sending the unique vertical marker $\alpha$ to itself.
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Given a covering translation \( g \) not acting freely in \( \mathcal{H} \), there is a leaf \( F \) with \( g(F) = F \), so there is a geodesic axis \( l \) in \( F \) invariant under \( g \). Since \( g(\alpha_F) = \alpha_F \), then \( \alpha_F \) is one of the ideal points of \( l \). Let now \( f \) be any other covering translation with \( f(F) \) and \( F \) distinct. Then

\[
(fg^{-1})(f(F)) = f(F) \quad \text{and} \quad (fg^{-1})(f(l)) = f(l)
\]

so \( f(l) \) has an ideal point in \( \alpha_{f(F)} \). This implies that \( l \) and \( f(l) \) are asymptotic in \( \tilde{M} \), which is impossible since \( \pi(l) \) is a closed curve in \( M \) and there is a minimum distance between any two distinct lifts to \( \tilde{M} \). We conclude that there are at least two markers connecting any two leaves. \( \square \)

**Proposition 3.13.** (local density of markers) Given \( F \) in \( \tilde{\mathcal{F}} \), \( Y \) any open set in \( S^1_\infty(F) \), and \( A \) any component of \( A - S^1_\infty(F) \), there is a marker with an endpoint in \( Y \) and contained in the closure of \( A \).

**Proof.** This shows that there are markers on “both” sides of \( Y \) in \( A \). The key property is that \( \mathcal{F} \) is minimal. Suppose the result is not true. Let

- \( F \) be a leaf of \( \tilde{\mathcal{F}} \), \( Y \) an open set in \( S^1_\infty(F) \);
- \( A \) component of \( A - S^1_\infty(F) \), so that there is no marker in \( A \) with an endpoint in \( Y \) and contained in the closure of \( A \).

Choose points \( p_i \) in \( F \) with \( p_i \) converging to \( p \) in \( Y \). Fix a small transversal \( \mu \) to \( \tilde{\mathcal{F}} \), let \( E, L \) be the leaves of \( \mathcal{F} \) through the endpoints of \( \mu \). Since \( \mathcal{F} \) is minimal, there is a positive constant \( a_0 \) so that any point in a leaf of \( \mathcal{F} \) is within \( a_0 \) (in its leaf) of another point very near the center of \( \pi(\mu) \). Lifting to \( \tilde{M} \), there are \( q_i \) in \( F \) with \( d_{\mathcal{F}}(q_i, p_i) \) smaller than \( a_0 \) and covering translations \( g_i \) with \( g_i(q_i) \) in \( \mu \). Up to subsequence

\[
g_i(q_i) \quad \text{converges to} \quad q_0 \in \mu, \quad q_0 \in F_0 \in \tilde{\mathcal{F}}.
\]

Notice that in \( F \cup S^1_\infty(F) \), \( q_i \) also converges to \( p \) in \( Y \), hence the visual measure of \( Y \) (in \( S^1_\infty(F) \)) as measured from \( q_i \) is \( \theta_i \) with \( \theta_i \) converging to \( 2\pi \). Hence from \( g_i(q_i) \), the visual measure of \( g_i(Y) \) in \( S^1_\infty(g_i(F)) \) is \( \theta_i \) also.

By lemma 3.12 there are at least two markers

\[
\zeta_1, \zeta_2 \quad \text{from} \quad S^1_\infty(E) \to S^1_\infty(L).
\]

Use the parametrization of the circles at infinity between \( E \) and \( L \) given by \( T_i \tilde{\mathcal{F}} | \mu \). The markers \( \zeta_1, \zeta_2 \) intersect \( S^1_\infty(F_0) \) in angles \( \delta_1, \delta_2 \) as measured by this identification. Since the markers are continuous in \( A \) and disjoint, there is positive \( a_1 \) so that for any \( G \) in \( \tilde{\mathcal{F}} \) intersecting \( \mu \) the markers \( \zeta_1, \zeta_2 \) define directions in \( G \) which are at least \( a_1 \) angles apart — as measured in \( T_i \tilde{\mathcal{F}} | (G \cap \mu) \). But \( g_i(F) \) converges to \( F_0 \) as \( i \) converges to infinity and the markers in one side of \( S^1_\infty(g_i(F)) \) are restricted to have an endpoint in \( S^1_\infty(g_i(F)) - g_i(Y) \). This set has visual measure smaller than \( 2\pi - \theta_i \) which converges to zero with \( i \). These two facts contradict each other. This shows the local density of markers. \( \square \)

Markers were introduced by Thurston in [Th7]: he showed that markers are (locally) dense in \( A \) (also in the non \( \mathbf{R} \)-covered case). We will show a much stronger fact in our setting: there is a dense set of contracting directions between any two leaves of \( \tilde{\mathcal{F}} \). The markers will be the skeleton of the vertical foliation in \( A \). It is fundamental for all the analysis that markers are continuous curves in \( A \). The continuity of markers can be strengthened to a property that says markers are not too horizontal:

**Lemma 3.14.** Let \( L \) a leaf of \( \tilde{\mathcal{F}} \) and \( Z \) a closed subset of \( S^1_\infty(L) \). For any open neighborhood \( N \) of \( Z \) in \( A \), there are neighborhoods \( V \) of \( L \) in \( \mathcal{H} \) defined by transversal \( \mu \) to \( \tilde{\mathcal{F}} \) (\( \tilde{\mathcal{F}}_{\mu} = V \)) and \( W \) of \( Z \) in \( A \), so that any marker \( \zeta \) which intersects \( W \) then its intersection with \( A_{\mu} \) is contained in \( N \).
Proof. If there were a horizontal marker - that is contained in some $S_\infty^1(F)$ - it would clearly fail the lemma. This is because no matter how small a neighborhood of $S_\infty^1(F)$ in $A$, this curve still moves a fixed amount in the horizontal direction. Still this is a continuous curve. The aim is to show that markers cannot even get too close to horizontal arcs.

If the lemma is not true there are 1) a leaf $L$ in $F$, 2) a closed subset $Z$ of $S_\infty^1(L)$, and 3) an open neighborhood $N$ of $Z$ in $A$ satisfying: there are shrinking neighborhoods $V_i$ of $L$ in $H$ (that is $\cap V_i = L$) defined by transversals $\mu_i$ (that is $F_{\mu_i} = V_i$), there are shrinking open neighborhoods $W_i$ of $Z$ in $A$ (that is $\cap W_i = Z$) and markers $\zeta_i$ with

$$\zeta_i \cap W_i \neq \emptyset \quad \text{but} \quad \zeta_i \cap A_{\mu_i} \not\subset N.$$

Choose points $x_i$ in the intersection of $W_i$ and $\zeta_i$. As the $W_i$ shrink to $Z$ assume up to subsequence that $x_i$ converges to $x_0$ with $x_0$ in $Z$. There are

$$y_i \in \zeta_i \quad \text{with} \quad y_i \in A_{\mu_i} \quad \text{but not in} \quad N.$$ 

Since $A_{\mu_i}$ shrinks to $S_\infty^1(L)$, we can choose another subsequence so that $y_i$ converges to $y_0$ a point in $S_\infty^1(L)$. But $y_i$ is not in $N$, so $y_0$ is not in $Z$, hence $y_0, x_0$ are different points. For simplicity assume $y_i$ are points in $S_\infty^1(F_i)$, with $F_i$ above $L$ and only consider the part of the markers on the corresponding side of $S_\infty^1(L)$ in $A$. Since the markers are continuous curves in $A$, then up to another subsequence the markers $\zeta_i$ have to limit in at least one of the segments in $S_\infty^1(L)$ defined by $x_0$ and $y_0$. Let $B$ be this segment. If $B$ has a marker $\zeta$ on that side of $S_\infty^1(L)$ in $A$, then because the $\zeta_i$ limit on $B$, it follows that $\zeta_i$ will intersect $\zeta$ for $i$ big enough. Lemma 3.8 shows that for each such $i$, $\zeta$ and $\zeta_i$ are subpieces of a possibly bigger marker $\zeta'$. Hence for $i$ big enough the intersection of $\zeta'$ and $A_{\mu_i}$ is equal to the intersection of $\zeta_i$ and $A_{\mu_i}$. The marker $\zeta'$ is a continuous curve in $A$, transverse to the horizontal foliation, so for $i$ big enough $i$,

$$\zeta' \cap A_{\mu_i} \subset N \quad \text{which implies} \quad \zeta_i \cap A_{\mu_i} \subset N,$$

contradiction to assumption. We conclude that this is impossible.

The remaining option is that there are no markers on that side of $S_\infty^1(L)$ with endpoint in $B$. This is disallowed by the previous proposition. The proof is complete. $\square$

The following lemma says that if a sequence of markers converges to a point in a marker $\zeta$ then the whole markers also converge to $\zeta$. It is needed later for the analysis of global density of markers.

Lemma 3.15. Let $S,S'$ leaves of $F$ and $\{\zeta_i\}$ with $i \geq 0$ a sequence of markers from $S_\infty^1(S)$ to $S_\infty^1(S')$. If the intersection $a_i$ of $\zeta_i$ and $S_\infty^1(S)$ converges to $a_0$ with $i$, then the $\zeta_i$ converge to $\zeta_0$ in $A$, that is, for any $Z$ in $[S,S']$, the intersection $b_i$ of $\zeta_i$ and $S_\infty^1(Z)$ converges to $b_0$, the intersection of $\zeta_0$ and $S_\infty^1(Z)$.

Proof. Else there are $\{\zeta_i\}, Z$ as above so that $b_i$ does not converge to $b_0$. For simplicity suppose the sequence $a_i$ for $i$ bigger than 0 is nested (with $i$) in $S_\infty^1(S)$. The non transversal intersection of markers implies that the $b_i$ are also nested in $S_\infty^1(Z)$. Let $r$ be a geodesic ray in $S$ with ideal point $a_0$ and let $v$ be a geodesic ray in $S$ with ideal point $b_0$. Let $p_j$ be a sequence in $r$ converging to $a_0$. We can choose $q_j$ in $v$ with $d(q_j,p_j)$ converging to zero since $a_0$ defines a contracting direction from $S$ to $Z$. For each positive $i$ let $r_{j,i}$ (respectively $v_{j,i}$) be the ray in $S$ starting in $p_j$ with ideal point $a_i$ (respectively in $Z$ starting in $q_j$ with ideal point $b_i$). For each $j$ we can choose $i(j)$ big enough so that the directed angle in $S$ at $p_i$ between $r$ and $r_{j,i(j)}$ is $\theta_i$ and $\theta_i$ converges to 0. Directed means it is measured from $r$ to $r_{j,i(j)}$ in the side the $r_{j,i}$ accumulates on $r$ (when $i$ grows). Since the $b_i(j)$ do not converge to $b_0$, then as seen from $q_0$ the visual angle of the segment in $S_\infty^1(Z)$ from $b_0$ to $b_i(j)$ does not converge to 0. It follows that the directed angle $\beta_j$ at $q_j$ between the rays

$$v \quad \text{and} \quad v_{j,i(j)}$$
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does not converge to zero (in fact it converges to \( \pi \)). Then choose covering translations \( f_j \) so that \( f_j(p_i) \)
converges to a point \( p_0 \), hence \( f_j(q_i) \) converges to \( p_0 \) as well. At \( f_j(p_i) \) the angle between
\[
f_j(r) \quad \text{and} \quad f_j(r_{j,i,j})
\]
converges to 0, but at \( q_j \) the angle between \( f_j(v) \) and \( f_j(v_{j,i,j}) \) does not converge to zero. This shows
that least one of the markers \( f_j(\zeta) \) or \( f_j(\xi_{i,j}) \) moves a definite amount horizontally in arbitrarily small
vertical displacement. For \( j \) big enough this contradicts lemma 3.14. This finishes the proof.

**Definition 3.16.** (invariant curves) An invariant curve in \( A \) is an embedded curve intersecting each
circle at infinity exactly once and invariant under all covering homomorphisms of \( A \). An invariant
curve which is a limit of longer and longer markers is called a limit invariant curve.

For instance if \( F \) is the stable foliation of a suspension Anosov flow, form the curve of all the positive
ideal points of leaves of \( F \). This is continuous in \( A \) and invariant. This foliation is \( R \)-covered and not
uniform. The analysis of \( R \)-covered non uniform foliations will go roughly as follows:

If the set of contracting directions between a pair of leaves is not dense then one produces a limit
invariant curve \( L \) in \( A \). One can show that the leaves are asymptotic away from the invariant curve - that
is all directions but one are contracting. So in any case one obtains a dense set of contracting directions.
The strategy here is to first analyse limit invariant curves in detail in lemmas 3.17 through 3.20 and
proposition 3.21 and then use that to produce the vertical foliation in proposition 3.22.

**Lemma 3.17.** Any limit invariant curve \( L \) has no points associated to contracting directions of \( F \).

**Proof.** Suppose the limit invariant curve line \( L \) has a point \( q \) associated to a contracting direction. Then
there is a marker \( \zeta_0 \) through \( q \). By hypothesis there are markers \( \zeta_j \) which converge pointwise to \( L \).
The previous lemma shows that these markers converge pointwise to \( \zeta_0 \) in the circles at infinity that \( \zeta_0 \)
intersects. This shows that \( L \) contains the marker \( \zeta_0 \) - that is, \( L \) coincides with \( \zeta_0 \) locally. We can map
any interval \([U, U']\) of \( H \) inside this small segment, hence the whole curve \( L \) is a marker. But since \( L \) is
\( \pi_1(M) \) invariant, the argument of lemma 3.12 shows that this is impossible. This finishes the proof.

We use the transversal flow distance between points and leaves: Fix a transversal line field to \( F \)
generating a foliation \( \tau \) with lift \( \tilde{\tau} \) to \( \tilde{M} \). Given \( G \) in \( \tilde{F} \) and \( z \) in \( \tilde{M} \), consider the transversal flow line
\( \tau_z \) through \( z \). As \( F \) is Reebless \( \tau_z \) can intersect \( G \) at most once. If they do not intersect let \( d_*(z, G) \) be
infinity. Otherwise let \( d_*(z, G) \) be the length of the segment of \( \tau_z \) from \( z \) to the intersection with \( G \). If
\( L \) is an invariant curve in \( A \) and \( L \) a leaf of \( \tilde{F} \) let \( \mathcal{L}_L \) be the intersection of \( S^1_\infty(L) \) and \( L \).

**Lemma 3.18.** Let \( L \) be a limit invariant curve. Given \( L \) in \( \tilde{F} \) and a side of \( L \) in \( \tilde{M} \) there is \( G \) of \( \tilde{F} \)
in that side so that: for any half plane \( H \) of \( L \) which does not limit on \( \mathcal{L}_L \) and any escaping sequence of
points \( z_i \) in \( H \) then the limsup of \( d_*(z_i, G) \) is bounded above (depending only on \( H \) and \( G \)).

**Proof.** This is stronger than limsup \( d(z_i, G) \) being bounded, which can occur even if \( d_*(z_i, G) \) is infinite
for all \( i \) - for example if \( F \) is the stable foliation of an Anosov geodesic flow and \( \tau \) is given by the strong
unstable foliation. We do the proof for \( G \) above \( L \), the same proof applies for \( G \) below \( L \).

Roughly the proof goes as follows: if there is \( u \) in \( S^1_\infty(L) \) distinct from \( \mathcal{L}_L \) so that \( d_* \) "blows up" near
\( u \), then one can map any transversal segment to one "near \( u \)". This produces covering translations with
invariant leaves in \( \tilde{F} \) and a contracting fixed point in \( L \) - contradicting the previous lemma.

Suppose the proposition is not true. Let \( G_i \) be a sequence in \( \tilde{F} \) converging to \( L \). Given \( i \) there
is a sequence \( z_{i,j} \) in \( H \) with \( d_*(z_{i,j}, G_i) \) bigger than \( j \) and \( z_{i,j} \) escapes in \( H \) (with \( j \) growing). Using
subsequences find

\[
z_i \in H \quad \text{with} \quad d_*(z_i, G_i) > i \quad \text{and} \quad z_i \to u \in su(L), \ u \ \text{distinct from} \ \mathcal{L}_L.
\]

Fix \( v \) in \( L \). Let \( \alpha \) be \( \tau_v \) and \( \alpha_i \) the subsegments of \( \alpha \) between \( L \) and \( G_i \), whose lengths converge to 0.
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Since $F$ is minimal any leaf is dense. Given positive $a_0$ there is positive $a_1$ so that if $\beta$ is a segment of the foliation $\tau$ of length bigger than $a_0$, $w$ any point in $M$ and $W$ the leaf of $F$ through $w$ then the following happens: $W$ intersects $\beta$ in a point $w'$ which is within $a_0/4$ of the midpoint of $\beta$ (in the flow length of $\beta$) and $w'$ is at most $a_1$ distant from $w$ in the path distance of $W$.

Also there is positive $a_2$ sufficiently small, so that for any segment $\beta'$ of $\tau$ of length smaller than $a_2$, then if it is moved by holonomy so that starting point moves a distance less than $a_1$ in its leaf (of $F$) then the final segment of $\tau$ has length bounded above by $a_0/4$. Hence any segment of $\tau$ of length bounded by $a_2$ can be moved by holonomy, with initial point moved a distance less than $a_1$ within its leaf to have a point in the segment $\beta$ within $a_0/4$ of the middle point of $\beta$. Since the length of the holonomy translate is less than $a_0/4$ the final holonomy translate is entirely contained in $\beta$.

By truncating finitely many terms assume length of $a_i$ is bounded above by $a_2$. Let $\beta_i$ be segments in leaves of $\tau$ of length $a_0$ with an endpoint in $z_i$ and contained in the positive side of $L$. The property of the $z_i$‘s implies that (at least for $i$ big enough) all $\beta_i$ are in the union of leaves $S$ of $F$ contained in the interval $[L, G_i]$ of $H$. Using the previous paragraph there are covering translations $h_i$ so that $h_i(v)$ is in a leaf $h_i(L)$ of $F$ intersecting $\beta_i$ within distance $a_0/4$ of the midpoint of $\beta_i$ and path distance from $h_i(v)$ to $\beta_i$ is less than $a_1$ in $h_i(L)$. By the previous paragraph the image of $h_i(\alpha_i)$ by holonomy will map into $\beta_i$. The endpoints of $\alpha_i$ are in $L, G_i$ and the endpoints of $\beta_i$ are in $L$ and in another leaf between $L$ and $G_i$. This implies that on the level of the leaf space $h_i$ sends the interval $[L, G_i]$ of $H$ into a subset of its interior $(L, G_i) - \alpha_i$ has a fixed point in $(L, G_i)$. Then $h_i^n(L)$ converges to a leaf $L_i$ of $F$ when $n$ converges to infinity (for each $i$!) and $L_i$ is invariant under $h_i$. Notice that $L_i$ converges to $L$ as $i$ converges to infinity, because $L_i$ is in $(L, G_i)$ and $G_i$ converges to $L$ in $H$.

Since $h_i(L_i) = L_i$ then $h_i$ acts as a hyperbolic isometry in $L_i$ and has two fixed points in $S^1_{\infty}(L_i)$. Let

$$h_i^+ = \lim_{n \to +\infty} h_i^n(x)$$

for any point $x$ of $L_i$. Let $h_i^-$ be the other fixed point of $h_i$. The key fact needed here is the following:

**Lemma 3.19.** $h_i^+$ converges to $u$ in $A$ when $i$ converges to infinity.

*Proof.* Let $N$ be a neighborhood of $u \in A$ in the top side of $S^1_{\infty}(L)$ ($u$ defined at the beginning of the proof lemma 3.18). Identify $N$ to a subset of $T_1\alpha$ using the ideal circles. Then $N$ contains an open segment $T$ in $S^1_{\infty}(L)$ with $u$ in $T$. As markers are locally dense in $S^1_{\infty}(L)$, there are markers

$$\xi_1, \xi_2$$

from $S^1_{\infty}(L)$ to $S^1_{\infty}(S_1)$

with $S_1$ above $L$, intersecting $S^1_{\infty}(L)$ in $\xi_1(L), \xi_2(L)$ respectively so that: the intersections with $S^1_{\infty}(L)$ are in $T$ and define a small segment in $S^1_{\infty}(L)$ with $u$ in the interior. Let $r, r_1, r_2$ be geodesic segments in $L$ starting in $v$ and with ideal points $u, \xi_1(L), \xi_2(L)$ respectively. Notice that $r_1, r_2$ are contracting directions between $L$ and $S_1$. Let $a_3$ positive, very small. Since $r_1, r_2$ are contracting directions between $L$ and $S_1$, there is $S_2$ in $(L, S_1)$ so that any point in $r_1, r_2$ is within $a_3$ of $S_2$, and hence within $a_3$ of any $S$ between $L$ and $S_2$. For any such $S$ we can move $r_1$ and $r_2$ to $S$ using the transversal flow - if $a_3$ is sufficiently small. The geodesic curvature of the pushed curves in $S$ is small tending to zero as $a_3$ tends to zero hence they are quasigeodesics in $S$ and their initial directions give arbitrarily close estimates of the direction defined by the lifts of $r_1, r_2$ to $S$. Hence these directions are in $N$ and are close to the direction of $r$ in $T_1\alpha$ (if $N$ is small). The markers $\xi_1, \xi_2$ and the circles $S^1_{\infty}(L), S^1_{\infty}(S_2)$ define a small neighborhood $N_1$ of $u$ in $A$ in that side of $S^1_{\infty}(L)$ in $A$. We can choose $N_1$ to be a subset of $N$.

Let $v_i$ be the intersection of $\alpha$ and $L_i$. Then for $i$ big $v_i$ is in the wedge of $L$ defined by $r_1, r_2$, so the intersection $b_i$ of $\beta_i$ and $L_i$ is in the wedge defined by the images of $r_1, r_2$ in $L_i$. The direction of the geodesic segment in $L_i$ from $v_i$ to $b_i$ is within this wedge and defines a point in $N_1$ and hence in $N$. As $h_i(v_i)$ is boundedly close to $b_i$ in $L_i$, then the direction of the geodesic segment in $L_i$ from $v_i$ to $h_i(v_i)$ also defines a point in $N$ for $i$ big enough. The points $v_i$ are in the fixed transversal $\alpha$ and very close to
v, hence they are in a compact subset of \( L_i \). The points \( h_i(v_i) \) are boundedly close to \( \beta_i \) hence also from \( z_i \). As \( d_L(v, z_i) \) converges to infinity then

\[
d_L(w_i, h_i(v_i))
\]

is also converging to infinity. Since \( h_i \) is a hyperbolic isometry of \( L_i \), this now implies that \( h_i(v_i) \) is close to \( h_i^\pm \) in the compactification \( L_i \cup S^1_\infty(L_i) \). Notice this argument does not give any information about \( h_i^+ \). This shows that the direction in \( L_i \) defined by \( h_i^+ \) is in \( N \). As \( N \) is arbitrary this shows that \( h_i^+ \) converges to \( u \). This finishes the proof. 

\[ \square \]

Remarks: 1) These arguments in fact show: if there is positive \( c_0 \) and there are \( z_i \) in \( L \) converging to \( u \), \( G_i \) in \( \tilde{\mathcal{F}} \) converging to \( L \) to that \( d_r(z_i, G_i) \) bigger than \( c_0 \), then one obtains \( h_i \) in \( \pi_1(M) \) with fixed points \( h_i^+ \) in \( \mathcal{A} \) converging to \( u \).

