# Study Questions Homework 2 Introduction to Computational Finance Spring 2023

**These are study questions. You are not required to submit solutions (even though the problems are worded like graded assignment problems).**

**Written Study Exercises**

# Problem 2.1

## 2.1.a

Consider the polynomial

$$p(x) = x^3 - x - 5$$

The following three iterations can be derived from $p(x)$.

$$\phi_1(x) = x^3 - 5$$
$$\phi_2(x) = \sqrt[3]{x + 5}$$
$$\phi_3(x) = 5/(x^2 - 1)$$

The polynomial $p(x)$ has a root $r > 1.5$. Determine which of the $\phi_i(x)$ produce an iteration that converges to $r$.

## 2.1.b

Consider the function

$$f(x) = xe^{-x} - 0.06064$$

**i**. Write the update formula for Netwon's method to find the root of $f(x)$.

**ii**. $f(x)$ has a root of $\alpha = 0.06469263\ldots$. If you run Newton's method with $x_0 = 0$ convergence occurs very quickly, e.g., in double precision $|f(x_4)| \approx 10^{-10}$. However, if $x_0$ is large and negative or if $x_0$ is near 1 convergence is much slower until there is a very rapid improvement in accuracy in the last one or two steps. For example, if $x_0 = 0.98$ then

$$|f(x_{47})| \approx 0.6 \times 10^{-2}$$
$$|f(x_{49})| \approx 1.5 \times 10^{-5}$$
$$|f(x_{50})| \approx 2.8 \times 10^{-10}$$

Explain this behavior. You might find it useful to plot $f(x)$ and run a few examples of Newton's method.

## Problem 2.2

Consider the fixed point iteration by the function

$$\phi(x) = x - \frac{(x^2 - 3)}{(x^2 + 2x - 3)}$$

The value $\alpha = \sqrt{3}$ is a fixed point for this iteration. Provide justification to all of your answers for the following:

(2.2.a) Show that there exists a nontrivial interval $\alpha - \delta < x < \alpha + \delta$ with $\delta > 0$ such that the iteration defined by $\phi(x)$ converges to $\alpha$ for any $x_0$ in the interval.

(2.2.b) Is the order of convergence on this interval linear ($p = 1$) or higher ($p \geq 2$)?

(2.2.c) Show that for $x > \alpha$ we have

$$\alpha < \phi(x) < x$$

and use this to explain the convergence of the iteration when $x_0 > \alpha$.

(2.2.d) Plot or otherwise enumerate values of the curves $y = \phi(x)$ and $y = x$ on the interval $0 < x < \beta$ for $\beta > \alpha$. Use the information to examine the behavior of the iteration for $0 < x_0 < \alpha$. For what subinterval, if any, do you expect convergence to $\alpha$?

## Problem