2) Similarly if \( u \) in \( S^1_\infty(L) \) is a contracting direction on the positive side (of \( \tilde{\mathcal{F}} \)), one switches the roles of \( c_0 \) and \( \beta_i \) to get: let \( G_i \) in \( \tilde{\mathcal{F}} \) converging to \( L \) all in the domain of contraction of holonomy in the direction \( u \). Fix geodesic ray \( r \) in \( L \) with ideal point \( u \). Fix \( i \) and let \( c_0 \) be the length of \( \alpha_i \). As above there is positive but very small \( c_1 \) so that any segment of \( \tilde{\mathcal{F}} \) of length smaller than \( c_1 \) can be transported by a bounded distance holonomy to be in the interior of a covering translate of \( \alpha_i \). As \( u \) is contracting direction choose \( z_i \) in \( r \) with \( d_r(z_i, G_i) \) smaller than \( c_1 \). Let \( \beta_i \) defined as before, now with length less than \( c_1 \). This produces \( g_i \) in \( \pi_1(M) \) with \( g_i(\beta_i) \) contained in the interior of the set of leaves of \( \tilde{\mathcal{F}} \) intersected by \( \alpha_i \). The \( g_i^{-1} \) acting on \( \mathcal{A} \) have (positive) fixed points \( c_i \) which converge to \( u \) in \( \mathcal{A} \). This shows that arbitrarily near any contracting direction there are contracting fixed points of covering homeomorphisms.

Conclusion of the proof of lemma 3.18. Let \( \gamma_i \) be the geodesic in \( L_i \) which is the axis of \( h_i \) in \( L_i \) so \( h_i(\gamma_i) = \gamma_i \). The ideal points of \( \gamma_i \) are \( h_i^+, h_i^- \). Then

\[ h_i \text{ sends } [L, L_i] \text{ inside } L, L_i \]

and has no other invariant leaf in \( (L, L_i) \). Hence \( h_i \) contracts the leaf space near \( L_i \) and therefore the direction of \( \gamma_i \) associated to \( h_i^+ \) is an expanding direction for \( \tilde{\mathcal{F}} \): nearby leaves of \( \tilde{\mathcal{F}} \) diverge from \( L_i \) in this direction.

This implies that the direction of \( \gamma_i \) associated to \( h_i^- \) is a contracting direction (or equivalently \( h_i^{-1} \) expands the leaf space near \( L_i \)). But \( h_i(\mathcal{L}) = \mathcal{L} \), so one of the ideal points of \( \gamma \) is in \( \mathcal{L} \). As \( h^+ \) converges to \( u \) and \( u \) is not \( \mathcal{L}_L \) then for \( i \) big enough \( h_i^+ \) is not \( \mathcal{L}_L \). So for \( i \) big enough, \( h_i^- \) is \( \mathcal{L}_L \). But this would imply \( \mathcal{L} \) has a point \( h_i^- \) associated to a contracting direction. This contradicts lemma 3.17 and finishes the proof of lemma 3.18.

\[ \square \]

With more work we can show \( d_r(z_i, G) \) converges to 0:

**Lemma 3.20.** Suppose there is a limit invariant curve \( \mathcal{L} \). For any \( L \) in \( \tilde{\mathcal{F}} \) and a side of \( L \) in \( \tilde{\mathcal{M}} \) there is \( G \) of \( \tilde{\mathcal{F}} \) in that side so that: for any \( u \) in \( S^1_\infty(L) \) and distinct from \( \mathcal{L}_L \) and any sequence \( z_i \) in \( L \) converging to \( u \) then \( d_r(z_i, G) \) converges to 0. In particular \( u \) is a contracting direction between \( L \) and \( G \).

**Proof.** Given \( L \) and a side of it pick a \( G \) as given by lemma 3.18. Suppose the proposition is not true. Then there is \( u \) in \( S^1_\infty(L) \) distinct from \( \mathcal{L}_L \) and sequence \( z_i \) with \( d_r(z_i, G) \) not converging to 0. By lemma 3.18 the limsup of \( d_r(z_i, G) \) is bounded above by a constant \( a_4 \) which depends only on \( L, G \) and \( u \). Since \( d_r(z_i, G) \) does not converge to 0, up to subsequence assume \( d_r(z_i, G) \) converges to \( a_5 \) positive. Up to another subsequence choose \( f_i \) in \( \pi_1(M) \) with \( f_i(z_i) \) converging to a point \( z_0 \). Then \( f_i(L) \) converges to \( L_0 \) containing \( z_0 \) and \( f_i(G) \) converges to a leaf \( G_0 \) because \( d_r(z_i, G) \) converges to \( a_5 \). Here \( G_0, L_0 \) are distinct leaves because \( a_5 \) is positive. For any \( w \) in \( L_0 \), \( d_{L_0}(w, z_0) \) is finite, so \( w \) is the limit of \( w_i \) with \( w_i \) in \( f_i(L) \) and \( d_{f_i(L)}(w_i, f_i(z_i)) \) bounded (the bound depends on \( d_{L_0}(w, z_0) \)). The points \( f_i^{-1}(w_i) \) of \( L \) are a bounded distance from \( z_i \) and in particular
\[ f_i^{-1}(w_i) \to u \in S_\infty^1(L) \text{ when } i \to +\infty. \]

Therefore the limsup of \( d_r(f_i^{-1}(w_i), G) \) is less than \( a_4 \). Up to subsequence \( d_r(f_i^{-1}(w_i), G) \) converges to \( a_6 \), which is not 0 because \( d_L(f_i^{-1}(w_i), z_i) \) is bounded above and \( d_r(z_i, G) \) is bounded below by a positive constant. There are \( y_i \) in \( G \) with \( y_i, f_i^{-1}(w_i) \) in the same leaf of \( \overline{\tau} \) and \( d_r(y_i, f_i^{-1}(w_i)) \) converging to \( a_6 \). Then \( f_i(y_i) \) converges to a point \( y \) in \( \tau_w \) and \( d_r(w, y) \) is \( a_6 \). But \( f_i(y_i) \) is in \( f_i(G) \) and \( f_i(G) \) converges only to \( G_0 \) hence \( y \) is in \( G_0 \). This produces a map \( \varphi \) from \( L_0 \) to \( G_0 \) given by \( \varphi(w) = y \). Notice that for any \( w \) in \( L_0 \), the \( w, \varphi(w) \) are in the same leaf of \( \overline{\tau} \) and \( d_r(w, \varphi(w)) \) is less than \( a_4 \).

The map \( \varphi \) from \( L_0 \) to \( G_0 \) is injective because \( \mathcal{F} \) is Reebless and hence it is a homeomorphism onto its image. If \( \varphi(L_0) \) is not all of \( G_0 \) then there is \( b \) in \( G_0 \) with \( b \) in the boundary of \( \varphi(L_0) \) (as a subset of \( G_0 \)). Choose \( s_j \) in \( \varphi(L_0) \) converging to \( b \). Let \( x_j \) in \( L_0 \) with \( \varphi(x_j) = s_j \). Then

\[ d(x_j, s_j) \leq d_r(x_j, s_j) \leq a_4 \]

hence \( d(x_j, b) \) is bounded and so \( d(x_j, x_1) \). As \( \mathcal{F} \) is \( \mathbb{R} \)-covered this implies that \( d_{L_0}(x_j, x_1) \) is bounded too (this is a key point!). Up to subsequence assume that \( x_j \) converges to \( x_0 \). Then \( s_j = \varphi(x_j) \) converges to \( \varphi(x_0) \) - a point in \( G_0 \). But \( b \) is equal to \( \varphi(x_0) \) and is in \( \varphi(L_0) \) contradicting the hypothesis.

We conclude that \( \varphi \) is surjective and in fact for every point \( s \) in \( G_0 \), \( d_r(s, L_0) \) is less than \( a_4 \). Using the fact that \( \mathcal{F} \) is minimal this quickly shows that any two leaves of \( \overline{\mathcal{F}} \) are a bounded distance from each other, contradicting the non-uniform hypothesis. This finishes the proof of lemma 3.20.

Since \( L \) in \( \mathcal{H} \) and \( u \) in \( S_\infty^1(L) - L \) are arbitrary, lemma 3.20 shows that every point \( u \) of \( A - L \) is an interior point of some marker \( \zeta \) in \( A \). If two markers intersect then their union is a marker. This produces a 1-dimensional foliation \( \mathcal{N} \) in \( A - L \) consisting of the collection of all markers. The goal is to show that any leaf of \( \mathcal{N} \) intersects all circles at infinity.

**Proposition 3.21.** Suppose there is a limit invariant curve \( \mathcal{L} \). For any \( E, F \) in \( \overline{\mathcal{F}} \) and any \( v \) in \( S_\infty^1(E) \) distinct from \( \mathcal{L}_E \) then \( v \) is a contracting direction with \( F \) - that is every direction but those in \( \mathcal{L} \) are contracting direction between arbitrary leaves.

**Proof.** Let \( v \) in \( A - \mathcal{L} \) be a fixed point of a covering homeomorphism \( f; \zeta \) the leaf of \( \mathcal{N} \) through \( v \) and \( R_0 \) in \( \overline{\mathcal{F}} \) with \( v \) in \( S_\infty^1(R_0) \). For simplicity assume that \( \zeta \) misses some \( S_\infty^1(R) \) with \( R \) above \( R_0 \). Let \( R_1 \) above \( R_0 \) the smallest such that \( \zeta \) misses - the set of \( R \) such that \( \zeta \) intersects \( S_\infty^1(R) \) is open in \( \mathcal{H} \) because every point in \( A - \mathcal{L} \) is in the interior of a marker. Then

\[ f(\zeta) = \zeta \quad \text{implies} \quad f(R_1) = R_1 \]

because \( \mathcal{F} \) is \( \mathbb{R} \)-covered. Also for any \( R \) in the interval \( (R_0, R_1) \) of \( \mathcal{H} \) one has \( f(R), R \) distinct, because all such \( R \) are asymptotic to \( R_0 \) in the \( v \) direction and cannot be left invariant by covering translations associated to that direction. If needed switch \( f, f^{-1} \) so that \( v \) is the attracting fixed point of \( f \) in \( S_\infty^1(R_0) \). Then \( R_0 \) is an expanding point for the action of \( f \) in \([R_0, R_1] \), see fig. ??, a. The action of \( f \) on the closed interval

\[ B = \mathcal{L} \cap (\cup_{R \in [R_0, R_1]} S_\infty^1(R)) \]

has an expanding fixed point in \( \mathcal{L}_{R_0} \). As \( f \) has no invariant leaf in \( (R_0, R_1) \), the action of \( f \) on \( B \) has \( \mathcal{L}_{R_1} \) as an attracting point, see fig. ??, a. As \( f(R_1) = R_1 \) this shows that \( \mathcal{L}_{R_1} \) corresponds to a contracting direction in \( R_1 \), contradicting lemma 3.17. We conclude that \( \zeta \) intersects all circles at infinity.

If a leaf \( \nu \) of \( \mathcal{N} \) intersecting \( S_\infty^1(S_0) \) does not intersect all circles at infinity; assume there is (say) a top limit \( S_\infty^1(S_1) \). Then \( \nu \) limits to \( \mathcal{L} \) near \( S_\infty^1(S_1) \), see fig. ??, b. It follows that \( \nu, S_\infty^1(S_0) \) and \( \mathcal{L} \) bound a region \( Z \) which does not intersect \( S_\infty^1(S_1) \), see fig. ??, b. Any marker intersecting \( Z \) is bounded above. Now just choose \( u \) in \( Z \) which is a fixed point of some covering translation – we showed before that any
contracting direction is the limit of fixed points of covering translations. Let $v'$ be the leaf of $N$ through $v'$. As $v'$ is in $Z$ then $v'$ is bounded above, which was previously disallowed.

The conclusion is that for any marker $\zeta$ in $\mathcal{A}$, then $\zeta$ intersects all $S^1_{\infty}(R)$. In particular given $E,F$ in $\tilde{\mathcal{F}}$ and $u \in S^1_{\infty}(E) - L_E$, then the marker $\zeta$ through $u$ intersects $S^1_{\infty}(F)$ - that is $u$ is a contracting direction between $E$ and $F$. This finishes the proof of proposition 3.21.

Using these results, we can now finish the analysis of the non uniform case:

**Proposition 3.22.** (vertical foliation - non uniform case) Let $\mathcal{F}$ be a minimal, non-uniform $\mathbf{R}$-covered foliation with hyperbolic leaves. Given any $F,E$ of $\tilde{\mathcal{F}}$, there is a dense set of directions in $F$ contracting towards $E$. The set of markers extends to a natural vertical foliation in $\mathcal{A}$ which is group invariant.

**Proof.** The argument goes like this: If markers are not dense, zoom in towards an interior point of the markerless set. This pushes markers to the opposite end and defines an ideal point of the leaf - all markers between sufficiently spaced leaves of $\tilde{\mathcal{F}}$ have to pass near this point. This collection of points at infinity produces a curve $\mathcal{L}$ in $\mathcal{A}$ which is a limit invariant curve. Then appeal to the previous proposition.

Suppose the proposition is not true. Then there are $F$ in $\tilde{\mathcal{F}}$ and $E$ in $\tilde{\mathcal{F}}$ (say above $E$) and not a dense set of contracting directions from $F$ to $E$ - this is from the point of view of $F$! Hence there is an open interval $J_0$ in $S^1_{\infty}(F)$ so that no point in $J_0$ corresponds to a contracting direction from $F$ to $E$. Let $q_0$ in $J_0$. Since $q_0$ is not a contracting direction between $F$ and $E$, there is positive $\epsilon$ and $p_i$ in $F$ converging to $q_0$ along a geodesic ray $l$ and so that $d(p_i,E)$ is bigger than $\epsilon$.

In the leaf $F$, the visual measure of $J_0$ from the point of view of $p_i$ is $\theta_i$ with $\theta_i$ converging to $2\pi$ as $i$ converges to infinity. Up to a subsequence of $p_i$ choose covering translations $g_i$ with $g_i(p_i)$ converging to $p_0$ and $g_i(l)$ converging to a geodesic ray $l_0$. Let $F_0$ in $\tilde{\mathcal{F}}$ containing $p_0$. Let

$$\mathcal{O} = \{ G \in \tilde{\mathcal{F}} \mid G = f(F_0), \text{ for some } f \in \pi_1(M) \} \subset \mathcal{H}.$$

We will define a function $\eta$ from $\mathcal{O}$ to $\mathcal{A}$ which picks out the “limit” marker direction and will produce a limit invariant curve. Since $d(g_i(E), g_i(p))$ is bigger than $\epsilon$ then $g_i(E)$ does not have any subsequence converging to $F_0$. A marker from $g_i(F)$ to $g_i(E)$ must start in the set

$$U_i = S^1_{\infty}(g_i(F)) - g_i(J_0).$$

From the point of view of $g_i(p)$ in $g_i(F)$, the visual measure of $U_i$ is $2\pi - \theta_i$ which converges to 0. Also visually from $g_i(p)$ the set $U_i$ is very close to the direction of the segment of $g_i(l)$ from $g_i(p_1)$ to $g_i(p_i)$. Because the directions of $g_i(l)$ converge to that of $l_0$ and the topology of $\mathcal{A}$ is given by the visual topology from transversals to $\tilde{\mathcal{F}}$, it follows that the segments $U_i$ converge to a unique point in $S^1_{\infty}(F_0)$.

**Definition 3.23.** (function $\eta$) Define $\eta : \mathcal{O} \to \mathcal{A}$ by

$$\eta(F_0) = \lim_{i \to \infty} U_i = \lim_{i \to \infty} (S^1_{\infty}(g_i(F)) - g_i(J_0))$$

and for any covering translation $f$ define $\eta(f(F_0)) = f(\eta(F_0))$.

The leaves $F, E, F_0$ of $\tilde{\mathcal{F}}$ as well as the covering translations $g_i$ will be fixed in this proof.

**Lemma 3.24.** The function $\eta$ from $\mathcal{O}$ to $\mathcal{A}$ extends to an embedding $\eta : \mathcal{H} \cong \mathbf{R} \to \mathcal{A}$.

**Proof.** Roughly the idea is: if the lemma is not true we produce spaced enough leaves $A, B$ in $\tilde{\mathcal{F}}$ with no markers between $S^1_{\infty}(A)$ and $S^1_{\infty}(B)$, contradiction.

Suppose the lemma is not true. There is $L$ in $\tilde{\mathcal{F}}$ and two sequences $L_j, H_j$ converging to $L$ with $\eta(L_j)$ converging to $a$, $\eta(H_j)$ converging to $b$, with $a, b$ distinct points in $S^1_{\infty}(L)$. The $L_j, H_j$ are covering translates of $F_0$:

$$L_j = f_j(F_0), \quad H_j = h_j(F_0) \quad \text{for chosen } f_j, h_j \in \pi_1(M).$$
Choose small disjoint open neighborhoods $V_a, V_b$ of $a, b$ respectively in $A$. Lemma 3.14 shows that there are small disjoint open neighborhoods $V'_a, V'_b$ of $a, b$ respectively in $A$, and a small neighborhood $Y$ of $L$ in $H$ defined by transversal $\mu$, so that any marker through $V'_a$ and contained in $A_\mu$ is contained in $V_a$ and similarly any marker intersecting $V'_b$ is contained in $V_b$. In particular the two sets of markers contained in $A_\mu$ through $V'_a$ and $V'_b$ are disjoint from each other. Lemma 3.8 implies that any marker in $A$ through $V'_a$ does not intersect $V'_b$.

Fix $j$ big enough so that $\eta(L_j)$ is in $V'_a$ and $\eta(H_j)$ is in $V'_b$. As $L_j = f_j(F_0)$ then

$$\eta(L_j) = f_j(\eta(F_0)) = f_j(\lim_{i \to \infty} U_i) = \lim_{i \to \infty} f_j(U_i) \in V'_a.$$

Similarly $\eta(H_j)$ is the limit of $h_j(U_i)$ with $i$ converging to infinity. Now fix $i$ big enough so that $f_j(U_i)$ is contained in $V'_a$ and $h_j(U_i)$ is contained in $V'_b$. By the property of $V'_a$ and $V'_b$, this implies that any marker through $f_j(U_i)$ is disjoint from a marker through $h_j(U_i)$, see fig. ??.

Choose $A$ in $\tilde{F}$ with $A$ less than $L$ in the linear ordering of $H$. By taking $j, i$ big enough we can assume that $L_j, H_j$ are bigger than $A$ and so are

$$f_j(g_k(F)), \quad h_j(g_k(F)).$$

Also choose $B$ in $\tilde{F}$ with $B$ bigger than both $f_j(g_k(E))$ and $h_j(g_k(E))$ in $H$. A marker from $S^1_\infty(A)$ to $S^1_\infty(B)$ has to pass through $S^1_\infty(f_j(g_k(F)))$ and through $S^1_\infty(h_j(g_k(E)))$, since the leaves $f_j(g_k(F))$ and $h_j(g_k(E))$ separate $A$ from $B$. By the property of $U_i$, it follows that the marker has to pass through $f_j(U_i)$ contained in $V'_a$. Similarly any such marker has to pass through $S^1_\infty(h_j(g_k(F)))$ and $S^1_\infty(h_j(g_k(E)))$ hence it has to pass through $h_j(U_i)$ contained in $V'_b$. But we just showed no marker can pass through both $V'_a$ and $V'_b$. This would imply there is no marker from $S^1_\infty(A)$ to $S^1_\infty(B)$ which contradicts proposition 3.11. These arguments show that $\eta$ can be extended to a continuous function from the closure of $O$ to $A$. But $\tilde{F}$ is a minimal foliation so $O$ is dense in $H$ so there is a continuous extension $\eta : H \cong \mathbb{R} \to A$. The image is a curve $L$ which is transverse to the horizontal foliation and intersects every circle at infinity. This finishes the proof of the lemma.

Conclusion of the proof of proposition 3.22.

The set

$$\{\eta(f(F_0))\} \quad f \in \pi_1(M)$$

is an equivariant subset of $A$. By the previous lemma the curve $L$ is left invariant by every covering homeomorphism - it is an invariant curve. Also given any covering translation $g$ with an invariant leaf $L$, $g(L) = L$, then one of the fixed points of $g$ in $S^1_\infty(L)$ is in $L$.

Let $L$ in $O$, $L = g(F_0)$. Then $L$ is the limit of $g(g_k(F))$ as $i$ converges to infinity. Any marker from $S^1_\infty(g(g_k(F)))$ to $S^1_\infty(g(g_k(E)))$ has to start in in $g(U_i)$. Recall that $g(U_i)$ converges to $\eta(L)$ as $i$ converges to infinity. Choose a collection of leaves $G_k, R_k$ in $\tilde{F}$, escaping to opposite ends of $H$ and $G_k$ always smaller than $R_k$ in $H$. For each $k$ choose a marker $\zeta_k$ from $S^1_\infty(G_k)$ to $S^1_\infty(R_k)$. Fix a neighborhood $N$ of $\eta(L)$ in $A$. Choose $i$ big enough so that $g(U_i)$ is contained in $N$. For $k$ big enough the leaves $G_k, g(g_k(F)), g(g_k(E))$ and $R_k$ are linearly ordered in increasing order in $H$, hence $\zeta_k$ has to pass through some point $z_k$ in $g(U_i)$, so $z_k$ is in $N$. Therefore

$$\text{for all } L \text{ in } O, \quad \eta(L) = \lim_{k \to \infty} (\zeta_k \cap S^1_\infty(L)) \quad (*).$$

As $O$ is dense in $H$, and $\eta$ is continuous in $H$, lemma 3.14 implies that equation $(*)$ holds for any $G$ in $\tilde{F}$. We conclude that $L$ is the limit of the sequence of longer and longer markers $\zeta_k$ and $L$ is a limit
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invariant curve. But in that case proposition 3.22 implies that given any \(G, H\) leaves in \(\tilde{F}\) and any \(u\) in \(S^1(\gamma - \mathcal{L}_G)\) then \(u\) is a contracting direction between \(G\) and \(H\). This now contradicts the assumption in the proof of proposition 3.22 that there is not a dense set of contracting directions from \(F\) to \(E\). So in any case for any \(G, H\) in \(\tilde{F}\) there a dense set of contracting directions between \(G\) and \(H\).

We now finish the proof of proposition 3.22. Given arbitrary \(G, H\) in \(\tilde{F}\), the dense set of markers between \(S^1(\gamma)\) and \(S^1(\zeta)\), extends uniquely to a vertical foliation of the region of \(\mathcal{A}\) between \(S^1(\gamma)\) and \(S^1(\zeta)\). This is because it is dense from the point of view of both \(G\) and \(H\)! In addition if \(G', H, H'\) are linearly ordered in \(\mathcal{H}\) and one does the same operation using \(G', H'\), the resulting foliation is an extension of the foliation between \(S^1(\zeta)\) and \(S^1(\gamma)\). This is because markers from \(S^1(\gamma)\) to \(S^1(\zeta)\) produce markers from \(S^1(\gamma)\) to \(S^1(\zeta)\) and there is a unique extension of the foliation to the bigger annulus. Consequently there is a well defined vertical foliation in \(\mathcal{A}\). Since the collection of markers in \(\mathcal{A}\) is invariant under covering homeomorphisms, the vertical foliation also is and is a natural foliation associated to \(\tilde{F}\). This finishes the construction of the vertical foliation in the non uniform case. 

4 The uniformly quasisymmetric case

As shown in the previous section, if \(F\) is \(\mathbb{R}\)-covered, with hyperbolic leaves, then both in the uniform or non uniform cases there is a vertical foliation in \(\mathcal{A}\) which is equivariant. No need of transverse orientability for these results. The leaf space of the vertical foliation is a circle, which is the universal circle of the foliation as defined by Thurston [Th9, Th10] and is denoted by \(\mathcal{U}\). For the arguments in this and the following sections, fix once and for all a leaf \(F^* \in \tilde{F}\) and identify \(\mathcal{U}\) to the circle at infinity \(S^1_\infty(F^*)\) for each point in \(\mathcal{U}\) associate the intersection of the corresponding vertical leaf with \(S^1_\infty(F^*)\). The leaf \(F^*\) is isometric to the hyperbolic plane \(\mathbb{H}^2\) and we use the model of \(F^*\) as the unit disk in the plane, hence \(S^1_\infty(F^*)\) homeomorphic to \(U\) is the unit circle \(S^1\) in the complex plane \(\mathbb{C}\).

**Notation:** If \(g\) is in \(\pi_1(M)\), let \(\theta(g)\) denote the induced homeomorphism of \(\mathcal{U} \cong S^1\).

The transverse change in geometry of leaves of \(\tilde{F}\) is encoded by how the hyperbolic metrics vary from leaf to leaf. The distortion can also be measured in the ideal circles in the following way: We say that a homeomorphism of \(\mathcal{U} \cong S^1\) is Möbius if it continuously extends to an isometry of \(\mathbb{H}^2\). Since hyperbolic isometries act freely and transitively on triples of distinct points in \(S^1\), one cannot verify directly whether \(f\) in Homeo0\((S^1)\) is Möbius by looking at the action on triples of points. However one can do that by considering the action on quadruples of points. Given 4 distinct points \(Z = \{z_1, z_2, z_3, z_4\}\) in \(S^1\), that follow each other in the positive counterclockwise direction, recall that the cross ratio of the set is

\[
C(Z) = \frac{z_1 - z_2}{z_3 - z_1} \frac{z_4 - z_2}{z_4 - z_3}.
\]

Then \(C(Z)\) is always real and in \((0, 1)\). The 4 points in \(Z\) define an unique ideal quadrilateral in \(\mathbb{H}^2\) with ideal points in \(Z\), which is regular if and only if \(C(Z)\) is equal to \(1/2\). For any homeomorphism \(f\) of \(\mathcal{U}\) let \(fZ = \{f(z_1), f(z_2), f(z_3), f(z_4)\}\). Let \(K\) bigger than \(1\). Then \(f\) is said to be \(K\)-quasisymmetric if

\[
(2K)^{-1} \leq C(fZ) \leq 1 - (2K)^{-1},
\]

whenever \(C(Z) = 1/2\) [Hi1, Hi2]. This means regular quadrilaterals do not get too distorted. The notation is \(f\) is \(K\)-qs [Hi1]. It is easy to see that \(f\) is Möbius if and only if \(C(Z) = C(fZ)\) for all sets of 4 distinct points – equivalently \(f\) is \(1\)-qs. This definition is the analogue in dimension 1 of the concept of a quasiconformal map in a complex domain of dimension \(\geq 2\).

There is a rich theory of quasisymmetric maps [Hi1, Hi2, Le]. A group \(\Gamma\) acting on \(S^1\) is uniformly quasisymmetric if there is \(K\) so that for any \(f \in \Gamma\), then \(f\) is a \(K\)-quasisymmetric homeomorphism of \(S^1\). We denote this by \(\Gamma\) is \(K\)-qs [Hi1]. In this section we deal with following the situation:

**Case 1** – \(\pi_1(M)\) acts on \(\mathcal{U}\) as a uniformly quasisymmetric group.
§ 4. THE UNIFORMLY QUASISYMMETRIC CASE

This is the rigid case and it implies that the action is always topologically conjugate to a Möbius action in $S^1$; there is $f$ in Homeo$(S^1)$ so that for every $g$ in $\pi_1(M)$, $f \circ \theta(g) \circ f^{-1}$ is Möbius on $S^1$. This has already been done in the literature using works of various authors. We just outline the possibilities.

Suppose first that $\pi_1(M)$ acts a non discrete group on $\mathcal{U}$. Given that $\pi_1(M)$ acts a $K$-qs group, it was proved by Hinkkanen in chapter 9 of [Hi1] that the action of $\pi_1(M)$ is conjugate to a Möbius group.

Suppose now that $\pi_1(M)$ acts as a discrete group of homeomorphisms of $\mathcal{U}$. This means that given a sequence $g_n$, $n \in \mathbb{N}$ in $\pi_1(M)$ with $g_n = \theta(g_n)$ and $f_n$ converging to the identity then $f_n$ is the identity for $n$ big enough. The idea is to first prove that $\pi_1(M)$ is a convergence group: that is, $\pi_1(M)$ acts discretely in the triple space which is the set of triples $\{a, b, c\}$ in $S^1 \times S^1 \times S^1$ with $a, b, c$ distinct [Ge-Ma, Ga4]. In general the convergence group property is stronger than discreteness of the group, but not in the case of uniformly quasisymmetric groups. This is because there is a normal property associated to $K$-qs groups: Let $g_n$ in $\pi_1(M)$, with $g_n = \theta(g_n)$ and suppose there are $a, b, c$ distinct in $\mathcal{U}$ so that

$$f_n(a), f_n(b), f_n(c)$$

converge to 3 distinct points. It follows that there is a subsequence which converges uniformly to a $K$-qs homeomorphism. Gehring and Martin [Ge-Ma] do this is detail for uniformly quasiconformal maps in higher dimension and the result for $K$-qs groups acting in the circle is mentioned by Hinkkanen in [Hi1], page 62, even though the proof is not written down there. The key ideas are well known, for instance: choose $z$ in $\mathcal{U}$ so that $Z = \{a, b, c, z\}$ form the vertices of an ideal quadrilateral in $F^*$. The key fact is that $C(f_n(Z))$ is bounded away from 0 and 1 so that the quadrilaterals associated to these points are never too thin (meaning that two opposite sides of the quadrilateral do not have points very close to each other). Hence there is a subsequence $f_{n_i}$ with $f_{n_i}(z)$ converging to $w$ different from the limits of $f_n(a), f_n(b), f_n(c)$. Starting with the initial ideal triangle in $F^*$ with ideal points $a, b, c$ we can tessellate $F^* \cong H^2$ with ideal triangles so that any two adjacent ones form a regular ideal quadrilateral. Then as above there is a subsequence $f_{n_i}$ which converges in all the ideal points of the triangles, hence in a dense set of the circle $\mathcal{U}$. Again using the $K$-qs property of the action of $\pi_1(M)$ one shows that the limit map extends to a continuous map $h$ from $\mathcal{U}$ to $S^1$, which is a homeomorphism and the convergence is in fact uniform. Also the inverses converge to $h^{-1}$. This implies that the compositions $f_{n_i}^{-1} \circ f_{n_i}$ converge uniformly to the identity. As the group is discrete $f_{n_i}$ are all equal for $i$ sufficiently big. This shows that $\pi_1(M)$ acts discretely in the triple space and is a convergence group. Fundamental work of Tukia [Tu], Gabai [Ga4] and Casson-Jungreis [Ca-Ju] then implies that $\pi_1(M)$ is conjugate to a Möbius group.

Let $f$ in Homeo$(S^1)$ be the conjugating homeomorphism. We prove a rigidity result. First produce a transversely hyperbolic [Th3, Ep] 1-dimensional foliation in a manifold $M'$ as follows:

Identify the leaf space $\mathcal{H}$ to $\mathbb{R}$ and parametrize it as $\{F_t\}$ with $t$ a real number and $F_0 = F^*$. Given $g$ in $\pi_1(M)$, let $g^*$ be the induced homeomorphism of $\mathcal{H}$. Let $\tilde{M}'$ be the product $H^2 \times \mathbb{R}$. Let $\pi_1(M)$ act on $\tilde{M}'$ as follows. By hypothesis for any $g$ in $\pi_1(M)$, the homeomorphism $f \circ \theta(g) \circ f^{-1}$ is Möbius and it extends to an isometry of $H^2$, still denoted by $f \circ \theta(g) \circ f^{-1}$. Define the action on $\tilde{M}'$ by

$$\tilde{g}(u, t) = (f \circ \theta(g) \circ f^{-1}(u), g^*(t)), \quad u \text{ in } H^2, \quad t \text{ in } \mathbb{R}.$$ 

We analyse properties of this action to prove the rigidity result.

Claim 1 – The action is free.

Suppose there is $g$ in $\pi_1(M)$ and $(u, t)$ in $\tilde{M}'$ with $\tilde{g}$ fixing $(u, t)$. Then $g^*$ fixes $t$ so $g$ leaves $F_t$ invariant. If $g$ is not the identity in $F_t$, then $g$ is a non trivial isometry in $F_t$ which must be of hyperbolic type. Hence $g$ acts on $S\infty(F_t)$ with two fixed points, one contracting one expanding and the same is true for the action of $\theta(g)$ on $\mathcal{U}$ and the action of $f \circ \theta(g) \circ f^{-1}$ in $S^1$. As the extension of $f \circ \theta(g) \circ f^{-1}$ to $H^2$ is an isometry it has to have hyperbolic type and has no fixed points in $H^2$. This contradiction shows that $g$ acts as the identity in $F_t$ and hence $g$ is the identity. This proves claim 1.
At this point we need the following simple but extremely useful continuity property which relates curves in $\mathcal{A}$ with geodesics in leaves of $\tilde{\mathcal{F}}$. We establish notation which will be used often: if $a, b$ are two ideal points of a leaf $L$ of $\tilde{\mathcal{F}}$, let $ab$ be the geodesic in $L$ defined by the ideal points $a, b$ if they are different and let this be the emptyset if $a, b$ are equal. We show a basic continuity property of geodesics:

**Lemma 4.1.** Let $L_i, i \in \mathbb{N}$ be a sequence of leaves of $\tilde{\mathcal{F}}$ converging to $L_0$. Let $p_i, q_i$ distinct points in $S^1_\infty(L_i)$ with $p_i$ converging to $p_0$, $q_i$ converging to $q_0$ in $\mathcal{A}$. Then $\overline{p_i q_i}$ converges in $\tilde{\mathcal{M}}$ to $\overline{p_0 q_0}$.

**Proof.** First suppose that $p_0, q_0$ are equal. Let $\alpha$ be a transversal to $\tilde{\mathcal{F}}$ through $x_0$ in $L_0$, which intersects $L_i$ in a point $x_i$. Identify the unit tangent bundle to the leaves $T^1_{\tilde{\mathcal{F}}} \alpha$ to the union of circles at infinity near $S^1_\infty(L_0)$. From the point of view of $\alpha$, the visual angle seen by $\overline{p_i q_i}$ is converging to $0$, because $p_i, q_i$ are both converging to $p_0$. Hence $d_{L_i}(x_i, \overline{p_i q_i})$ converges to infinity. As $\mathcal{F}$ is $\mathbb{R}$-covered $d(x_i, \overline{p_i q_i})$ also converges to infinity. Therefore the geodesics $\overline{p_i q_i}$ escape in $\tilde{\mathcal{M}}$ and have no limit point in $L_0$.

Suppose now that $p_0, q_0$ are distinct points. With the notation as in option 1, the visual angle of $\overline{p_i q_i}$ as seen from $x_i$ converges to the visual angle of $\overline{p_0 q_0}$ as seen from $x_0$ – this last one is not 0, so we may assume all of them are bounded away from 0. Hence the geodesics $\overline{p_i q_i}$ have points $y_i$ a bounded distance from $x_i$. There is a subsequence $y_{i_n}$ converging to $y_0$ which is in $L_0$ and so that directions at $y_{i_n}$ also converge. Choose a transversal $\alpha'$ to $\tilde{\mathcal{F}}$ through $y_0$ and containing $y_{i_n}$. From the new point of view still $p_{i_n}$ converges to $p_0$ and $q_{i_n}$ converges to $q_0$ in $\mathcal{A}$. Hence the two rays of $\overline{p_{i_n} q_{i_n}}$ defined by $y_{i_n}$ converge to two rays in $L_0$ starting in $y_0$ and with ideal points $p_0, q_0$. In addition the angle between the rays of $\overline{p_{i_n} q_{i_n}}$ is always equal to $\pi$, hence so is the angle between the two rays in $L_0$ starting at $y_0$. This means that the union of these two rays in $L_0$ is a geodesic, which is none other than $\overline{p_0 q_0}$. So $\overline{p_i q_i}$ converges to $\overline{p_0 q_0}$. Since any such sequence has a convergent subsequence to $\overline{p_0 q_0}$ this proves the lemma.

It follows that if $\beta, \gamma$ are continuous curves in $\mathcal{A}$ which are transverse to the horizontal foliation, then the geodesics in $L$ defined by the intersections of $\beta$ and $\gamma$ with $S^1_\infty(L)$ vary continuously in $\tilde{\mathcal{M}}$ as $L$ varies in $\tilde{\mathcal{F}}$. In particular this occurs if $\beta, \gamma$ are contained in leaves of the vertical foliation of $\mathcal{A}$.

**Claim 2.** The action of $\pi_1(M)$ on $\tilde{\mathcal{M}}'$ is properly discontinuous.

Let $C$ compact in $\tilde{\mathcal{M}}$. Let $g \in \pi_1(M)$ so that there are $x_i \in C$ with $x_i$ also in $(\overline{g})^{-1}(C)$. Let $y_i$ be $\overline{g}(x_i)$. Up to subsequence $x_i$ converges to $x_0$ and $y_i$ converges to $y_0 = (u', t')$. Let

$$x_i = (u_i, t_i) \text{ with } u_i \in \mathbb{H}^2, \quad t_i \in \mathbb{R}, \quad u_i \to u_0, t_i \to t_0.$$ 

Choose triples of points $(z_i, w_i, v_i)$ in $\partial \mathbb{H}^2 = S^1$ with $u_i$ the barycenter of the ideal triangle in $\mathbb{H}^2$ defined by these 3 points. Assume that $z_i$ converges to $z_0$ which implies $w_i, v_i$ also converge to distinct points $w_0, v_0$. Up to another subsequence $f \circ \theta(g_i) \circ f^{-1}(z_i)$ converges to $z_0'$, so also $f \circ \theta(g_i) \circ f^{-1}(w_i)$ converges to $w_0'$ and $f \circ \theta(g_i) \circ f^{-1}(v_i)$ converges to $v_0'$, distinct in $S^1$. Hence $y_i$ is equal to $(s_i, t_i)$ with $s_i$ converging to the barycenter of the triangle defined by $z_0', w_0', v_0'$. Using the conjugacy by $f$, it follows that in $\mathbb{U}$ the sequences

$$f^{-1}(z_i), \quad f^{-1}(w_i), \quad f^{-1}(v_i) \text{ converge to distinct points } \quad f^{-1}(z_0), \quad f^{-1}(w_0), \quad f^{-1}(v_0)$$

respectively. Let $a_i$ in $F_{t_i}$ which are the barycenters of the ideal triangles in $F_{t_i}$ defined by the points $b_i^1, b_i^2, b_i$ in $S^1_\infty(F_{t_i})$ associated to $f^{-1}(w_i), f^{-1}(z_i), f^{-1}(v_i)$ of $\mathbb{U}$. As these points converge to 3 distinct points in $\mathcal{A}$, the lemma above implies that $b_i^1, b_i^2, b_i$ converge to geodesics in the limit leaf. The associated barycenters $a_i$ also converge to the barycenter $a_0$ in $F_{t_0}$ of the limit ideal triangle. But

$$\theta(g_i)(f^{-1}(z_i)) = f^{-1}(f \circ \theta(g_i) \circ f^{-1}(z_i)) \to f^{-1}(z_0') \text{ in } \mathbb{U},$$

similarly for $w_i, v_i$. The $g_i(a_i)$ are barycenters of ideal triangles in $F_{g_i(t_i)}$ with ideal points associated to
\[ \theta(g_i(f^{-1}(z_i)), \theta(g_i(f^{-1}(w_i)), \theta(g_i(f^{-1}(v_i)) \]

and they converge to the barycenter \( c_0 \) of the ideal triangle in \( F_0 \) defined by \( f^{-1}(z'_0), f^{-1}(w'_0), f^{-1}(v'_0). \) That means \( g_i(a_i) \) converges to \( c_0. \) As the action of \( \pi_1(M) \) on \( \tilde{M} \) is properly discontinuous there are only finitely many distinct \( g_i. \) This proves claim 2.

Let \( \Gamma \) be the action of \( \pi_1(M) \) induced in \( \tilde{M}' \). Claims 1) and 2) imply that \( M' = \tilde{M}' / \Gamma \) is a manifold.

Claim 3 — \( M' \) is a compact manifold.

Let \( x_i \) in \( M' \) and lift to \( \pi_i \) in \( \tilde{M}' \). Similarly to arguments in claim 2, find associated points \( a_i \) in \( \tilde{M}. \) Up to subsequence there are covering translates \( g_i(a_i) \) converging in \( \tilde{M}. \) Again similarly to claim 2 show that \( \tilde{g}_i(\pi_i) \) is in a compact subset of \( \tilde{M}' \), implying compactness of \( M' \). We leave the details to the reader.

Notice that \( M \) is homotopy equivalent to \( M' \).

There are two product foliations in \( \tilde{M}' \) one by leaves \( \mathbb{H}^2 \times \{ t \} \) and another by vertical lines \( \{ x \} \times \mathbb{R}. \) Both of these foliations are invariant by the action of \( \Gamma \) producing two transverse foliations in \( M'. \) The two dimensional foliation implies that \( M' \) is irreducible. The foliation by vertical lines \( \{ x \} \times \mathbb{R} \) induces a 1-dim \( \mathcal{V} \) foliation in \( M' \) which is transversely hyperbolic: there is a transversal \( \mathbb{H}^2 \) structure which is preserved by holonomy [Th3, Ep]. Under these circumstances Thurston [Th3, Ep] showed that either

1) \( M' \) is a Seifert fibered manifold with \( \mathcal{V} \) a Seifert fibration, or

2) \( M' \) is a torus bundle over \( S^1 \) with Anosov monodromy and \( \mathcal{V} \) is (say) the strong unstable foliation of the corresponding suspension Anosov flow.

In case 2) \( M' \) is Haken and as \( M \) is homotopy equivalent to \( M' \), then \( M \) is in fact homeomorphic to \( M' \) [He, Wa]. In case 1) \( M' \) is homotopy equivalent to a Seifert fibered space and since \( M \) is irreducible, Scott [Sc] proved that \( M \) is homeomorphic to \( M' \).

Using averaging techniques one can show that \( \mathcal{F} \) is topologically conjugate to \( \mathcal{F}' \). These techniques have been used for instance by Ghys [Gh] and others in the one dimensional setup. The two dimensional case is more involved and for brevity we only do the following: In situation 1) \( M \) is a Seifert fibered space, and work of Brittenham [Br] (see also [Th1]), implies that \( \mathcal{F} \) is either vertical (a union of circle fibers in the Seifert fibration) or horizontal (transverse to the circle fibers). The first option cannot occur because the leaves of \( \mathcal{F} \) are hyperbolic. Hence \( \mathcal{F} \) can be put transverse to the Seifert fibration and by careful choices, the transversal flow in \( M \) lifts to a flow in \( \tilde{M} \) which produces global homeomorphisms between leaves of \( \tilde{\mathcal{F}} \) which are isometries of the hyperbolic metric. No change in geometry! In any case it is easy to see that in this case there is a \( \mathbb{Z} \oplus \mathbb{Z} \) subgroup of \( \pi_1(M). \)

In case 2), \( M \) fibers over \( S^1 \) with Anosov monodromy. One can put the incompressible torus \( T \) transverse to \( \mathcal{F} \) [Rou, Th1, Ga5] and hence there is an induced foliation in \( T. \) This foliation is invariant by the monodromy of the fibration and hence has to be the stable or unstable foliation of the monodromy. Hence \( \mathcal{F} \) is conjugate to (say) the (weak) stable foliation of the associated Anosov flow and the transversal flow can be chosen to be the strong unstable foliation of this flow. As in case 1) no transversal change of the leaf metrics. In any case there is a \( \mathbb{Z} \oplus \mathbb{Z} \) subgroup of \( \pi_1(M). \)

This finishes the proof of the main theorem in the case \( \pi_1(M) \) acts by uniformly quasisymmetric homomorphisms of \( \mathcal{U}. \)

5 The non uniformly quasisymmetric case

For the rest of the article we we analyse the following situation:

Case 2 — The action of \( \pi_1(M) \) on \( \mathcal{U} \) is not uniformly quasisymmetric.

**Theorem 5.1.** If the action of \( \pi_1(M) \) on \( \mathcal{U} \) is not uniformly quasisymmetric and \( \mathcal{F} \) is transversely orientable, then there is a laminating \( \mathcal{G} \) transverse to \( \mathcal{F} \) intersecting leaves of \( \mathcal{F} \) in geodesics.
\[\text{Step 1 - Constructing ideal quadrilaterals which get stretched in opposite directions.}\]

By hypothesis there are quadruples \(Z_i\) of points in \(U\) with \(C(Z_i) = 1/2\) and \(g_i\) in \(\pi_1(M)\) with the cross ratio \(C(\theta(g_i)(Z_i))\) arbitrarily close to 0 or 1. There is a way to produce the transverse laminations to \(F\) using earthquake maps on the hyperbolic plane [Th6] as explained by Thurston [Th9]. Here we use simple properties of the cross ratio to obtain the laminations.

If \(Z = \{z_1, z_2, z_3, z_4\}\) is a positively oriented quadruple of points in \(S^1\), then \(C(Z)\) is very near 0 or 1 if and only if the ideal quadrilateral in \(H^2\) associated to it is very thin: there are two opposite sides of the quadrilateral which are very close to each other in the hyperbolic metric. This obviously implies that the other two opposite sides are very far from each other and is equivalent to it. Given an ideal quadrilateral we define the waist to be the minimum distance between opposite sides. Using the formula for \(C(Z)\) it is very easy to verify that \(C(Z)\) is very near 0 if and only if the geodesic \(\overline{z_1z_2}\) of \(H^2\) defined by \(z_1, z_2\) has a point very near \(\overline{z_1z_2}\) and that \(C(Z)\) is very near 1 if and only if \(\overline{z_2z_3}\) has a point very near \(\overline{z_1z_2}\).

**Definition 5.2.** Given a quadruple \(U\) of points in a circle at infinity \(S^1_{\infty}(L)\) of a leaf \(L\) of \(\bar{F}\), let \(W(U)\) denote the ideal quadrilateral in \(L\) with endpoints in \(U\).

Let \(Z\) be a quadruple in \(S^1_{\infty}(F^*)\). For \(g\) in \(\pi_1(M)\) the map \(\theta(g)\) acts on \(U\) (identified to \(S^1\)). This defines an action on geodesics and ideal quadrilaterals of \(F^*\), for simplicity of notation also denoted by \(\theta(g)\). In particular

\[\theta(g)(W(Z)) = W(\theta(g)(Z)).\]

We stress that

\[\theta(g)(W(Z)) \subset F^*, \text{ usually not isometric to } W(Z),\]

whereas \(g(W(Z)) \subset g(F^*)\) is always isometric to \(W(Z)\).

First check the action in \(F^*\). Let \(Z_i\) be a sequence of quadruples in \(U\) and \(g_i\) in \(\pi_1(M)\) with \(C(Z_i) = 1/2\) (that is, the \(W(Z_i)\) are regular ideal quadrilaterals in \(F^* = H^2\)) but the cross ratios \(C(\theta(g_i)(Z_i))\) converge to either 0 or 1. Circularly rename the points in \(Z_i\) so that these cross ratios converge to 0.

**Lemma 5.3.** There are ideal quadrilaterals \(C_i\) in \(H^2\) defined by quadruples \(Y_i\) in \(U\) and covering translations \(g_i\) with \(C(Y_i)\) converging to 0, but \(C(\theta(g_i)(Y_i))\) converging to 1.

**Proof.** This means that with the ordering in the quadruples, the map \(\theta(h_i)\) sends quadrilaterals very thin in one direction (\(C(Z)\) near 0) to quadrilaterals very thin in the other direction (\(C(Z)\) near 1). Given \(n\) in \(N\) and any waist size \(b_0\) sufficiently small, there is a waist size \(b_1\) much smaller than \(b_0\) so that any quadrilateral of waist \(b_1\) can be covered by \(n\) quadrilaterals.
\( \{ \mathcal{E}_m \}, \quad 1 \leq m \leq n, \)

of waist smaller than \( b_0 \) (the associated cross ratio is very close to 0) satisfying: the interiors of the \( \mathcal{E}_m \) are disjoint from each other, consecutive quadrilaterals are adjacent, see fig. ??, a.

Now if \( \mathcal{W}(\theta(g_i)\mathcal{Z}_i) \) has waist size less than \( b_1 \), cover it by quadrilaterals \( \{ \mathcal{E}_m \} \) with \( m \in [1,n] \) all of waist size smaller than \( b_0 \) as above, see fig. ??, a. Let \( \theta(g_i^{-1}) \) act on this. The union of

\[ \theta(g_i)^{-1}(\mathcal{E}_m) \]

will cover the regular quadrilateral \( \mathcal{W}(\mathcal{Z}_i) \). Since the quadrilaterals \( \theta(g_i)^{-1}(\mathcal{E}_m) \) are restricted in one direction to be inside the regular quadrilateral \( \mathcal{W}(\mathcal{Z}_i) \) (see fig. ??, b) then: If \( n \) is sufficiently big, at least one of the quadrilaterals \( \theta(g_i)^{-1}(\mathcal{E}_m) \) is very thin in the other direction, that is the associated cross ratio is very close to 1. Let \( \mathcal{C}_i \) be one such quadrilateral \( \mathcal{E}_m \). This finishes the proof of lemma 5.3. \( \square \)

**Definition 5.4.** If \( I \) is a geodesic in a leaf a leaf \( F \) of \( \tilde{\mathcal{F}} \), it has two ideal points in \( S^1_{\infty}(F) \) and therefore two distinct points in \( \mathcal{U} \). The set \( l \times \mathbb{R} \) consists of the union of the geodesics in leaves of \( \mathcal{F} \) associated to the same points in \( \mathcal{U} \) defined by \( l \). The curves in \( \mathcal{A} \) defined by each point in \( \mathcal{U} \) are continuous hence the set \( l \times \mathbb{R} \) is a topological plane which is properly embedded in \( \tilde{\mathcal{M}} \). In the same way if \( V \) is a convex set in a leaf of \( \tilde{\mathcal{F}} \) bounded by geodesics \( s_i \), one forms \( s_i \times \mathbb{R} \) and jointly they bound the set \( V \times \mathbb{R} \).

**Step 2 - The distortion parallelepipeds.**

We will use the thin quadrilaterals

\[ \mathcal{W}(\mathcal{Y}_i), \quad \mathcal{W}(\theta(g_i)(\mathcal{Y}_i)) \]

of the previous step to produce immersed transverse laminations to \( \mathcal{F} \). First construct a distorted ideal parallelepiped in \( \tilde{\mathcal{M}} \) as follows. For simplicity suppose that \( F^* \) is in the back of \( g_i^{-1}(F^*) \). Notice that \( \theta(g_i)(\mathcal{Y}_i) \) defines an ideal quadrilateral in \( F^* \) with cross ratio very close to 1. Since \( g_i \) acts as isometries between leaves of \( \tilde{\mathcal{F}} \), then

\[ Z_i = g_i^{-1}(\mathcal{W}(\theta(g_i)(\mathcal{Y}_i))) \subset g_i^{-1}(F^*) \]

is isometric to \( \mathcal{W}(\theta(g_i)(\mathcal{Y}_i)) \) and has cross ratio very close to 1. The ideal points of

\[ g_i^{-1}(\mathcal{W}(\theta(g_i)(\mathcal{Y}_i))) \text{ in } S^1_{\infty}(g_i^{-1}(F^*)) \]

define the same points \( b_1, b_2, b_3, b_4 \) in \( \mathcal{U} \) that \( Y_i \) does. For each \( F \) in \( \mathcal{H} \) between \( F^* \) and \( g_i^{-1}(F^*) \) form the ideal quadrilateral with the ideal points corresponding to \( b_1, b_2, b_3, b_4 \) in \( \mathcal{U} \). The 4 curves in \( \mathcal{A} \) defined by these points in \( \mathcal{U} \) are continuous curves in \( \mathcal{A} \), hence lemma 4.1 implies that the sides of the ideal quadrilaterals in \( F \) vary continuously with the leaf \( F \). The union of these ideal quadrilaterals between \( F^* \) and \( g_i^{-1}(F^*) \) is a parallelepiped \( \mathcal{P}_i \) in \( \tilde{\mathcal{M}} \), see fig. ??, b. That is

\[ \mathcal{P}_i = \bigcup \{ (\mathcal{W}(\mathcal{Y}_i) \times \mathbb{R}) \cap F \mid F \in [F^*, g_i^{-1}(F^*)] \}. \]

The bottom of \( \mathcal{P}_i \) is the quadrilateral \( \mathcal{W}(\mathcal{Y}_i) \) in \( F^* \); the top is the quadrilateral \( Z_i \) in \( g_i^{-1}(F^*) \) and there are 4 sides which are transverse to \( \tilde{\mathcal{F}} \), which are \( \mathcal{B}_1 \mathcal{B}_2 \mathcal{B}_3 \mathcal{B}_4 \) and so on. The tops of the parallelepipeds will shrink to geodesics producing one lamination and likewise for the bottoms. We will change the parallelepipeds \( \mathcal{P}_i \) in the next step.

**Remark:** Lemma 4.1 implies that for any geodesic \( \alpha \) in a leaf of \( \tilde{\mathcal{F}} \), then the geodesics in \( \alpha \times \mathbb{R} \) vary continuously in \( \tilde{\mathcal{M}} \). Hence \( \alpha \times \mathbb{R} \) is an embedded plane in \( \tilde{\mathcal{M}} \). It follows that all objects constructed here \( (\mathcal{D}_- \times \mathbb{R}, \mathcal{D}_+ \times \mathbb{R}, \mathcal{G}, \mathcal{G}_-, \mathcal{G}_+, \text{etc.}) \) are continuous.
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**Step 3 - Convergence of the bottoms of the parallelepipeds.**

We use the distortion parallelepipeds $P_i$ from step 2. We take limits: First project to leaves of $\mathcal{F}$ in $M$. The quadrilaterals $W(Y_i)$ have associated cross ratios $C(W_i)$ converging to 0. Let $x_i$ in the boundary of the waist of $W(Y_i)$. Up to subsequence assume that $\pi(x_i)$ converges to $x_0$ in $M$ and the directions of the geodesic sides of $\pi(W_i)$ at $\pi(x_i)$ also converge. In $M$ the quadrilaterals $\pi(W(Y_i))$ shrink to geodesics in leaves. Lift $x_0$ to $\tilde{x}_0$ in $\tilde{M}$ with the limit geodesic in the leaf of $\mathcal{F}$ lifting to a geodesic $l_0$ in a leaf $F'$ of $\tilde{\mathcal{F}}$ through $\tilde{x}_0$. This geodesic defines two points in $\mathcal{U}$ and hence a geodesic $l_1$ in $F^*$ — the geodesic $l_1$ is exactly $l_0 \times R$ intersected with $F^*$. We define two important sets:

$$D_+ = \bigcup_{\theta \in \pi_1(M)} \{\theta(h)(l_0)\} \subset F^* \quad \text{and} \quad D_+ \times R = \bigcup_{\ell \in D_+} (\ell \times R) \subset \tilde{M}.$$ 

We now change the $P_i$ so that bottoms converge to $l_1$. Up to covering translations we may assume that the bottoms $B_i$ of $P_i$ converge to the geodesic $l_0$ in a leaf $F'$ of $\tilde{\mathcal{F}}$. This changes the $P_i$ so the bottoms are not in $F^*$ anymore - we adjust that as follows: For $F$ in $\tilde{\mathcal{F}}$ between $F^*$ and $F'$ let

$$A_i(F) = (B_i \times R) \cap F.$$ 

These are ideal quadrilaterals in $F$. The ideal endpoints of $B_i$ converge to the 2 ideal points of $l_0$ as $i$ grows, so the ideal points of $A_i(F^*)$ collapse to the 2 ideal points of $l_1$ — because the leaves of the vertical foliation in $A$ vary continuously. This produces a thin wall from $F^*$ to $F'$. Since the horizontal quadrilaterals in $P_i$ eventually have cross ratio close to 1 (when going up), the $P_i$ extends beyond $F'$. We can extend or contract the parallelepipeds $P_i$ so that the bottoms are now always in $F^*$ and they converge to $l_1$.

**Conclusion 1** - Up to subsequences, covering translations and extension or contractions we may assume that the parallelepipeds $P_i$ have bottoms $B_i$ which are ideal quadrilaterals in $F^*$ with cross ratio converging to 0 and $B_i$ converging to the geodesic $l_1$ of $D_+$, so that the waists of $B_i$ converge to a fixed point of $l_1$. The tops of $P_i$ have cross ratio converging to 1.

**Step 4 - Convergence of the tops of the parallelepipeds.**

We want to do the same approach for the tops of $P_i$. Since the bottoms of the $P_i$ will stay in $F^*$ clearly the tops cannot do the same. As in step 3, up to another subsequence the waists of $\pi(T_i)$ converge to a point in $\tilde{M}$ and so do the directions of the sides of $\pi(T_i)$. Lifting to $\tilde{M}$ this defines a geodesic in a leaf of $\tilde{\mathcal{F}}$ and using the vertical foliation this defines a geodesic $l_2$ in $F^*$. Define

$$D_+ = \bigcup_{\theta \in \pi_1(M)} \{\theta(h)(l_2)\} \subset F^* \quad \text{and} \quad D_+ \times R = \bigcup_{\ell \in D_+} (\ell \times R) \subset \tilde{M}.$$ 

In the same way as above we can extend or contract the tops $P_i$ so that:

**Conclusion 2** - In addition to conclusion 1: There are covering translations $g_i$ in $\pi_1(M)$ so that the tops $g_i(T_i)$ are contained in $F^*$ and converge to a geodesic $l_2$ of $F^*$. The waists of $g_i(T_i)$ converge to a single point of $l_0$. Finally in the case there is a leaf of $D_-$ transversely intersecting a leaf of $D_+$, then up to renaming $l_2$ by covering translations we can assume that these are $l_1$ and $l_2$. If this does not happen but $D_-$ and $D_+$ share a leaf assume that $l_1$ is equal to $l_2$. Fix the $P_i, B_i, T_i, g_i$ from now on.

Using covering translations one gets the same conclusions for any leaves of $D_-$ and $D_+$. In fact using a diagonal process on sequences the same is true for any limit of leaves of $D_-$ or $D_+$. We stress this:

**Lemma 5.5.** Using covering translations, extensions/contractions of parallelepipeds and limits; the following happens: suppose that $l$ is either the intersection of a leaf of $D_- \times R$ with a leaf $F$ of $\tilde{\mathcal{F}}$ (a geodesic in a leaf of $\tilde{\mathcal{F}}$) or a limit of such intersections. Then there is a sequence $Q_i$ of parallelepipeds so that $2$
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opposite sides of bottoms converge to \( l \) and so that cross ratio of bottoms (respectively tops) converges to 0 (respectively 1). The same holds if \( l \) comes from \( \mathcal{D}_+ \times \mathbb{R} \) in which case the tops converge to \( l \).

**Step 5** - Producing the (a priori only immersed) laminations.

We will eventually prove (in the next section) that each of \( \mathcal{D}_- \times \mathbb{R} \), \( \mathcal{D}_+ \times \mathbb{R} \) does not have transverse self intersections. But the first step is to obtain some embedded lamination which may not be one of these two a priori. There are 3 cases to consider (we will keep coming back to these options in the next section):

**Option A** - No leaf of \( \mathcal{D}_- \) transversely intersects another leaf of \( \mathcal{D}_- \) (similarly for \( \mathcal{D}_+ \)).

Then \( \mathcal{D}_- \times \mathbb{R} \subset \mathcal{M} \) is a collection of properly embedded planes without any transverse intersections which is invariant under covering translations. Its closure is a \( \pi_1(M) \) invariant lamination in \( \mathcal{M} \) which intersects leaves of \( \tilde{\mathcal{F}} \) in a union of geodesics. The image in \( \mathcal{M} \) is a lamination transverse to \( \mathcal{F} \).

**Option B** - No leaf of \( \mathcal{D}_- \) transversely intersects a leaf of \( \mathcal{D}_+ \).

If no two leaves of \( \mathcal{D}_- \) intersect transversely, then as in option A, we produce a lamination in \( \mathcal{M} \) transverse to \( \mathcal{F} \). Otherwise by option B, no leaf of \( \mathcal{D}_- \) is also a leaf of \( \mathcal{D}_+ \). Consider a connected component in \( F^* \) of the union of leaves in \( \mathcal{D}_- \). Then the convex hull \( C \) of this set (in the hyperbolic metric of \( F^* \)) is not all of \( \mathbb{H}^2 \). Let \( B \) be the boundary of \( C \). The translates of \( B \) under \( \theta(g) \) with \( g \) in \( \pi_1(M) \) do not intersect \( B \) transversely. Therefore \( \pi(B \times \mathbb{R}) \) produces a lamination as in option A above. Notice that in this case maybe all leaves of the lamination are neither in \( \mathcal{D}_- \) nor \( \mathcal{D}_+ \).

**Option C** - There is a leaf of \( \mathcal{D}_- \) transversely intersecting a leaf of \( \mathcal{D}_+ \).

This is the most interesting case. The rest of the proof of theorem 5.1 is devoted to an analysis of this case. By conclusion 2, here we can choose the \( l_1 \) in \( \mathcal{D}_- \) and \( l_2 \) in \( \mathcal{D}_+ \) with a transverse intersection. The goal is to show that leaves of \( \mathcal{D}_- \) do not intersect transversely and likewise for \( \mathcal{D}_+ \), that is, option C implies option A. We stress that options A and B can happen concurrently, but B and C are contradictory.

An important remark here is that in all options A, B, C, these laminations are obtained as a union of \( r \times \mathbb{R} \) for a collection of geodesics \( r \) in \( F^* \). If \( r_1 \times \mathbb{R} \) intersects \( r_2 \times \mathbb{R} \) then there is \( F \) in \( \mathcal{F} \) so that

\[
u_1 = (r_1 \times \mathbb{R}) \cap F \quad \text{intersects} \quad \nu_2 = (r_2 \times \mathbb{R}) \cap F,
\]

both geodesics in \( F \). If \( \nu_1 = \nu_2 \), then \( r_1 \times \mathbb{R} = r_2 \times \mathbb{R} \). In particular \( r_1 \times \mathbb{R} \) cannot be tangent to \( r_2 \times \mathbb{R} \) at level \( F \) and then cross from one side of \( r_2 \times \mathbb{R} \) to the other when passing through \( F \). If on the other hand \( \nu_1 \) and \( \nu_2 \) intersect transversely in \( F \) then \( r_1 \times \mathbb{R} \) and \( r_2 \times \mathbb{R} \) will have transverse intersection for all \( G \) in \( \mathcal{F} \). This is one big advantage of producing these laminations using the universal circle.

To prove that option C implies option A, then by way of contradiction suppose there is \( l_3 \) in \( \mathcal{D}_- \) which transversely intersects \( l_1 \). There is a covering translation \( h \) with \( \theta(h)(l_1) = l_3 \). Fix \( h \) for the rest of the analysis of option C. It follows that \( \theta(h^{-1})(l_1) \) also intersects \( l_1 \) transversely. We use the setup in much more detail in this case. Notice that the bottoms \( B_1 \) of the parallelepipeds \( P_i \) converge to \( l_1 \), in fact two opposite sides of \( B_1 \) do and likewise two opposite sides of \( \theta(g_1(T_1)) \) converge to \( l_2 \) of \( \mathcal{D}_+ \).

For the rest of the proof fix \( i' \) big enough so that: If \( p_1, p_2, p_3, p_4 \) in \( U \) are the ideal points of \( B_1 \) then \( \overline{p_1p_2} \) and \( \overline{p_3p_4} \) are very close to \( l_1 \) and so

\[
\overline{p_1p_2}, \quad \overline{p_3p_4} \quad \text{intersect} \quad l_2, \quad l_3 = \theta(h)(l_1) \quad \text{and} \quad \theta(h^{-1})(l_1) \quad \text{transversely}.
\]

Also \( \theta(g)(T_i') \) has two opposite sides very close to \( l_2 \). Let

\[
g = g_i^2, \quad h_1 = \theta(h), \quad g_1 = \theta(g) \quad \text{and} \quad q_j = g_1(p_j) = \theta(g)(p_j) \in U.
\]

For simplicity we omit the notational dependence of \( p_j, q_j, h, h_1, g, g_1 \) on the index \( i' \).
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**Step 6** - Analysing the dynamics of \( h_1 = \theta(h) \) and \( g_1 = \theta(g) \) in \( \mathcal{U} \).

First we explain why we consider \( g \) the square of \( g_\varepsilon \) instead of just \( g_\varepsilon \). Let \( \mathcal{I} \) be the open interval of \( \mathcal{U} \) defined by \( p_1, p_2 \) and not containing \( p_3 \) and similarly \( \mathcal{J} \) defined by \( p_3, p_4 \) and not containing \( p_1 \). One possibility is that \( \theta(g_\varepsilon)(p_1) \) and \( \theta(g_\varepsilon)(p_2) \) are in \( \mathcal{I} \). Another possibility is that the quadrilateral \( \theta(g_\varepsilon)(\mathcal{T}_\varepsilon) \) is rotated 180 degrees, that is, \( \theta(g_\varepsilon)(p_1) \) and \( \theta(g_\varepsilon)(p_2) \) are in \( \mathcal{J} \). In any case \( \theta(g_\varepsilon^2)(p_1) \) (equal to \( q_1 \)) and \( q_2 \) are in \( \mathcal{I} \) and \( q_1 \) is the one closest to \( p_1 \), see fig. ??, a. Also

\[
g_1(\mathcal{T}_\varepsilon) = \theta(g_\varepsilon^2)(\mathcal{B}_\varepsilon)
\]

is even thinner than \( \theta(g_\varepsilon)(\mathcal{B}_\varepsilon) \). Similarly \( q_3, q_4 \) are in \( \mathcal{J} \) and \( q_3 \) is closest to \( p_3 \). The dynamics of \( g_1 \) in \( \mathcal{U} \) is as follows: \( g_1(\mathcal{U}) \) is contained in \( \mathcal{I} \) producing at least one fixed point in \( \mathcal{I} \) and similarly \( g_1(\mathcal{J}) \) a subset of \( \mathcal{J} \) yields a fixed point in \( \mathcal{J} \), where here \( \mathcal{J} \) denotes the closure in \( \mathcal{U} \). Similarly there are at least two fixed points outside \( \{ \mathcal{I} \cup \mathcal{J} \} \): one near \( p_1, p_2 \) and another near \( p_3, p_4 \). In any case \( g_1 \) has at least 4 fixed points in \( \mathcal{U} \). It follows that \( g \) acts freely in \( \mathcal{H} \): if \( g(F) = F \) for some leaf \( F \) of \( \mathcal{F} \), then \( g \) fixes only two points in \( S(F) \) and similarly for \( g_1 \) acting in \( \mathcal{U} \).

We define 8 points in \( \mathcal{U} \) from the dynamics of \( \theta(g) \): Let

\[
a_j = \lim_{n \to -\infty} g_1^n(p_j), \quad b_j = \lim_{n \to +\infty} g_1^n(p_j).
\]

Notice \( b_1, b_2 \) are in \( \mathcal{I} \); \( b_3, b_4 \) are in \( \mathcal{J} \) and none of the \( a_j \) are in \( \mathcal{I} \) or \( \mathcal{J} \). Let

- \( \gamma = \frac{a_1a_2}{a_3a_4} \) which is very close to \( \frac{b_1b_2}{b_3b_4} \) and to \( l_1 \),
- \( \beta_1 = \frac{b_1b_4}{b_2b_3} \) and \( \beta_2 = \frac{b_2b_1}{b_3b_4} \), both both very close to \( \frac{b_4b_3}{b_1b_2} \).

We define 8 closed intervals in \( \mathcal{U} \): first \( I_1 \) is the interval of \( \mathcal{U} \) from \( a_4 \) to \( a_1 \) not containing the other points \( a_j, b_j \), then similarly

\[
I_2 : \text{from } a_1 \to b_1, \quad I_3 : b_1 \to b_2; \quad I_4 : b_2 \to a_2; \quad I_5 : a_2 \to a_3; \quad I_6 : a_3 \to b_3; \quad I_7 : b_3 \to b_4 \text{ and } I_8 : b_4 \to a_4,
\]

see fig. ??, b. Notice that \( g_1(I_{1k}) = I_k \) for all \( k \). In addition \( g_1 \) acts as a homeomorphism with only two fixed points in \( I_3 \), the repelling is \( a_1 \) and the attracting is \( b_1 \). Similarly for \( I_4, I_6, I_8 \). Any of the intervals \( I_1, I_3, I_5, I_7 \) may be a single point — that is, if \( a_1 = a_4 \) then \( I_1 \) is a single point. On the other hand none of \( I_2, I_4, I_6, I_8 \) is a single point.

One key point here is that by choice of \( i' \), none of the endpoints of \( h_1(l_1) = l_3 \) are in \( I_1 \) or in \( I_5 \) and we may assume the same happens for the endpoints of \( h_1(\gamma) \) and for \( h_1(p_3) \) (make \( i' \) bigger if necessary). Let

\[
A = \gamma \times \mathbb{R}, \quad B = h(A) = l_3 \times \mathbb{R}.
\]

Then \( g(A) = A \) and \( A \) is a properly embedded plane in \( \tilde{\mathcal{M}} \). Also \( B \) transversely intersects \( A \), because \( l_3 \) transversely intersects \( l_1 \). Using the dynamics of \( g_1 \) we show that the intersections of the surfaces \( \pi(A), \pi(B) \) stay in compact parts of both \( \pi(A) \) and \( \pi(B) \) and derive a contradiction.

**Step 7** - Analysing intersection of walls \( A = \gamma \times \mathbb{R} \) and \( B = h(A) \).

Consider \( \beta_1 \times \mathbb{R} \) a properly embedded plane which intersects \( A \) in an infinite curve \( \zeta_1 \). Then

\[
g(\beta_1 \times \mathbb{R}) = \beta_1 \times \mathbb{R}, \quad g(A) = A, \quad \text{so} \quad g(\zeta_1) = \zeta_1.
\]

Hence \( \zeta_1 \) projects to a closed curve \( \alpha_1 \) in the annulus \( A/g \). Similarly \( \beta_2 \) produces a closed curve \( \alpha_2 \) in \( A/g \). Notice \( \alpha_1 \) is equal to \( \alpha_2 \) if and only if \( \beta_1 = \beta_2 \). Let \( \tilde{N} \) be the annulus (possibly degenerate) in \( A/g \) bounded by \( \alpha_1 \) and \( \alpha_2 \) and let \( \tilde{N} \) be its lift to \( A \). Recall the parametrization of \( \tilde{\mathcal{F}} \) as \( \{ F_t \} \) with \( t \) in \( \mathbb{R} \) and \( F^* \) equal to \( F_0 \). The curve
\( \mu = (B \cap A) \cap \left( \bigcup_{t \geq 0} F_t \right) \)

is an infinite curve of transverse intersection of \( A \) and \( B \).

The geodesic \( l_3 \) of \( \mathbb{H}^2 \) has one endpoint \( c_1 \) in \( I_2, I_3 \) or \( I_4 \) and the other \( c_2 \) in \( I_6, I_7 \) or \( I_8 \). Suppose first the endpoints are in \( I_2 \) and \( I_8 \) respectively. Notice that \( g^{-1}(\mathcal{T}^*) \) is contained in \( F^* \), so \( g \) sends a leaf above \( F^* \) to \( F^* \), or \( g^{-1} \) acts as an increasing homeomorphism in \( \mathcal{H} \). Let

\[
 s = h_1(\gamma) \cap \gamma = (B \cap A) \cap F_0 \in \mu.
\]

We consider how the points \( \mu \) move in \( A \) as \( t \) increases. The action by \( g^{-1} = (\theta(g))^{-1} \) brings the endpoints of \( l_3 \) closer to \( I_1 \) because the endpoints of \( l_3 \) are in \( I_2 \) and \( I_8 \), see fig. ??, b. Looking at the action \( g^{-1} \) in \( A \), it sends \( s \) to a point with same distance from \( \tilde{N} \), see fig. ??, a \((\tilde{N} \text{ is invariant under } g^{-1}) \). Hence \( s_1 = \mu \cap g^{-1}(F^*) \) is closer to \( \tilde{N} \) than \( g^{-1}(s) \) is, see fig. ??, a. This means that going up brings the intersection of \( B \) and \( A \) closer to \( \tilde{N} \) in \( A \). In the same way let

\[
 s_n = \mu \cap g^{-n}(F^*)
\]

Then \( g^n(s_n) \) is in a geodesic in \( F^* \) with endpoints \( \theta(g^n)(c_1) \) and \( \theta(g^n)(c_2) \). But

\[
 \theta(g^n)(c_1) \to b_1, \quad \theta(g^n)(c_2) \to b_1 \quad \text{when} \quad n \to +\infty.
\]

As \( \beta_1 \) has ideal points \( b_1, b_4 \), the above convergence implies that \( \mu \) is actually asymptotic to \( \zeta_1 \) going up. Hence \( \mu \) projects in \( A/g \) to a curve asymptotic to the closed curve \( \zeta_1/g \) see fig. ??, b.

Let \( \alpha \) be the projection of \( \zeta_1 \) to \( M \), that is, \( \alpha = \pi(\zeta_1) \). Since \( \zeta_1 \) is invariant under \( g \), it follows that \( \alpha \) is a closed curve in \( M \) and since \( \zeta_1/g \) is already closed it follows that

\[
 \alpha = g^n, \quad \text{for some} \quad n \neq 0.
\]

This means that the curve \( \alpha \) represents the element \( g^n \) in the fundamental group.

Now reverse the roles of \( A \) and \( B \). The points in \( \mathcal{U} - \{ I_1 \cup I_5 \} \) get contracted towards \( I_3 \) and \( I_7 \) under the action of \( g_1 = \theta(g) \). Let

\[
 \delta = hgh^{-1}.
\]

Notice that the points \( a_j \) are not in the union of \( h_1(I_1) \) and \( h_1(I_5) \), by choice of \( \mathcal{V} \). Hence the \( a_j \) are in the regions of \( \mathcal{U} \) which get contracted by the action of \( \theta(\delta) \) towards \( h_1(I_3) \) and \( h_1(I_7) \). From the point of view of \( B \) the same arguments as above show that the intersection of \( B \) with \( A \) going up (in the positive direction) is also trapped closer to a band of \( B \) invariant under \( \delta \). Here we use the fact that \( \mathcal{F} \) is transversely orientable - \( h \) preserves orientation in \( \mathcal{H} \), so going up in \( A \) (action by \( g^{-1} \)) corresponds to going up in \( B \) (action by \( \delta^{-1} \)) as well. An argument as the one done in for the curve \( \mu \) as seen in \( A \) shows that there is a curve \( \zeta_2 \) in this band invariant under \( \delta \) and so that \( \mu \) is asymptotic to \( \zeta_2 \) in the positive direction. Let \( \alpha_1 \) be the projection of \( \zeta_2 \) in \( M \). Similarly as above one shows that

\[
 \delta = hgh^{-1} = \alpha_1^m \quad \text{for some} \quad m \neq 0.
\]

Now \( \mu \) is asymptotic to \( \zeta_1 \) and to \( \zeta_2 \) both of which project to closed curves in \( M \). Therefore \( \alpha \) is equal to \( \alpha_1 \). Since the intersection of \( B \) and \( A \) is a single curve, then in fact \( \zeta_1 \) is equal to \( \zeta_2 \).

**Step 8** - Incompatible actions in \( \mathcal{U} \).

In the previous step we proved that

\[
 g = \alpha^n \quad \text{and also} \quad hgh^{-1} = \alpha^m, \quad n, m \neq 0.
\]
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It follows that

$$g^m = \alpha^m = (hg^{-1})^n = hg^n h^{-1}.$$  

This is obtained when \(l_3\) has endpoints in \(I_2, I_8\). The other cases are similar. An argument as above shows that the curve \(\mu\) when viewed in \(A\) is always asymptotic to a curve \(\zeta\) which is invariant under \(g\). The curve \(\zeta\) is obtained as the intersection

$$\zeta = (l \times \mathbb{R}) \cap A,$$

where \(l\) is a geodesic in \(F^*\) with endpoints

$$a = \lim_{i \to +\infty} \theta(g^i)(c_1), \quad b = \lim_{i \to +\infty} \theta(g^i)(c_2).$$

Recall that \(c_1, c_2\) are the endpoints of \(l_3\). This occurs because \(c_1\) is in the union of \(I_2, I_3\) and \(I_4\) and so is \(a\). Similarly for \(b\). Now the same arguments as above imply the same conclusion. The important fact is that the endpoints of \(l_3\) are not in \(I_1\) or \(I_5\)!

**Conclusion:** In all cases there are \(n, m\) non zero integers with \(g^n = h^{-1} g^m h\).

We now prove this is impossible. Notice that \(g\) and hence \(h^{-1}g h\) act freely in \(H\) and both act as decreasing homeomorphisms of \(H\). This again uses the fact that \(h\) preserves orientation of \(H\)! Hence if \(n\) is positive then \(m\) is positive as well. Assume this is the case. Given \(u\) in \(U\), if

$$g_1(u) = u, \quad \text{then} \quad g_1^n(u) = u = h_1^{-1} g_1^m h_1(u) \quad \text{or} \quad g_1^n h_1(u) = h_1(u).$$

Since \(g_1\) has fixed points in \(U\) this implies \(g_1 h_1(u) = h_1(u)\). The same applies to \(h^{-1}_1\) so \(h_1\) leaves the set of fixed points of \(g_1\) invariant. These fixed points are in \(I_1, I_3, I_5\) and \(I_7\). By construction \(h_1(I_1)\) is disjoint from \(I_1\) and \(I_5\) therefore it is a subset of \(I_3\) or \(I_7\) and likewise for \(h_1(I_5)\). Similarly \(h^{-1}_1(I_1), h^{-1}_1(I_5)\) are subsets of \(I_3\) or \(I_7\), so \(h_1(I_3), h_1(I_7)\) are contained in \(I_1\) or \(I_5\). Together these imply that

$$h_1(I_j) = I_{k(j)} \quad \text{for any} \quad j \in [1, 8].$$

There are 4 cases to consider all similar. Suppose first that \(h_1(I_1) = I_3\) and \(h_1\) preserves orientation of \(U\). Since \(h_1(I_j) = I_{k(j)}\), this implies the very important consequence that \(h_1(I_2) = I_4\).

Now consider the action of \(g_1^n\) and \(h^{-1}_1 g_1^m h_1\) in \(I_2\). The key is that both \(n, m\) are positive!

- \(g_1^n\) only fixes \(\partial I_2\) in \(I_2\), with \(a_1\) repelling fixed point for \(g_1^n\) and \(b_1\) attracting,

- \(h^{-1}_1 g_1^m h_1\) conjugates the action of \(g_1^m\) in \(I_1\) to act in \(I_2\). As \(h_1\) preserves orientation in \(U\) then \(h_1(a_1) = b_2\) and \(h_1(b_1) = a_2\). In \(I_4\), \(g_1^n\) fixes only \(\partial I_4\) and \(b_2\) is attracting, \(a_2\) is repelling. Hence the action of \(h^{-1}_1 g_1^m h_1\) on \(I_2\) fixes only \(\partial I_2\) and has \(a_1\) attracting, \(b_1\) repelling.

Hence the actions of \(g_1^n\) and \(h^{-1}_1 g_1^m h_1\) are incompatible in \(I_2\) and therefore they cannot be equal. Consider the other 3 cases: When \(h_1(I_1) = I_3\) but \(h_1\) reverses orientation in \(U\), then \(h_1(I_2) = I_2\) flipping the endpoints. The same argument produces a contradiction. When \(h_1(I_1) = I_7\) and \(h_1\) preserves the orientation then \(h_1(I_2) = I_8\) but it sends the attracting fixed point in \(I_2\) (of \(g_1\)) to the repelling one in \(I_8\) (of \(g_1\)) again contradiction. Finally if \(h_1(I_1) = I_7\) and \(h_1\) reverses orientation in \(U\) then \(h_1(I_8) = I_8\), flipping the endpoints, again a contradiction.

As all cases are impossible this finally shows that \(h_1(l_1)\) and \(l_1\) intersecting transversely is impossible. The same proof applied to \(D_+\) shows that \(\theta(h')(l_2)\) transversely intersecting \(l_2\) for some \(h'\) in \(\pi_1(M)\) is impossible. Hence both \(D_-\) and \(D_+\) generate laminations in \(M\):

**Lemma 5.6.** If a leaf of \(D_-\) transversely intersects a leaf of \(D_+\) then both the sets \(\mathcal{C}(\pi(D_- \times \mathbb{R}))\) and \(\mathcal{C}(\pi(D_+ \times \mathbb{R}))\) are embedded laminations in \(M\) which are transverse to each other — here \(\mathcal{C}\) denotes closure in \(M\). In particular option \(C\) implies option \(A\) for both \(D_-\) and \(D_+\).
§6. The two transverse laminations

This finishes the proof of theorem 5.1

6 The two transverse laminations

In the previous section we proved that if $\mathcal{F}$ is $\mathbb{R}$-covered with hyperbolic leaves, then either $M$ is Seifert fibered, or a torus bundle over $S^1$ or there is a lamination transverse to $\mathcal{F}$. We use the constructions and notations of the previous section. In this section we show that in the atoroidal case both $\mathcal{D}_-$ and $\mathcal{D}_+$ produce laminations transverse to $\mathcal{F}$ which are also transverse to each other. Unless otherwise stated, from now on assume that $M$ is homotopically atoroidal. We first obtain some general results about laminations transverse to $\mathbb{R}$-covered foliations and then use these results to study the laminations constructed in the previous section.

We say that a lamination $\mathcal{G}$ transverse to a foliation $\mathcal{F}$ is a lamination by geodesics if leaves of $\mathcal{G}$ intersect leaves $F$ of $\mathcal{F}$ in geodesics of $F$. Now restrict to $\mathbb{R}$-covered foliations with hyperbolic leaves. If in addition for each leaf $G$ of $\mathcal{G}$, the ideal points of $G \cap F$ as $F$ varies in $\mathcal{F}$ define two leaves of the vertical foliation of $\mathcal{A}$, then we say that $\mathcal{G}$ is a universal lamination by geodesics - like the laminations constructed in the last section. First we analyse complementary regions of general universal geodesic laminations in a series of results from proposition 6.1 till lemma 6.4.

If $F$ is a leaf of $\bar{\mathcal{F}}$ (or of $\mathcal{F}$) let $\bar{\mathcal{G}}_F$ ($\mathcal{G}_F$ respectively) denote the lamination by geodesics induced by $\bar{\mathcal{G}}$ (or $\mathcal{G}$) in $F$.

Proposition 6.1. Let $\mathcal{F}$ be $\mathbb{R}$-covered with hyperbolic leaves and $M$ homotopically atoroidal. Suppose that $\mathcal{G}$ is an universal lamination by geodesics transverse to $\mathcal{F}$. Then for any leaf $F$ of $\bar{\mathcal{F}}$, the complementary regions of $\bar{\mathcal{G}}_F$ (that is, $\bar{\mathcal{G}}_F \cap F$) are all finite sided ideal polygons in $\mathcal{F}$ with an upper bound on the number of boundary sides. Complementary regions of $\mathcal{G}$ in $M$ are either solid tori or solid Klein bottles bounded by finitely leaves of $\mathcal{G}$.

Proof. We first show that geodesics which are boundary leaves of $\bar{\mathcal{G}}_F$ and which get sufficiently close in $F$ are asymptotic in $F$:

Lemma 6.2. There is positive $\epsilon$ so that for any $F$ in $\bar{\mathcal{F}}$ then any neck of size smaller than $\epsilon$ in a complementary region of $\mathcal{G}_F$ will produce asymptotic leaves in $F$. Similarly for $\mathcal{G}$ and $\bar{\mathcal{F}}$.

Proof. Here a neck is a geodesic segment $\alpha$ in a leaf $F$ of $\bar{\mathcal{F}}$ so that its boundary is in $\bar{\mathcal{G}}_F$ but the interior of $\alpha$ is disjoint from $\bar{\mathcal{G}}_F$. There are two leaves $u_1, u_2$ of $\bar{\mathcal{G}}_F$ through the endpoints of $\alpha$ and the goal is to show that if $\alpha$ has small length then $u_1, u_2$ are asymptotic. Suppose the lemma is not true. Find necks of size smaller than $1/i$ from points in leaves

$$s_i, r_i \text{ of } \bar{\mathcal{G}}_{L_i}, \text{ with } L_i \in \bar{\mathcal{F}}, \text{ but } s_i, r_i \text{ not asymptotic in } L_i.$$ 

Since $s_i, r_i$ are not asymptotic we may assume these are the closest points in $L_i$ from $s_i$ to $r_i$. Since $s_i, r_i$ are distinct they eventually diverge (in some direction), so find necks $\xi_i$ of size 1 between $s_i, r_i$ with angles between the necks and the geodesics $s_i, r_i$ bounded away from 0 and $\pi$. Let $p_i$ be the middle points of $\xi_i$. Let $f_i$ in $\pi_1(M)$ with $f_i(p_i)$ converging to $p$ with necks $f_i(\xi_i)$ also converging, and so that the geodesics $f_i(s_i), f_i(r_i)$ converge to leaves $s, r$ in $L_0$ of $\bar{\mathcal{F}}$. Here $L_0$ is the limit of $f_i(L_i)$.

If $s, r$ are not asymptotic $L_0$, there is a minimum positive distance $b_0$ between them and they diverge from each other in each direction. For nearby $f_i(p_i)$ the leaves $f_i(s_i), f_i(r_i)$ also get roughly $b_0$ away from each other and then start to diverge from each other - this is all happening in a compact set near the leaf $L_0$. But in hyperbolic geometry, once a pair of geodesics starts to diverge from each other they will never get close anymore. Hence for $i$ big the minimum distance between $f_i(s_i)$ and $f_i(r_i)$ in $f_i(L_i)$ is close to $b_0$. This contradicts $d_{L_i}(s_i, r_i)$ converging to 0. We conclude that $s, r$ are in fact asymptotic in $L_0$.

The leaves $f_i(s_i), f_i(r_i)$ are boundary leaves of $\bar{\mathcal{G}}_{f_i(L_i)}$. Using the universal circle identification,

$$(f_i(s_i) \times \mathbb{R}) \cap L_0, \quad (f_i(r_i) \times \mathbb{R}) \cap L_0$$
are also boundary leaves of \( \tilde{\mathcal{G}}_{L_0} \) with necks of size very close to 1 near \( p_0 \). But the only boundary leaves of \( \tilde{\mathcal{G}}_{L_0} \) near \( p_0 \) and with neck size near 1 are \( s, r \), hence \( f_i(s_i) \) is contained in say \( (s \times \mathbb{R}) \) and \( f_i(r_i) \) is contained in \((r \times \mathbb{R})\) for \( i \) big enough.

Equivalently this argument says that nearby gaps of \( \tilde{\mathcal{G}} \) in fact map to each other by moving transversally to \( \mathcal{F} \). This uses the fact that the laminations are made of sets \( l \times \mathbb{R} \) associated to the universal circle - a universal geodesic lamination. This argument that if necks are very near, then the geodesics through the endpoints are contained in the same leaves of \( \tilde{\mathcal{G}} \) will be often used here – we call it the matching boundary effect.

But now the leaves \( s, r \) define the same ideal point in \( S^1_\infty(L_0) \) and again by hypothesis of the proposition this implies that leaves \( f_i(s_i), f_i(r_i) \) also define the same ideal point in \( S^1_\infty(f_i(L_i)) \), that is, \( f_i(s_i), f_i(r_i) \) are asymptotic for \( i \) big enough. This contradicts the fact that \( s_i, r_i \) are not asymptotic in \( L_i \) and finishes the proof of lemma 6.2.

We stress that this works for boundary leaves. In general there are infinitely many pairs of leaves \( s_i, r_i \) of \( \tilde{\mathcal{G}}_F \) which have necks of arbitrarily small size but are not asymptotic in \( F \). But if they are boundary leaves of the same complementary region of \( \tilde{\mathcal{G}}_F \) then they have to be asymptotic.

We now return to the proof of proposition 6.1. If \( \mathcal{G} \) is a foliation proposition 6.1 is trivial. Otherwise consider \( \epsilon' \) much smaller than \( \epsilon/2 \) (\( \epsilon \) as in lemma 6.2). We define an open set \( B_{\epsilon'} \) in \( M \): let \( u \) be a point in a leaf \( E \) of \( \mathcal{F} \). Then

\[ u \in B_{\epsilon'} \quad \text{if} \quad d_E(u, \mathcal{G}_E) < \epsilon', \]

where \( d_E \) is measured in \( E \). Choose \( \epsilon' \) small enough so that \( B_{\epsilon'} \) is not \( M \) and let \( Z \) be a component of the boundary of \( B_{\epsilon'} \). We consider how \( Z \) intersects the foliation \( \mathcal{F} \). Let \( p \) be a point in \( Z \) which is in a leaf \( E \) of \( \mathcal{F} \). If \( p \) is \( \epsilon' \) away from two leaves \( l_1, l_2 \) of \( \mathcal{G}_E \), then \( l_1, l_2 \) are \( 2\epsilon' \) away from each other, which is less than \( \epsilon \). By lemma 6.2, \( l_1 \) and \( l_2 \) are asymptotic in \( E \) and in that direction their distance decreases: any point between them is less than \( \epsilon' \) from at least one of \( l_1 \) and \( l_2 \) and therefore \( Z \) does not intersect that direction anymore. Now consider the opposite direction: in that direction \( l_1, l_2 \) diverge from each other and become more than \( 2\epsilon' \) from each other, this means that the intersection of \( Z \) and \( E \) has a corner at \( p \) and in the diverging direction two arcs of

\[ Z \cap E \quad \text{emanate from} \quad p. \]

On the other hand if \( p \) is not a corner then \( p \) is \( \epsilon \) distant from a single boundary leaf \( l_1 \) of \( \mathcal{G}_E \) and the intersection of \( Z \) with \( E \) tracks this leaf \( l_1 \) nearby. Conclusion: the intersections of \( Z \) with \( E \) track boundary leaves of \( \mathcal{G}_E \) until they hit a corner and start to track another leaf of \( \mathcal{G}_E \) (they can also never hit a corner).

What happens transversely to \( \mathcal{F} \) in a nearby leaf \( L \)? If \( p \) is not a corner point, then for nearby \( L \), the set \( Z \) intersects \( L \) in a curve \( \epsilon' \) away from \( \mathcal{G}_L \) and near the one in \( E \). This means that \( Z \) is a two dimensional manifold near \( p \). If on the other hand \( p \) is a corner point, then \( l_1, l_2 \) are asymptotic in \( E \). For nearby \( L \) there are unique boundary leaves of \( \mathcal{G}_L \) associated to \( l_1, l_2 \) – by the matching boundary effect, and these leaves in \( L \) are also asymptotic. The associated corner point in \( L \) is near the corner point in \( E \), which shows that near \( p \), the set \( Z \) is a two dimensional set. This also shows that \( Z \) is transverse to \( \mathcal{F} \).

This is a crucial point: if boundary leaves of \( \mathcal{G}_E \) could get close to each other without being asymptotic, there are two corners associated to these leaves. Moving transversely to \( \mathcal{F} \) could push those boundary leaves apart from each other. In terms of \( Z \) this would mean two corners coming together and splitting to two curves without corners tracking the two boundary leaves, producing a saddle tangency of \( Z \) and \( \mathcal{F} \).

In addition to being two dimensional it is easy to see by definition that \( Z \) cannot limit on itself transversely: one side would have to be closer to \( \mathcal{G} \). Hence \( Z \) is compact surface transverse to \( \mathcal{F} \), so \( Z \) is either the torus or the Klein bottle. Let \( \mathcal{F}_Z \) be the induced foliation in \( Z \) by \( \mathcal{F} \).
Lemma 6.3. The leaves of $\mathcal{F}_Z$ are closed curves which are null homotopic in their respective leaves of $\mathcal{F}$. The set $Z$ bounds a solid torus or solid Klein bottle in $M$.

Proof. If $Z$ is $\pi_1$-injective then there is a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup in $\pi_1(M)$, contrary to the atoroidal hypothesis. Hence there is a simple closed curve $\gamma$ in $Z$ which is null homotopic in $M$.

First we show there are no Reeb annuli in $\mathcal{F}_Z$. Suppose there is a Reeb annulus $C$ bounded by leaves $\alpha, \beta$, which are the limit of $\alpha_x$ with $x$ converging to infinity. Consider lifts

$$C, \tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}_x, \text{ to } \tilde{M}.$$ 

Since $C$ is a Reeb annulus, the curves $\tilde{\alpha}, \tilde{\beta}$ are in leaves of $\tilde{\mathcal{F}}$ which are not separated from each other. But $\mathcal{F}$ is $\mathbb{R}$-covered, so they are in the same leaf of $\tilde{\mathcal{F}}$, let it be $E$. Then $\tilde{\beta}, \tilde{\alpha}$ do not track the same geodesics of $\tilde{\mathcal{G}}_F$, because they are distinct curves. Now look at nearby leaves $\tilde{\alpha}_x$: by construction they track a chain of consecutively asymptotic boundary leaf geodesics of $\tilde{\mathcal{G}}$ in the respective leaves of $\tilde{\mathcal{F}}$. By the matching boundary effect this chain pulls to $F$ to a chain of consecutively asymptotic geodesics in $\tilde{\mathcal{G}}_F$. But some of them are tracked by $\tilde{\beta}$ and some are tracked by $\tilde{\alpha}$. This can only happen if $\tilde{\beta}$ is equal to $\tilde{\alpha}$ which is impossible. As in the previous lemma this is basically saying that the combinatorics of the intersections of $Z$ with $\mathcal{F}$ do not change transversely to $\mathcal{F}$.

Given that there are no Reeb annuli in $\mathcal{F}_Z$ the curve $\gamma$ is homotopic to one which is either a leaf of $\mathcal{F}_Z$ or transverse to it. A transversal to $\mathcal{F}_Z$ is transverse to $\mathcal{F}$ and as $\mathcal{F}$ is Reebless, the transversal is not null homotopic in $M$. Hence we can assume that $\gamma$ is a leaf $\mathcal{F}_Z$ - by Reebless again, $\gamma$ is null homotopic in its leaf. Nearby leaves of $\mathcal{F}_Z$ are also closed since $\gamma$ has no holonomy. The limit of compact leaves is compact [Ha], so all leaves of $\mathcal{F}_Z$ are closed and bound disks in their respective leaves of $\mathcal{F}$. It now follows that $Z$ bounds a solid torus or solid Klein bottle in $M$. This finishes the proof.

We now finish the proof of proposition 6.1. If $U$ is a complementary region of $\mathcal{G}$, let $\epsilon'$ be small enough so that $B_{\epsilon'}$ does not contain $U$ and let $Z$ be a component of $\partial B_{\epsilon'}$ contained in $U$. Let $E$ be a leaf of $\mathcal{F}$ intersecting $Z$ and $\beta$ a component of the intersection of $E$ and $Z$. By lemma 6.3, $\beta$ is a closed curve in $E$ which is null homotopic in $E$ and tracks boundary leaves of $\mathcal{G}_E$. Hence the associated complementary region in $E$ is a finite sided ideal polygon in $E$. Moving transversely does not change the combinatorics or the number of boundary sides in this polygon and since $Z$ is closed, it follows that the complementary region $U$ is a solid torus or solid Klein bottle. This also shows that for any $F$ in $\tilde{\mathcal{F}}$ any complementary region of $\tilde{\mathcal{G}}_F$ is a finite sided ideal polygon.

We know there are finitely many complementary regions of $\mathcal{G}$. Do the argument in $\tilde{M}$. A complementary region of $\tilde{\mathcal{G}}$ contains an ideal triangle in a leaf of $\tilde{\mathcal{F}}$. Suppose for a moment there are infinitely many complementary regions $\mathcal{V}_i$ which are not equivalent under covering translations. Let $v_i$ be the barycenter of an ideal triangle contained in a leaf $L_i$ of $\tilde{\mathcal{F}}$ and also in $\mathcal{V}_i$. Up to covering translations and a subsequence the $v_i$ converge to a point $v_0$ in a leaf $L_0$. Also

$$d_{L_i}(v_i, \tilde{g}_{L_i}) > c_0,$$

for some positive constant $c_0$, because $v_i$ is the barycenter of an ideal triangle in $L_i - \tilde{g}_{L_i}$. By continuity $v_0$ is not in $\tilde{\mathcal{G}}$ so there is a complementary component $\mathcal{V}_0$ of $\tilde{\mathcal{G}}$ with $v_0$ in $\mathcal{V}_0$. As $\mathcal{V}_0$ is open then $\mathcal{V}_i$ is equal to $\mathcal{V}_0$ for $i$ sufficiently big, contradiction.

Hence there are only finitely many complementary regions of $\mathcal{G}$ in $M$ and there is an upper bound to the number of sides in any complementary region of $\tilde{\mathcal{G}}_F$ for any $F$ in $\tilde{\mathcal{F}}$. This finishes the proof of proposition 6.1.

Lemma 6.4. Let $\mathcal{F}$ be a transversely oriented foliation with hyperbolic leaves in $M$ orientable and $\mathcal{G}$ a minimal, universal lamination by geodesics transverse to $\mathcal{F}$. If $s_1, s_2$ are asymptotic leaves of $\mathcal{G}_F$ then $s_1, s_2$ are in the boundary of a complementary region of $\mathcal{G}_F$. Similarly for asymptotic leaves of $\mathcal{G}$. 


Proof. Here we use the notation of the previous section where $F^*$ is the distinguished leaf. In addition if $g$ is in $\pi_1(M)$ then $g$ acts in $\mathcal{U}$ and geodesics of $F^*$ by $\theta(g)$. We also use the notation $l \times \mathbb{R}$ for any geodesic $l$ in a leaf of $\mathcal{F}$. Let $V_0$ be $\tilde{\mathcal{G}}_{F^*}$. It suffices to prove the lemma for $V_0$.

If the lemma is false find $s_1$ boundary leaf of a complementary region $Q$ of $V_0$ and $s_2$ not in $\partial Q$ but asymptotic to $s_1$, see fig. ??, a. Assume that $s_1$ separates $s_2$ from $Q$. Let $u_1$ be the common ideal point of $s_1$, $s_2$ and $u_2$ the other ideal point of $s_1$. Let $s_3$ be the other boundary leaf of $Q$ with ideal point $u_2$.

Fix $z$ in $s_1$. Let $\epsilon$ given by lemma 6.2. Here $s_1$ is not isolated in the $s_2$ side, since it is isolated in the $s_3 \times \mathbb{R}$ and $\mathcal{G}$ is minimal. Let $w_i$ be a sequence in $s_1$ converging to $u_2$. As $\mathcal{G}$ is minimal, there are $v_i$ in $(s_1 \times \mathbb{R})$ which are a bounded distance (depending on $\epsilon$) from $w_i$ in $(s_1 \times \mathbb{R})$ and

$$f_i \text{ in } \pi_1(M) \text{ with } d(f_i(v_i), z) < \epsilon/2, \text{ but } f_i(u_i) \text{ not in } (s_1 \times \mathbb{R}).$$

As $s_1$, $s_3$ are asymptotic and $d(v_i, w_i)$ is bounded, then there are $y_i$ in $(s_3 \times \mathbb{R})$ with $d(y_i, v_i)$ converging to zero. So $d(f_i(y_i), z)$ is smaller than $\epsilon/2$ for $i$ big enough. We may assume that $f_i(y_i)$ and $f_i(v_i)$ are in $V_0$. By the property of $\mathcal{U}$, the leaves of $V_0$ through $f_i(y_i)$ and $f_i(v_i)$ are asymptotic in the $u_1$ direction, see fig. ??, a. Since $s_1$, $s_3$ are asymptotic in the $u_2$ direction this implies that

$$\theta(f_i)(u_2) = u_1 \text{ for } i \text{ big enough}.$$

To simplify notation fix $i$ sufficiently big and let $h = \theta(f_i)$. By assumption $h$ preserves the orientation of $\mathcal{U}$. It follows that $h(Q)$ is asymptotic to $Q$ along $s_1$, $h(s_1)$ and in the same way $h^2(Q)$ is asymptotic to $h(Q)$ along $h(s_1)$, $h^2(s_1)$, see fig. ??, b. This shows that $h(s_1)$ is asymptotic to both $s_1$ and $h^2(s_1)$. Recall that $h(s_1)$ is not isolated in $V_0$. But it is isolated on the $h(Q)$ side. On the other side it is asymptotic to $s_1$, $h^2(s_1)$ both in $V_0$, so again it is isolated. This contradiction shows that $s_1$, $s_2$ cannot be asymptotic and finishes the proof of the lemma.

Notice that $M$ atoroidal is not needed for this lemma.

**Definition 6.5.** Let $Q_1$ be a sequence of distortion parallelepipeds in $\mathcal{M}$ so that bottoms are ideal quadrilaterals in leaves of $\mathcal{F}$ with cross ratios converging to 0, while tops are ideal quadrilaterals in leaves of $\mathcal{F}$ with cross ratios converging to 1. We call $Q_1$ a shrinking sequence of distortion parallelepipeds. For simplicity we sometimes omit the word shrinking and refer to $Q_1$ as a sequence of distortion parallelepipeds.

We now analyse the laminations constructed in the last section in detail in particularly in relation to the options in step 5 of the proof of theorem 5.1. The eventual goal of this section is

**Goal:** Show that option $B$ does not occur.

Hence option C will occur and as shown in the previous section option A holds for both $\mathcal{D}_-$ and $\mathcal{D}_+$ and there will be two transverse laminations in $M$. We use the notations and constructions of section 5. In order to analyse this situation recall the options in step 5 to produce laminations by geodesics in $M$:

- If leaves of $\mathcal{D}_-$ do not self intersect transversely (option A which can also happen when option B occurs) then let $\mathcal{G}_-$ be the closure of $\pi(\mathcal{D}_- \times \mathbb{R})$ and this is a lamination.

- Suppose there are transverse self intersections of leaves of $\mathcal{D}_-$. Then since option C implies option A, we have that no leaf of $\mathcal{D}_-$ transversely intersects a leaf of $\mathcal{D}_+$. Let $A$ be the connected component of the union of leaves of $\mathcal{D}_-$ containing $l_1$ and let $C$ be its convex hull. Then $C$ is not all of $\mathcal{H}^2$ by hypothesis here. Let $B$ be $\partial C$ and in this case let $\mathcal{G}_-$ be the closure of $\pi(C \times \mathbb{R})$, also a lamination. Here we are in option B.

Similarly for $\mathcal{D}_+$ producing a lamination $\mathcal{G}_+$. Hence there are always two laminations, which a priori may be equal. In addition let

- $\mathcal{G}_m^-$ be a minimal sublamination of $\mathcal{G}_-$ and
\[ \mathcal{G}_m \text{ a minimal sublamination of } \mathcal{G}_+. \]

The 4 laminations \( \mathcal{G}_-, \mathcal{G}_+, \mathcal{G}^m_- \) and \( \mathcal{G}^m_+ \) will be fixed from now on. As we will see later \( \mathcal{G}^m_- \) and \( \mathcal{G}^m_+ \) are uniquely defined and in the end we will prove that \( \mathcal{G}^m_- \) is equal to \( \mathcal{G}_- \) and similarly for \( \mathcal{G}^m_+ \). Let

\[
V_\pm = \mathcal{G}_\pm \cap F^\ast, \quad V^m_\pm = \mathcal{G}^m_\pm \cap F^\ast,
\]

all laminations in \( F^\ast \) (could be foliations too). Also \( V^m_- \) is contained in \( V_- \) and \( V^m_+ \) is contained in \( V_+ \).

We first derive general properties of leaves of \( V^m_-, V^m_+ \). Let \( l \) be a leaf of \( V^m_- \). If \( \mathcal{D}_- \) does not transversely self intersect, then \( \mathcal{G}^m_- \) is contained in \( \mathcal{D}(\pi(\mathcal{D}_- \times \mathbb{R})) \) so there are leaves \( u_i \) in \( \mathcal{D}_- \) converging to \( l \). On the other hand, if \( \mathcal{D}_- \) has transverse self intersections, say \( l_1 \) with \( l_2 \); then \( l_1 \) does not intersect \( \mathcal{G}^m_- \), so \( l_1 \) has to be in a complementary region of \( V^m_- \) – a finite sided ideal polygon. Therefore a ray of \( l_1 \) is asymptotic to a leaf of \( V^m_- \). But since \( \mathcal{G}^m_- \) is minimal this implies that \( \pi(l_1 \times \mathbb{R}) \) limits on every leaf of \( \mathcal{G}^m_- \), so again there are leaves \( u_i \) of \( \mathcal{D}_- \) converging to \( l \). The same happens for \( \mathcal{D}_+ \) so if \( l \) is a leaf of \( V^m_+ \) there are leaves \( v_i \) of \( \mathcal{D}_+ \) converging to \( l \). Lemma 5.5 now implies:

**Conclusion 3:** If \( l \) is a leaf of \( V^m_- \) there is a shrinking sequence of parallelepipeds \( Q_i \) with bottoms in \( F^\ast \) so that two opposite sides of the bottoms converge to \( l \), the cross ratio of bottoms (respectively tops) converges to 0 (respectively 1). In the same way if \( l \) is a leaf of \( V^m_+ \) there are parallelepipeds \( Q_i \) with the same cross ratio characteristics with tops converging to \( l \). The same holds for intersections of \( \mathcal{G}^m_- \), \( \mathcal{G}^m_+ \) with any leaf \( F \) of \( \mathcal{F} \).

If \( Q \) is an ideal polygon in \( F^\ast \) (or \( \mathbb{H}^2 \)) let \( \partial_{\infty} Q \) be the ideal points of \( Q \).

We will now prove a crucial technical lemma which will help in analysing option B later on and also help produce a transverse pseudo-Anosov flow to \( \mathcal{F} \). Consider the case that \( \mathcal{G}^m_- \) is not a foliation. Recall that if \( g \) is in \( \pi_1(M) \), then \( \theta(g) \) acts in \( \mathcal{U} \) and in convex sets of \( F^\ast \). Let \( C_1 \) be a complementary region of \( V^m_- \). By proposition 6.1, \( \pi(C_1 \times \mathbb{R}) \) is a solid torus or solid Klein bottle, and the core is a curve transverse to \( \mathcal{F} \). So there is a non trivial \( g \) in \( \pi_1(M) \) with \( \theta(g)(C_1) \) equal to \( C_1 \). Taking powers we may assume \( \theta(g) \) fixes all points of \( \partial_{\infty} C_1 \). Here we are identifying \( \hat{S}_{\infty}^1(F^\ast) \) with \( U \) so \( \theta(g) \) acts on \( \hat{S}_{\infty}^1(F^\ast) \). There are at least 3 points on \( \partial_{\infty} C_1 \), therefore \( g \) acts freely in \( \hat{H} \). Up to taking inverse assume that \( g \) is monotone decreasing in \( \hat{H} \), that is \( F^\ast \) is in the front of \( g(F^\ast) \).

**Lemma 6.6.** Suppose that \( \mathcal{G}^m_- \) is not a foliation and \( C_1 \) is a complementary region of \( V^m_- \). Let \( g \) in \( \pi_1(M) \) non trivial with \( \theta(g)(C_1) \) equal to \( C_1 \) and \( \theta(g) \) fixing all ideal points of \( C_1 \). Suppose that \( g \) is monotone decreasing in \( \hat{H} \). Let \( J_1 \) be a component of \( \mathcal{U} - \partial C_1 \). Then \( \theta(g) \) acts as a contraction in \( J_1 \) with a single fixed point. Similarly if \( \mathcal{G}^m_+ \) is not a foliation and \( C_1 \) is a complementary region of \( V^m_+ \) with \( g \) in \( J_1 \) as above \( (g \text{ acts decreasing in } \hat{H}) \) then \( \theta(g^{-1}) \) acts as a contraction in \( J_1 \) with a single fixed point.

**Proof.** First we do the proof for \( \mathcal{G}^m_- \). Let \( s_0 \) be a side of \( C_1 \), hence \( \theta(g)(s_0) \) is equal to \( s_0 \). Let \( e_1, e_2 \) be the ideal points of \( s_0 \) and

\[
J_1 = \text{closure of interval of } \mathcal{U} - \{ e_1, e_2 \} \text{ not containing other ideal points of } C_1.
\]

We analyse the action of \( \theta(g) \) in \( J_1 \). Notice \( \theta(g) \) fixes \( C_1 \) hence fixes \( e_1, e_2 \). Notice \( s_0 \) is isolated on the \( C_1 \) side, so not isolated on the other side. Choose \( s_1 \) in \( V^m_- \) arbitrarily close to \( s_0 \) hence with ideal points in \( J_1 \). By conclusion 3 above there are parallelepipeds \( Q_i \) with bottoms \( \mathcal{R}_i \) having two opposite sides converging to \( s_1 \) and tops \( S_i \) with cross ratios of bottoms converging to 0 and cross ratios of tops converging to 1. Let \( F_i \) in \( \mathcal{F} \) with \( S_i \) contained in \( F_i \), the \( t \) here depends on \( i \). Since \( g \) is monotone decreasing in \( \hat{H} \), there is a unique positive \( n \) so that either

\[
F_i = g^{-n}(F_0) \quad \text{or} \quad F_i \text{ is between } g^{-n}(F_0) \text{ and } g^{-(n+1)}(F_0).
\]

(notice that \( g^{-n}(F_0) \) is above \( F_0 \)). The quadrilateral \( \mathcal{R}_i \) has ideal points \( e_3, e_4, e_5, e_6 \) in \( \mathcal{U} \), so that \( e_3 e_4, e_5 e_6 \) are the sides close to each other and close to \( s_1 \), see fig. ??, a. By lemma 6.4, if \( s_1 \) is
asymptotic to $s_0$ then $s_0$ and $s_1$ are boundary sides of a complementary region of $V_0$. But then $s_0$ would be isolated on both sides, contradiction to minimality of $\mathcal{G}$. Hence once $s_1$ is fixed, then for $i$ big enough all the ideal points $e_1, e_4, e_5, e_6$ are in $J_1 - \partial J_1$. Choose the points so that $s_2 = \frac{1}{2} s_6$ is farther from $s_0$ than $\frac{1}{2} s_4$. Notice $t, n, e_3, \ldots, e_6$ all depend on $i$, which is omitted for simplicity.

Map $\mathcal{S}_i$ back by $g^n$, pushing it down in the leaf space. Then $g^n(\mathcal{S}_i)$ is either in $F^*$ or is between $F^*$ and $g^{-1}(F^*)$. Also $g^n(\mathcal{S}_i)$ is an ideal quadrilateral with cross ratio very close to 1. Suppose that both $\theta(g^n)(e_5)$ and $\theta(g^n)(e_6)$ are very close to $e_5, e_6$ respectively. Projecting $g^n(\mathcal{S}_i)$ to $F^*$ using $\mathcal{U}$, that is,

$$(g^n(\mathcal{S}_i) \times \mathbb{R}) \cap F^*,$$

we get an ideal quadrilateral in $F^*$ with cross ratio very close to 0. This is because $\theta(g^n)$ fixes $e_1$ and $e_2$ and $\theta(g^n)(e_3)$ is between $e_1$ and $\theta(g^n)(e_5)$ and $\theta(g^n)(e_4)$ between $e_2$ and $\theta(g^n)(e_6)$. The same will happen for the cross ratio of the intersection

$$(g^n(\mathcal{S}_i) \times \mathbb{R}) \cap F, \quad \text{for any } F \text{ between } F^* \text{ and } g^{-1}(F^*).$$

All of them are very near the intersection of $F$ and $s_0 \times \mathbb{R}$. This contradicts the fact that cross ratio of $g^n(\mathcal{S}_i)$ is very close to 1. Hence at least one of $\theta(g^n)(e_5)$ and $\theta(g^n)(e_6)$ is very far from $e_5$ or $e_6$ respectively.

One option is say that $\theta(g^n)(e_5)$ is very close to $e_2$. But if this keeps happening as $s_1$ gets closer to $s_0$ and $i$ converges to infinity, then the only possibility for the dynamics of $\theta(g^n)$ in $J_1$ is that $\theta(g^n)$ has a repelling fixed point in $e_1$, an attracting fixed point in $e_2$ and no other fixed points. In fact since $\theta(g)(J_1)$ is equal to $J_1$, this implies that $\theta(g)$ acts in $J_1$ with the same dynamics. Take $s$ leaf of $V^m$ very close to $s_0$ with ideal points $a, b, a$ close to $e_1$, $b$ close to $e_2$. By lemma 6.4, $a$ is not $e_1$ and $b$ is not $e_2$. Hence for $a, b$ sufficiently close to $e_1, e_2$ respectively, the points

$$e_1, a, \theta(g)(a), b, \theta(g)(b), e_2,$$

are circularly ordered in $J_1$. In other words the endpoints of $s$ separate the endpoints of $\theta(g)(s)$ in $\mathcal{U}$, or $s$ intersects $\theta(g)(s)$ transversely. This contradicts the fact that $\mathcal{G}^m_0$ is a lamination.

It follows that $J_1$ contains an interval $J_2$ bounded by $e_5, e_6$, with $\theta(g^n)(J_2)$ a subinterval of $J_2$. The cross ratio of $\theta(g^n)(\mathcal{S}_i)$ is very close to 1, so this quadrilateral is very thin in the other direction, see fig. ??, b. Since $\theta(g^n)(e^2), \theta(g^n)(e^6)$ cannot be near $e_5$ or $e_6$, it follows that $\theta(g^n)(J_2)$ is a very small interval contained in $J_2$.

This is the fundamental property of a curve which is in the boundary of a complementary region of $V^m$ so that it is the limit of bottoms of a shrinking sequence of distortion parallelepipeds. Choosing now $s_1$ closer and closer to $s_0$ and cross ratio of $R_i$ converging to 0, cross ratio of $S_i$ converging to 1, we can get $e_5$ arbitrarily close to $e_1$ and $e_6$ arbitrarily close to $e_2$. In addition the subinterval of $J_1$ from $\theta(g^n)(e_5)$ to $\theta(g^n)(e_6)$ is shrinking to a point (recall here that $n$ varies with $s_1$ and $i$). This yields the conclusion:

**Conclusion 4:** Using that $s_0$ is the limit of bottoms of distortion parallelepipeds, we obtain that $\theta(g)$ acts in $J_1 - \partial J_1$ as a contraction with a unique fixed point $y$ not in $\partial J_1$.

This proves the first part of the lemma. If on the other hand $C_1$ is a complementary component of $V^m_+$ (assumed not to be $F^*$) and $s_0, J_1$ are defined as above, the same analysis applies. But now there is a sequence of distortion parallelepipeds with the tops converging to $s_0$. The difference is that $R_i$ is below $S_i$, hence use a translate $g^n$ to bring $R_i$ closer to $F^*$, with $m$ negative. Using the same arguments as above, we conclude that $\theta(g^{-1})$ acts as a contraction in $J_1 - \partial J_1$ or that $\theta(g)$ acts as an expansion in $J_1 - \partial J_1$. This finishes the proof of lemma 6.6.

We will now show that option B does not occur.

**Proposition 6.7.** Suppose that $M$ is orientable and $\mathcal{F}$ is transversely orientable. There is a leaf of $\mathcal{D}_-$ intersecting a leaf of $\mathcal{D}_+$ transversely.
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Proof. Suppose that this is not true. Then option B holds and it implies that no leaf of \( \mathcal{G}_- \) transversely intersects a leaf of \( \mathcal{G}_+ \). If \( \mathcal{G}_- \) (or \( \mathcal{G}_+ \)) is a foliation then \( \mathcal{G}_- \) is minimal, for a non-trivial sublamination would have complementary regions in leaves of \( \mathcal{F} \) which are finite sided ideal polygons and could not be filled with a foliation by geodesics. This also implies \( \mathcal{G}_- \), \( \mathcal{G}_+ \) are equal. In this case all of the laminations

\[ \mathcal{G}_-, \mathcal{G}_+, \mathcal{G}_-^m \text{ and } \mathcal{G}_+^m \]

are the same.

If none of \( \mathcal{G}_-, \mathcal{G}_+ \) are foliations then a complementary region of \( V_m^* \) is a finite sided ideal polygon \( Q \) in \( F^* \). Since \( \mathcal{G}_-^m \) does not intersect \( \mathcal{G}_+^m \) transversely then \( V_m^* \) cannot intersect \( \partial Q \) transversely. So any intersection of \( V_m^* \) with \( Q \) is a geodesic in the interior of \( Q \). There are only finitely many such geodesics and these would be isolated contradicting minimality of \( \mathcal{G}_+^m \). Hence \( \mathcal{G}_-^m \) does not intersect the interior of \( Q \) and so \( \mathcal{G}_-^m \) is contained in \( \mathcal{G}_-^m \). Similarly \( \mathcal{G}_+^m \) is contained in \( \mathcal{G}_+^m \) and so they are equal. These arguments also show that in general \( \mathcal{G}_-^m \) and \( \mathcal{G}_+^m \) are uniquely defined laminations. In any case we proved:

Fact: In option B, then \( \mathcal{G}_-^m \) is equal to \( \mathcal{G}_+^m \).

This is the fact that will lead to a contradiction.

Suppose first that \( \mathcal{G}_- \) and \( \mathcal{G}_+ \) are not foliations. Then \( \mathcal{G}_-^m, \mathcal{G}_+^m \) are also not foliations. Let \( C_1 \) be a complementary region of \( V^m \) and \( s_0 \) a boundary leaf of \( C_1 \). Let \( g \) be nontrivial in \( \pi_1(M) \) with \( \theta(g) \) fixing all points in \( \partial \infty C_1 \) and let \( J_1 \) be a component of \( \mathcal{U} - \partial \infty C_1 \) with endpoints the ideal points of \( s_0 \). Since \( s_0 \) is a leaf of \( V^m \), the first part of lemma 6.6 shows that \( \theta(g) \) acts as a contraction in \( J_1 \). As \( \mathcal{G}_-^m \) is equal to \( \mathcal{G}_+^m \), then \( s_0 \) is also a leaf of \( V^m \). The second part of lemma 6.6 implies that \( \theta(g^{-1}) \) acts as a contraction in \( J_1 \).

These two conclusions are contradictory, so we obtain that \( D_- \) and \( D_+ \) not intersecting transversely is impossible when at least one of \( \mathcal{G}_- \) or \( \mathcal{G}_+ \) is not a foliation.

The next proposition shows that none of \( \mathcal{G}_-, \mathcal{G}_+, \mathcal{G} \) can be foliations, so this finishes the proof of proposition 6.7.

Proposition 6.8. The lamination \( \mathcal{G}_- \) (or \( \mathcal{G}_+ \)) cannot be a foliation in \( M \).

Proof. This is presented separately because here we do not a priori assume that \( D_- \) and \( D_+ \) have no transversal intersection. It will be used later as well, see remark 1. Also no orientability conditions here.

Suppose that say \( \mathcal{G}_- \) is a foliation in \( M \). Then as seen before \( \mathcal{G}_- \) is minimal, hence \( \mathcal{G}_-^m = \mathcal{G}_- \). Since

\[ V_- = \mathcal{G}_- \cap F \]

is a foliation by geodesics in \( F^* \) then its leaf space is Hausdorff and hence homeomorphic to the reals. We analyse this in detail. Fix \( q' \) in \( F^* \), let \( s' \) be the leaf of \( V_- \) through \( q' \). Let \( \gamma \) be a finite transversal to \( V_- \) in \( F^* \) starting at \( q' \) and look at all geodesics of \( V_- \) through \( \gamma \). Suppose one is asymptotic to \( s' \) in the direction of the ray \( r' \) of \( s' \). Then all leaves between these two are also asymptotic to \( s' \) and hence in any case there is a direction so no nearby leaf is asymptotic to \( s' \) (the opposite direction to \( r' \) if there is \( r' \)). As points in \( s' \) escape in that direction their distance to nearby leaves grows without bound. For any natural \( n \) there are points \( q_n \) in these nearby leaves of \( V_- \) which are centers of balls \( B_n \) of \( F^* \) of radius \( n \) so that all leaves in \( B_n \) will eventually intersect a subsegment of \( \gamma \) of length less than \( 1/n \). Up to subsequence choose \( f_n \) in \( \pi_1(M) \) with \( f_n(q_n) \) converging to \( q_0 \) and \( q_0 \) in a leaf \( L \) of \( \mathcal{F} \). Let \( s \) be the leaf through \( q_0 \) of the lamination \( \mathcal{G}_- \cap L \). Let \( v \) another leaf of this lamination. If \( s \) and \( v \) are not asymptotic there is a minimum distance between them which is achieved in a fixed distance from starting points. Pairs of geodesics limiting on \( s \) and \( v \) will also have a minimum distance between them close to this distance, contradiction to the construction.

Conclusion — All leaves of \( \mathcal{G}_- \cap L \) are asymptotic in \( L \) defining a unique ideal point in \( S^1_\infty(L) \).
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As \( \mathcal{G}_- \) is a universal geodesic lamination the same holds for all leaves of \( \tilde{\mathcal{F}} \) and let \( u \) in \( S^{1}_{\infty}(F^*) \) be the distinguished ideal point in \( F^* \). Hence leaves of \( \tilde{\mathcal{G}}_- \) are described exactly as \( s^* \times \mathbb{R} \), where \( s^* \) is an arbitrary geodesic in \( F^* \) with one ideal point \( u \). Every point in \( S^{1}_{\infty}(F^*) \) is an ideal point of some leaf in \( \mathcal{V}_- \).

We claim that this implies that \( \mathcal{G}_- \) and \( \mathcal{G}_+ \) are equal. Let \( l^* \) be a leaf of \( \mathcal{V}_+ \). Then \( l^* \) is asymptotic to a leaf \( l' \) of \( \mathcal{V}_- \) so the angle between \( l^*, l' \) converges to 0. Take a limit in \( M \) and obtain \( F \) in \( \tilde{\mathcal{F}} \) and a common leaf of \( \mathcal{G}_- \cap F \) and \( \mathcal{G}_+ \cap F \), so a common leaf of \( \mathcal{G}_+ \) and \( \mathcal{G}_- \). As \( \mathcal{G}_- \) is minimal, then \( \mathcal{G}_- \) is contained in \( \mathcal{G}_+ \), which implies that \( \mathcal{G}_- \) is equal to \( \mathcal{G}_+ \) (because \( \mathcal{G}_- \) is a foliation). Hence all laminations

\[ \mathcal{G}_-, \mathcal{G}_+, \mathcal{G}_m \text{ and } \mathcal{G}_m^+ \]

are equal and they are foliations. In that case let \( V_0 \) be \( \mathcal{V}_- \) (same as \( \mathcal{V}_+ \) and so on).

The proof now is similar to arguments in lemma 6.6. However in that situation there were complementary regions of \( \mathcal{V}_- \), and these naturally produced covering translations acting freely on \( \mathcal{H} \) and leaving invariant the complementary region and one needs to first find appropriate \( g \) acting freely in \( \mathcal{H} \). First lift to a finite regular cover so that the manifold is orientable and all foliations are transversely oriented and for simplicity in this proof we assume they are the original \( M, \mathcal{G}_-, \mathcal{G}_+ \). Notice \( \mathcal{G}_-, \mathcal{G}_+ \) are still minimal since they are foliations. Notice \( \mathcal{U} \) is still the same and the action of \( \pi_1(M) \) on \( \mathcal{U} \) is still not uniformly quasisymmetric because of compactness of the finite cover.

Let \( \zeta_i \) be a leaf of \( \mathcal{V}_- \). Then \( \zeta_i \) is the limit of bottoms \( \mathcal{R}_i \) of a shrinking sequence of distortion parallelepipeds with cross ratios of \( \mathcal{R}_i \) converging to 0. There are 2 points in \( \partial \mathcal{R}_i \) very close to the distinguished point \( u \) of \( \mathcal{U} \).

Suppose first that \( u \) is in the interior of the small interval of \( \mathcal{U} \) with these endpoints. We split the quadrilateral into thinner quadrilaterals: cover the quadrilateral \( \mathcal{R}_i \) by two thin quadrilaterals \( \mathcal{Q}_1, \mathcal{Q}_2 \) both of which have ideal point in \( u \) and another in \( v \), see fig. ??, a. There are two corresponding parallelepipeds \( \mathcal{V}_1, \mathcal{V}_2 \) with tops \( S_1, S_2 \) in a leaf \( L \) of \( \tilde{\mathcal{F}} \). For simplicity we omit the dependence of \( L \) on \( i \). We now choose the point \( v \) carefully so that at least one of \( S_1, S_2 \) has cross ratio very close to 1. Let \( u'_1, u'_2, u', v' \) be the points in \( S^{1}_{\infty}(L) \) corresponding to \( u_1, v_1, u, v \) respectively under the universal circle \( \mathcal{U} \), see fig. ??, b. The top of the original parallelepiped is a quadrilateral \( S \) very thin in the other direction, see fig. ??, b. If for example \( u' \) is not very near either of \( u'_1, u'_2 \), then choose \( v' \) very close to one of the other ideal points of \( \mathcal{T} \), see fig. ??, b. Then one of \( S_1 \) or \( S_2 \) is thin in the picture \( S_2 \) is thin. Conversely if \( u' \) is very near \( u'_1 \) or \( u'_2 \) choose \( v' \) in the middle between the other ideal points of \( \mathcal{T} \). Regardless one obtains a thin quadrilateral in the other direction.

If on the other hand the small interval from \( u_1 \) to \( u_2 \) does not contain \( u \), then enlarge \( \mathcal{R}_i \) to include \( u \). For high enough \( i \), the new \( \mathcal{R}_i \) will still have cross ratio very close to 0. But clearly the top of the associated parallelepiped is an ideal quadrilateral which is even thinner, so its cross ratio is even closer to 1.

A priori this process has changed \( \mathcal{G}_-, \mathcal{G}_+ \), so let us consider this carefully. The splitting or enlarging does not distort the bottoms \( \mathcal{R}_i \) substantially, so \( \mathcal{G}_- \) is not changed. Therefore \( \mathcal{G}_- \) is still a foliation. Since \( \mathcal{G}_- \) is a foliation, then regardless of what the new \( \mathcal{D}_- \) is, the first part of the proof implies that \( \mathcal{G}_+ \) is equal to \( \mathcal{G}_- \). Hence \( \mathcal{G}_+ \) is not changed either. This fact will be used in the arguments below.

**Conclusion** – Any leaf of \( \mathcal{V}_- \) is a limit of bottoms \( \mathcal{R}_i \) of a shrinking sequence of distortion parallelepipeds with all ideal quadrilaterals \( \mathcal{R}_i \) having a vertex in \( u \).

In order to mimic the proof of lemma 6.6 we first construct a suitable \( g \) in \( \pi_1(M) \) with at least 3 fixed points in \( \mathcal{U} \). Put any orientation in \( \mathcal{U} \). Let ideal points of \( \mathcal{R}_i \) as positively oriented in \( \mathcal{U} \) be denoted by

\[ u_1, u_2, u_3, u_4, \]

where \( u_1 \) is equal to \( u \). Up to subsequence and choosing orientation to \( \mathcal{U} \) assume \( u_2 \) is very close to \( u_1 \), see fig. ??, a. We are still using the fixed leaf \( \zeta_1 \) introduced before. The points \( u_j \) are fixed in this proof.
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Now choose a leaf $\zeta_2$ of $V_+$ having an ideal point in the interval defined by $u_2$ and $u_3$, not containing $u$ and not near $u_2$ or $u_3$, see fig. ??, a. This is possible since $G_+$ is equal to $G_-$. The tops $S_i$ of the sequence of distortion parallelepipeds have cross ratio very close to 1. For $i$ big enough there is a covering translation $g$ so that $\theta(g)(R_i)$ is very close to $\zeta_2$ and with cross ratio very close to 1. This is because $\zeta_2$ is a leaf of $G_+$. The element $g$ depends on $\zeta_1, \zeta_2$ and $i$. Notice that $\theta(g)$ fixes $u_1$ and preserves the orientation in $\mathcal{U}$. The fact that $\theta(g)(R_i)$ is very close to $\zeta_2$ and has cross ratio very close to 1 together with $\theta(g)(u_1) = u_1$ implies the following:

- $\theta(g)(u_1)$ is very close to $u_1$,
- $\theta(g)$ moves $u_1$ counterclockwise (as seen in fig. ??, a);
- $\theta(g)(u_2), \theta(g)(u_3)$ are very close to the ideal point of $\zeta_2$ (the other one besides $u_1$) and
- $\theta(g)$ moves $u_2$ counterclockwise, moves $u_3$ clockwise, see fig. ??, a.

This implies there are at least 3 fixed points of $\theta(g)$ in $\mathcal{U}$: $u_1$, plus one fixed point near $u_2$ and one near $\theta(g)(u_3)$. Hence $g$ acts freely in $\mathcal{H}$. Assume it is decreasing in $\mathcal{H}$.

Once the suitable $g$ is found the argument follows the analysis in the proof of lemma 6.6. We only sketch the main ideas. The element $g$ is fixed here. Let $\zeta_3$ be another a leaf in $V_-$ invariant under $\theta(g)$ with an ideal point $u_5$ not equal to $u$ and $u_5$ near $u_3$ ($u_3$ defined above). Let $J$ be the interval of $\mathcal{U} - \{u_1,u_5\}$ containing $\theta(g)(u_2)$. By the above there is a fixed point of $\theta(g)$ in $J$. There is a sequence of ideal quadrilaterals which are bottoms of distortion parallelepipeds, for simplicity still denoted by $R_i$, so that $R_4$ converges to $\zeta_3$ and $R_i$ has ideal points $u, v_2, v_3, v_4$ positively circularly oriented in $\mathcal{U}$ and $v_2, v_3, v_4$ all in $J$ ($v_2, v_3, v_4$ depend on $i$). This is possible because $u_2$ (in the prior construction) was chosen in $J$ and using covering translates we can map $\zeta_3$ to arbitrarily near $\zeta_1$ with ideal point in $J$. Going up transversely to $\mathcal{F}$ using the universal circle we obtain quadrilaterals $S_i$ with cross ratios very close to 1 and map back by $g^n, n$ positive.

We check the action of $\theta(g^n)$ in $J$: first of all it fixes the boundary of $J$. Notice $\theta(g^n)$ has a fixed point in $J$, hence $\theta(g^n)$ cannot map $v_2$ very near $u_5$ and cannot map $v_3, v_4$ very near $u$. Hence the arguments in lemma 6.6 show that $\theta(g^n)$ moves $v_2, v_3, v_4$ close together. In the limit one obtains that the action of $\theta(g)$ in $J$ is a contraction with a single fixed point.

Using the same arguments with $\zeta_3$ a leaf of $G_+$, that is $\zeta_3$ being the limit of tops of a sequence of distortion parallelepipeds we obtain $\theta(g)$ acts as an expansion in $J$.

This is contradiction and shows that $G_-$ (or $G_+$) cannot be a foliation. This finishes the proof of proposition 6.8.

These results imply the following:

**Corollary 6.9.** Suppose that $M$ is orientable and $\mathcal{F}$ transversely orientable. Neither $G_-$ nor $G_+$ can be a foliation. Also by proposition 6.7 there is a leaf of $D_-$ intersecting a leaf of $D_+$ transversely. This shows that option $C$ in step 5 of theorem 5.1 occurs. As seen in the proof of theorem 5.1 this implies that both $D_-$ and $D_+$ have no transverse self intersections. This means that option $A$ occurs for both of them. Therefore $G_-$ is $d(\pi(D_- \times R))$. Similarly $G_+$ is $d(\pi(D_+ \times R))$. In addition there is positive $r_1$ so that for every $G$ in $G_-$, $F$ in $\mathcal{F}$ and $p$ in the intersection $l$ of $G$ and $F$, there is a point in the intersection of $G_+$ and $l$ at most $r_1$ distant from $p$ in $l$. Otherwise taking limits we find one such $l$ not intersecting $G_+$ and a leaf of $G_-$ not intersecting $G_+$ which was disallowed. Finally if $l'$ is a leaf of the intersection of $G_+$ and $F$ and $l'$ intersects $l$ in $q$ then the angle between $l,l'$ in $F^*$ at $q$ is bounded away from 0 and $\pi$. Similarly for $G_{m-}, m^+$.

To sum up what we have obtained so far:

**Corollary 6.10.** Suppose that $\mathcal{F}$ is an $\mathcal{R}$-covered foliation with hyperbolic leaves, $M$ is atoroidal and not a Seifert fibered space. Suppose that $\mathcal{F}$ is transversely oriented and $M$ orientable. Let $G_-$ and $G_+$ be the universal geodesic laminations constructed in the previous section. Then neither $G_-, G_+$ is a foliation. They are transverse to each other and with solid torus complementary components.
Remark: At this point it is useful to make the following remark: In some situations it may seem at first that the main theorem is trivial: Consider \( \varphi \) an Anosov flow so that its stable foliation \( \mathcal{F}^s \) is \( \mathbb{R} \)-covered and transversely orientable (there are many examples [Fe2]). The flow \( \varphi \) is tangent to \( \mathcal{F}^s \) and since \( \mathcal{F}^s \) is transversely orientable one can perturb \( \varphi \) slightly to a new flow \( \varphi' \) transverse to \( \mathcal{F}^s \).

By Anosov's fundamental results \( \varphi' \) is also an Anosov flow [An]. However if in addition \( M \) is atoroidal then the flow \( \Phi \) transverse to \( \mathcal{F}^s \) which will be constructed here is not a perturbation of \( \varphi \) as above. It has substantially different properties. In particular \( \Phi \) is not an Anosov flow – it has singularities. This is because \( \Phi \) is obtained by blowing down complementary regions of \( \mathcal{G}_-, \mathcal{G}_+ \) and the solid torus complementary components will produce singularities. In fact this is the key property that implies the weak hyperbolization conjecture for \( M \). Also the flow \( \Phi \) will be regulating for \( \mathcal{G} \); every orbit of the flow \( \Phi \) in \( \mathcal{M} \) intersects every leaf of \( \mathcal{F} \) and vice versa, as opposed to what happens for small perturbations of the Anosov flows above [Fe2, Fe5].

We now relate \( \tilde{\mathcal{G}}_- \) and \( \tilde{\mathcal{G}}_+ \).

**Proposition 6.11.** For every complementary region \( Q \) of \( V_- \) there is an unique associated complementary region \( Q' \) of \( V_+ \) having the same number of sides as \( Q \). Let \( g \in \pi_1(M) \), non trivial with \( \theta(g) \) fixing all points in \( \partial_{\infty} Q \). Suppose that \( g \) acts as a decreasing homeomorphism in \( \mathcal{H} \). Then the fixed point set of \( \theta(g) \) in \( \mathcal{U} \) is exactly the union of \( \partial_{\infty} Q \) and \( \partial_{\infty} Q' \); the points in \( \partial_{\infty} Q \) are repelling fixed points, those in \( \partial_{\infty} Q' \) are attracting and they alternate in \( \mathcal{U} \). There is a unique compact complementary region of the union of \( Q \) and \( Q' \) in \( \mathcal{F}^s \) which is a compact finite sided polygon.

**Proof.** First we will prove this result for the minimal sublaminations \( \mathcal{G}^m_-, \mathcal{G}^m_+ \) of \( \mathcal{G}_-, \mathcal{G}_+ \). Then we show that \( \tilde{\mathcal{G}}_-, \tilde{\mathcal{G}}_+ \) minimal, that is \( \mathcal{G}^m_- \) is equal to \( \tilde{\mathcal{G}}_- \) proving the result for \( \mathcal{G}_- \) and \( \mathcal{G}_+ \) as well.

Let \( Q \) be a complementary region of \( V_-^m \) (which is the intersection of \( \tilde{\mathcal{G}}_-^m \) with \( \mathcal{F}^s \)) and \( j \) the number of ideal points of \( Q \). The proof of lemma 6.6 shows that the action of \( \theta(g) \) on each component of

\[ \mathcal{U} - \partial_{\infty} Q \]

is a contraction with a single fixed point. This is because each component of the boundary of \( Q \) (a geodesic in \( \mathcal{F}^s \)) is a leaf of \( V_-^m \). This shows that there are exactly \( 2j \) fixed points of \( \theta(g) \) in \( \mathcal{U} \). Let \( l_1 \) be a boundary leaf of \( Q \) and \( l_2 \) another leaf asymptotic to it, defining the common ideal point \( w_1 \) – a repelling fixed point of \( \theta(g) \), see fig. ???. Let \( J_1 \) be the complementary interval of \( \partial_{\infty} Q \) in \( \mathcal{U} \) defined by \( l_1 \) and similarly define \( J_2 \). Let

\[ p_j \text{ in } l_1 \cap V_+^m \text{ with } p_j \to w_1. \]

Let \( \gamma_j \) be the leaves of \( V_+^m \) through \( p_j \). Since the angle between \( \gamma_j \) and \( l_1 \) is bounded away from 0 and \( \pi \) the endpoints of \( \gamma_j \) are eventually in the union of \( J_1 \) and \( J_2 \) and close to \( w_1 \). Let \( \gamma \) be one such leaf. Then \( \theta(g^n)(\gamma) \) is a leaf of \( V_-^m \) and as \( n \) converges to \( +\infty \), the endpoints of \( \theta(g^n)(\gamma) \) converge to the fixed points \( y_1, y_2 \) of \( \theta(g) \) in \( J_1, J_2 \) respectively, see fig. ??, a. As \( V_-^m \) is a lamination in \( \mathcal{F}^s \) this produces \( s_1 \) a leaf of \( V_+^m \) with ideal points \( y_1, y_2 \). Also \( \theta(g)(s_1) \) is equal to \( s_1 \).

In the same way associated to any ideal point \( w_j \) of \( Q \) we find \( s_j \) leaf of \( V_+^m \). Consecutive \( s_j \)'s are asymptotic, creating a finite sided ideal polygon \( Q' \) with the same number of sides as \( Q \) and \( Q' \) invariant under \( \theta(g) \). If there is any leaf of \( V_-^m \) in the interior of \( Q' \) it would be isolated in \( V_-^m \) contradiction. Hence \( Q' \) is a complementary region of \( V_+^m \). This proves the conclusion of of lemma 6.11 for \( V_-^m \) and \( V_+^m \).

This analysis implies that \( \mathcal{G}_-^m \) is minimal and so equal to \( \mathcal{G}_+^m \). Otherwise there is \( Q \) complementary region of \( V_-^m \) left invariant under \( \theta(g) \) with \( g \) non trivial in \( \pi_1(M) \) and \( l \) a leaf of \( V_- \) in the interior of \( Q \). Hence \( Q \) has at least 4 ideal points. Let \( Q' \) be the associated complementary region of \( V_+^m \) – by construction it has at least two ideal points on each component of \( \mathcal{U} - l \), let \( p_1, p_2 \) in one component and \( p_3, p_4 \) in the other component. The \( p_j \) are all fixed by \( \theta(g) \). Consider a sequence of ideal quadrilaterals \( \mathcal{R}_i \) converging to \( l \) which are the bottoms of a sequence of distortion parallelepipeds. Then one can show that the associated tops \( S_i \) have cross ratio bounded away from 1 – the fixed points \( p_1, p_2 \) and \( p_3, p_4 \) of
\(\theta(g)\) in \(\partial_\infty Q'\) keep the quadrilateral \(S_i\) from being too thin in the other direction. The \(S_i\) are trapped between two walls

\[
\left(\mathbb{P}^1 \times \mathbb{R}\right) \quad \text{and} \quad \left(\mathbb{P}^1 \times \mathbb{R}\right).
\]

These walls are invariant under \(g\), so the cross ratios (with the correct order) cannot get too close to 1. This contradiction shows that \(g^m\) is equal to \(\tilde{g}_-\), that is \(\tilde{g}_-\) is a minimal lamination.

Similarly \(\tilde{g}_+\) is minimal. As we already proved the results for \(g^m\) and \(\tilde{g}_+^m\) this finishes the proof of proposition 6.11.

Before producing the translational flow we check the non orientable situations:

**Proposition 6.12.** Suppose that \(\mathcal{F}\) is transversely oriented and \(\mathbb{R}\)-covered with hyperbolic leaves and that \(M\) is homotopically atoroidal. Then there are laminations by geodesics \(\tilde{g}_+, \tilde{g}_-\) transverse to \(\mathcal{F}\) and transverse to each other and which satisfy the conclusions of lemma 6.11.

**Proof.** The difference here is that \(M\) may be non orientable. The covering translations reversing orientation in \(M\) are exactly those which reverse orientation in \(\mathcal{U}\), because \(\mathcal{F}\) is transversely orientable. If \(M\) is orientable previous results apply. Otherwise let \(M_2\) be the orientable double cover so \(\pi_1(M_2)\) is a normal subgroup of index 2 in \(\pi_1(M)\). Using that \(M_2\) is homotopically atoroidal construct the laminations \(\tilde{g}_+, \tilde{g}_-^*\) in \(M_2\) as before. Let \(f\) in \(\pi_1(M)\) which is orientation reversing. It induces an involution \(f_2\) of \(M_2\) so that \(M = M_2/f\).

We claim that \(f\) leaves invariant the laminations \(\tilde{g}_-, \tilde{g}_+^*\). Lift \(\mathcal{F}\) to \(\mathcal{F}_2\) in \(M_2\). Notice that the universal circle is the same for \(\mathcal{F}\) and \(\mathcal{F}_2\). Let

\[
V_- = \tilde{g}_-^* \cap F^*, \quad V_+ = \tilde{g}_+^* \cap F^*
\]

as before. Suppose that \(\theta(f)(V_-)\) is not equal to \(V_-\). Since \(V_-\) is a lamination in \(F^*\) with no isolated leaves and finite sided complementary regions, then \(\theta(f)(V_-)\) has some transverse intersection with \(V_-\). Let \(Q\) be a complementary region of \(V_-\) with a boundary leaf \(l\) and a leaf \(l'\) of \(\theta(f)(V_-)\) intersecting \(l\) transversely. Let \(g\) be a non trivial covering translation in \(\pi_1(M_2)\) with \(\theta(g)(Q)\) equal to \(Q\) and fixing all ideal points of \(Q\). Assume that \(g\) acts as a decreasing homomorphism of \(\mathcal{H}\). The arguments in the proof of proposition 6.11, show that the sequence of geodesics \(\theta(g^n)(l')\) converges to a geodesic \(l''\) which is asymptotic to a leaf \(r\) of \(V_+\). This is because the ideal points of \(l''\) are attracting fixed points of \(\theta(g)\), so the rays in \(l''\) are asymptotic to leaves in \(V_+\). We do not know a priori that \(l''\) is a leaf of \(V_+\), it could be a diagonal in a complementary component of \(V_+\).

But since \(\pi_1(M_2)\) is normal in \(\pi_1(M)\), then for any \(h\) in \(\pi_1(M_2)\) it follows that \(hf = fh'\) for some \(h'\) in \(\pi_1(M_2)\). Hence

\[
hf(\tilde{g}_-) = f(h')\tilde{g}_-^* = f(\tilde{g}_-^*).
\]

So \(\pi_1(M_2)\) preserves \(f(\tilde{g}_-)\) and therefore \(\theta(\pi_1(M_2))\) preserves \(f(V_-)\). Hence

\[
\theta(g^n)(l') \in f(V_-) \quad \text{and} \quad V_+ \in f(V_-)
\]

also. Since \(r\) in \(V_+\) and \(l''\) in \(\theta(f)(V_-)\) are asymptotic, taking limits this implies that \(\theta(f)(V_-)\) and \(V_+\) share a leaf. But \(f(\tilde{g}_-^*\) and \(\tilde{g}_+^*\) are minimal so \(\theta(f)(V_-)\) is equal to \(V_+\). However leaves in \(\theta(f)(V_-)\) are still limits of bottoms of sequences of distortion parallelepipeds. As in the proof of proposition 6.7 this contradicts the properties of leaves of \(V_+\) (that is they are limits of tops of sequences of distortion parallelepipeds), because \(\theta(f)(V_-)\) is equal to \(V_+\). We conclude that

\[
\theta(f)(V_-) = V_-, \quad \theta(f)(V_+) = V_+, \quad f(\tilde{g}_-) = \tilde{g}_-, \quad f(\tilde{g}_+^*) = \tilde{g}_+^*.
\]

Therefore \(\tilde{g}_-, \tilde{g}_+^*\) are invariant under \(f_2\) and induce laminations \(g_-, g_+^*\) in \(M\) which are transverse to \(\mathcal{F}\) and to each other and satisfy the properties of lemma 6.11. This finishes the proof of lemma 6.12. \(\square\)
Remark 1 — By proposition 6.8 the lamination $G_-$ is never a foliation. Also in the atoroidal case the complementary regions of $G_-$ are solid tori (or solid Klein bottles). Therefore $G_-$ is a an essential lamination [Ga-Oe] and it is a genuine lamination [Ga-Ka], that is, complementary regions are not $I$-bundles. A theorem of Gabai and Kazez [Ga-Ka] then implies that:

**Corollary 6.13.** If $M$ is aspherical and has an $R$-covered foliation, then $M$ satisfies the weak hyperbolicity conjecture: either there is a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(M)$ or $\pi_1(M)$ is Gromov negatively curved.

Since $M$ is irreducible [Ro] there are many important consequences for the geometry of $M$. In particular $M$ is conjecturally hyperbolic [Th4].

**Remark 2** — The final case to be considered is $F$ not transversely orientable. Lift to double cover $M'$ so that the lifted $F'$ is transversely orientable. Using the proposition 6.12 produce laminations $G_+, G_-$ in $M'$. Let $B$ be $\pi_1(M')$ and $f$ in $(\pi_1(M) - B)$. Then $f$ reverses orientation in $H$. Hence leaves of $f(V_-)$ are now the limit of thin quadrilaterals which get distorted when moving down transverse to $F$. The same arguments as in the previous proposition show that

$$\theta(f)(V_-) = V_+ \quad \text{and} \quad \theta(f)(V_+) = V_-,$$

that is, $f$ switches the invariant laminations. This is because $f$ reverses the orientation to $H$ so something which is limit of bottoms of sequences of distortions parallelepipeds, has image under $\theta(f)$ which is the limit of tops of sequences of distortion parallelepipeds. This would produce a nonorientable line field in the intersection of the two laminations in $M$, which is transverse to $F$. We think this situation in fact cannot occur, but at this point we cannot rule it out. In any case $M$ has a finite regular cover $M'$ with an essential lamination $G_-$. Remark 1 shows that $\pi_1(M')$ is negatively curved in the large and so is $\pi_1(M)$, because $\pi_1(M')$ has finite index in $\pi_1(M)$ [Gr].

**Remark 3** — If there is a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(M)$, it can be represented by an immersed incompressible torus $T$ which is in general position with respect to $F$. Following classical ideas of Thurston [Th1], Roussarie [Rou] and more recently Gabai [Ga5], it follows that $T$ can be put in tight position with respect to $F$. As $F$ is $R$-covered and Reebless it follows that $F$ is taut [Fe5, Go]. Given that $F$ is taut Gabai [Ga5] showed that $T$ can be homotoped to be either contained in a leaf of $F$ or transverse to $F$ (here $T$ may fail to be embedded!). Taut is used to avoid circles of tangency. In the first case not all leaves of $F$ are hyperbolic. In the second case $T$ represents a region in leaves whose geometry is only boundedly distorted moving transversely to the foliation — this explains the dichotomy mentioned in the introduction.

## 7 The transverse pseudo-Anosov flow

Here $F$ is a transversely oriented, $R$-covered foliation with hyperbolic leaves; $M$ homotopically atoroidal. By section 6 there are universal laminations by geodesics $G_+, G_-$, transverse to $F$ and to each other. We use the notations and constructions from the previous sections. A complementary region $Q$ of $V_-$ is an ideal polygon and has associated complementary region $Q'$ of $V_+$, producing a complementary region

$$P = Q \cap Q'$$

with compact closure. This region has at least 6 boundary sides, see fig. ??, a; and there is $g$ in $\pi_1(M)$ with $\theta(g)$ leaving both $Q$ and $Q'$ invariant, hence also leaving their intersection invariant. In $M$ these produce complementary regions of $G_+ \cup G_-$ which are solid tori or solid Klein bottles. They are homeomorphic to

$$P \times I/\eta,$$
where $\eta$ is a homeomorphism of $P$. All other complementary regions $P'$ of $V_\pm \cup V_+$ are in “cusps” of $V_-$ and $V_+$, hence are relatively compact quadrilaterals in $F^s$. Up to the action of $\pi_1(M)$ there are finitely many complementary components of $(V_\pm \cup V_+)$ in $F^s$ with 6 sides or more.

There is a flow transverse to $F$ defined in the intersection of $G_+$ and $G_-$: just consider the orientable line field which is the intersection of leaves of $G_+, G_-$. Now collapse the complementary regions of $G_+ \cup G_-$ along leaves of $F$ to produce 2 invariant singular foliations in $M$. In $F^s$ each closure of complementary region of $V_\pm \cup V_+$ collapses to a point. This produces a flow $F$ in $M$ which is transverse to $F$. The collapsing of $G_+$ produces the singular foliation $F^u$ (unstable) and $G_-$ produces $F^s$ (stable). This operation of collapsing along leaves was described in great detail in Mosher’s articles [Mo1, Mo2]. A complementary region of $V_-$ with $p$ sides (see fig. ??, a) blows down to a $p$-prong singular leaf of $F^s$ in $F^s$, see fig. ??, b. These complementary regions are periodic under some $\theta(g)$ with $g$ in $\pi_1(M)$ and therefore produce closed orbits of $F$ in $M$. The local cross section is given in figure ??, b. All other points in $M$ are topologically non singular for the flow $F$ and foliations $F^s, F^u$. The flow lines of $F$ are “tangent” to $F^u, F^s$. Since the laminations $G_-, G_+$ are minimal it follows that all leaves of $F^s$ and $F^u$ are dense in $M$. There is a homotopy equivalence $\xi : M \to \tilde{M}$ preserving leaves of $F$ and sending $G_+$ to $F^u$, $G_- \to F^s$. There is a lift $\tilde{\xi} : \tilde{M} \to \tilde{M}$ preserving leaves of $\tilde{F}$ and moving points a bounded distance. We now show that $F$ is a pseudo-Anosov flow.

**Definition 7.1.** (topological pseudo-Anosov) A flow $\varphi$ in a manifold $N^3$ is a topological pseudo-Anosov flow if there are no point orbits of $\varphi$ and orbits of $\varphi$ are contained in two (possibly singular) foliations $E^s, E^u$ stable and unstable satisfying:

1. All flowlines in a leaf of $E^s$ are forward asymptotic, all flow lines in a leaf of $E^u$ are backward asymptotic.
2. The (topological) singularities of $E^s, E^u$ are all of $p$-prong type. The singular locus is a finite union of closed orbits of $\varphi$ and $p$-local leaves of $E^s$ about this singular orbit and similarly for $E^u$.
3. The foliations $E^s, E^u$ are transverse to each other and intersect exactly along the flow lines of $\varphi$.

The flow $\tilde{F}$ constructed above is transverse to $F$ and its flow lines are contained in leaves of $F^s, F^u$. Under a small perturbation so that $\tilde{F}$ is still transverse to $F$ we can assume that: the orbits of $F$ are $C^1$ and leaves of $F^s, F^u$ are $C^1$ submanifolds in the complement of the singularities and in the singularities we have a standard topological $p$-prong picture. We stress that is not clear whether these flows can be made “smooth” pseudo-Anosov as defined by Mosher in [Mo2]. In particular it is not clear whether one can define the strong stable/unstable foliations associated to the flow.

Notice that for any $g$ in $\pi_1(M)$, $\theta(g)$ acts on $V_-$ and $V_+$ hence acts on the points of the intersection. This action is still denoted by $\theta(g)$. A leaf of $F^s$ or $\tilde{F}^s$ is periodic if it contains a periodic orbit of $\tilde{F}$ or the lift of a periodic orbit. Given $x$ in $M$ let $W^s(x)$ be the leaf of $F^s$ containing $x$ and likewise define $W^u(x)$. Let $\tilde{F}^s, \tilde{F}^u, \tilde{F}$ be the lifts to $\tilde{M}$. If $y$ in $\tilde{M}$ define $\tilde{W}^u(y), \tilde{W}^s(y)$ similarly to the above.

**Proposition 7.2.** (topological hyperbolicity) For any two points $x, y$ in a leaf $E$ of $\tilde{F}^s$ their orbits are asymptotic in future time. In negative time the distance between orbits converges to infinity, in the intrinsic metric of $E$. The opposite behavior occurs in leaves of $F^u$.

**Proof.** The dynamics of $\tilde{F}$ is entirely encoded by the dynamics of the orientable line field $G_+ \cap G_-$. This is what is going to be used here. We first analyse the case that $\pi(E)$ contains a singular orbit of $\tilde{F}$. Then $E$ is a blow down of $p$ leaves of $G_-$ in $\tilde{M}$. Let $L$ in $G_-$ be one of them. The intersection $L$ of $L$ and $E^s$ is in the boundary a complementary region $Q$ of $V_-$ and there is an associated complementary region $Q'$ of $V_+$. There is $g$ in $\pi_1(M)$ non trivial, with $g$ acting as a decreasing homeomorphism of $\mathcal{H}$ and $\theta(g)$ fixing only $\partial_\infty Q$, $\partial_\infty Q'$ in $S^\infty_\infty(F^s)$, so that points in $\partial_\infty Q'$ are attracting and points in $\partial_\infty Q$ are repelling. This dynamics of $\theta(g)$ is the fundamental point here. There are two boundary leaves of $Q'$ intersecting $L$, let $s$ be one of them. Let

$$S = (s \times \mathbb{R}), \quad \tilde{\alpha} = S \cap L \quad \text{and} \quad z = (S \cap L) \cap F^s = s \cap L.$$
Also

\[ l \subset \partial Q \] is a leaf of \( \mathcal{V}_- \), \( s \subset \partial Q' \) is a leaf of \( \mathcal{V}_+ \), \( s \cap l \neq \emptyset \).

Then \( \theta(g)(s) = s \) and \( g(S) = S \). The map \( g \) is associated to the closed orbit \( \alpha = \pi(\tilde{\alpha}) \) in \( \pi(L) \). Orbits of \( \Phi \) in \( L \) correspond to leaves of \( \mathcal{G}_+ \) intersecting \( L \), so let

\[ H \in \mathcal{G}_+ \text{ with } H \cap L = \tilde{\gamma}, \]

orbit of the flow. Start with the intersection \( a_0 \) of \( H \) and \( L \) at level \( F^s \) - which is a point in \( I \). Go up to \( g^{-1}(F^s) \) along the flow line \( \tilde{\gamma} \). Mapping the intersection of \( \tilde{\gamma} \) and \( g^{-1}(F^s) \) down by \( g \) produces a point \( a_1 \) in \( l \) - this is like the first return map associated to the closed orbit \( \alpha \). The action of \( \theta(g) \) in \( S^1_{\infty}(F^s) \) moves the ideal points of \( H \cap F^s \) closer to the ideal points of \( s \), because the ideal points of \( s \) are in \( \partial_{\infty}Q' \) and are attracting for \( \theta(g) \). This implies that \( a_1 \) is closer to \( z \) than \( a_0 \) is. This is exactly the same argument as in step 7 of the proof of theorem 5.1. Iterating this procedure the images in \( l \) converge to \( z \), that is,

\[ a_n = g^n(\tilde{\gamma} \cap g^{-n}(F^s)) \]

converges to \( z \). Hence in \( \bar{M} \) the orbit \( \tilde{\gamma} \) is asymptotic to the orbit \( \tilde{\alpha} \) in the forward direction as one moves up. All orbits in \( L \) on that side of \( S \) are asymptotic to the orbit \( \tilde{\alpha} \). Orbits through the corners of \( Q \cap Q' \) collapse to a single orbit in the blow down. Hence all orbits are asymptotic to the closed orbit in \( \pi(E) \) after the collapsing. This proves the result for singular leaves. The key is the action of \( \theta(g) \) in \( \mathcal{U} \).

If now \( \zeta \) is any periodic orbit of \( \Phi \), which is non singular then \( \zeta \) is the intersection of two leaves of \( \mathcal{F}^s \) and of \( \mathcal{F}^u \) which come from unique leaves

\[ L \text{ of } \mathcal{G}_- \text{ and } S \text{ of } \mathcal{G}_+. \]

Let

\[ l = L \cap F^s, \quad s = S \cap F^s, \quad \tilde{\alpha} = S \cap L \]

and let \( b_1, b_2 \) the ideal points of \( l \). Let \( g \) in \( \pi_1(M) \) non trivial with \( g(\tilde{\alpha}) \) equal to \( \tilde{\alpha} \) so that \( g \) leaves both components of \( L - S \) invariant. Then \( \theta(g) \) has at least 4 fixed points and therefore acts freely in \( H \), assume as a decreasing homeomorphism. Given that, use an analysis similar to the proof of lemma 6.6: let \( R \) be quadrilaterals converging to a leaf \( l_0 \) of \( V_- \) near but not equal to \( l \). The analysis of lemma 6.6 shows that

\[ \theta(g) \text{ is a contraction in that interval of } \mathcal{U} = \{b_1, b_2\}. \]

Therefore \( \theta(g) \) has exactly 4 fixed points in \( \mathcal{U} \) which are the ideal points of \( l \) and \( s \). Then the same analysis as in the singular case yields that all orbits in \( L \) are forward asymptotic to \( \tilde{\alpha} \) so after collapse all orbits are forward asymptotic to \( \zeta \).

This takes care of periodic leaves of \( \mathcal{F}^s, \mathcal{F}^u \). We now deal with general leaves. Notation: if \( x, y \) are in the same leaf of the intersection of \( \bar{F} \) and \( \bar{F}^s \), let \( d'(x, y) \) be their distance along that leaf. Consider \( x, y \) in the same leaf \( F \cap E \) where \( F \) is a leaf of \( \bar{F} \) and \( E \) a leaf of \( \bar{F}^s \). If the orbits of \( \tilde{\Phi} \) through \( x, y \) are not asymptotic in future time we can find positive \( a_0 \) so that

\[ \tilde{\Phi}_{t_i}(x), \tilde{\Phi}_{s_i}(y) \in F_i \subset \bar{F}, \quad d'((\tilde{\Phi}_{t_i}(x), \tilde{\Phi}_{s_i}(y))) > a_0 \text{ and } t_i, s_i \to +\infty. \]

Hence we can find segments \( \beta_i \) in the intersection of leaves of \( \mathcal{F}^s \) with leaves of \( \bar{F} \) with endpoints \( x_i, y_i \) in leaves of \( \bar{F} \) which have length of \( \beta_i \) converging to \( a_0 \) and so that

\[ \tilde{\Phi}_{-t_i}(x_i), \tilde{\Phi}_{-s_i}(y_i) \in F, \quad \text{and } d'((\tilde{\Phi}_{-t_i}(x_i), \tilde{\Phi}_{-s_i}(y_i))) < a_1, \text{ for some fixed } a_1 > 0. \]
Up to subsequence there are covering translations $h_i$ with $h_i(\beta_i)$ converging to $\beta_0$.

Since a periodic leaf of $\mathcal{F}^u$ is dense in $M$ let $\gamma$ orbit of $\hat{\Phi}$ with $\pi(\gamma)$ periodic and non singular so that $W^u(\gamma)$ intersects $\beta_0$, see fig. ?? Hence for $i$ big $W^u(\gamma)$ and $h_i(\beta_i)$ intersect. Making $\beta_i, \beta_0$ smaller if necessary we may assume that: for any $p$ in $\beta_1$, the leaf $\tilde{W}^u(p)$ does not intersect a singular orbit between $W^s(\beta_i)$ and $W^s(\gamma)$. Equivalently $W^u(\gamma)$ intersects $W^s(\gamma)$. Now flow back. The points in $\beta_0$ flow back very near $\tilde{W}^s(\gamma)$ and from then on always near $\tilde{W}^s(\gamma)$, see fig. ?? So for $i$ big $h_i(\beta_i)$ also does. Flowing backwards in $\tilde{W}^s(\gamma)$, the segments which intersect $\gamma$ in the interior blow up, because all orbits in $\tilde{W}^s(\gamma)$ are forward asymptotic to $\gamma$ and $\tilde{W}^s(\gamma)$ is periodic — hence the lengths will blow up past $2a_1$ and will never again be smaller than $2a_1$. Therefore nearby segments obtained flowing back pieces of $h_i(\beta_i)$ will also have big length. But the segments $h_i(\beta_i)$ flow back to segments of length smaller than $a_1$ for arbitrarily long time when $i$ is big — hence this is a contradiction. Hence orbits in $E$ are forward asymptotic.

The same argument shows that flowing in the negative direction blows up distance along stable leaves without bound — because this happens in periodic leaves and then use the argument above of the intersection with $\tilde{W}^u(\gamma)$. This finishes the proof of the proposition.

Finally we get the metric pseudo-Anosov property for the flow $\Phi$.

**Proposition 7.3. (metric hyperbolic) For every positive $a_2$, there is positive $a_3$ so that: let $\beta$ be a segment in the intersection of a leaf of $\hat{\mathcal{F}}$ with a leaf of $\mathcal{F}^u$ of length at least $a_2$. Flow forward every point of $\beta$ to obtain another segment $\beta'$ in another leaf of $\hat{\mathcal{F}}$. If every point of $\beta$ moves at least $a_3$ flow length, then the length of the final segment is at least double the length of $\beta$.

**Proof.** Otherwise get segments $\beta_i$ so that there are longer and longer times so that length of flow of $\beta_i$ is smaller than $2a_2$. Call these segments in leaves $\mathcal{F}_i$ of $\hat{\mathcal{F}}$ to be $\zeta_i$. Then the $\zeta_i$ have length less than $2a_2$ and it takes longer and longer for them in the negative direction to decrease to length $a_2$. The arguments in the previous proposition disallow this. This finishes the proof.

**Remark:** A very important question is to analyse geometric properties of the transverse flow $\Phi$. For instance is the flow quasigeodesic? That means flow lines of $\hat{\Phi}$ are uniformly efficient in measuring distance [Fe2]. This has several important consequences, for instance the continuous extension property for leaves of $\hat{\mathcal{F}}$ [Ca-Th]. When $\mathcal{G}$ is an uniform foliation, it is very easy to see that $\Phi$ is quasigeodesic, because $\Phi$ is regulating [Th7]. In general this is an open question.

**References**


REFERENCES


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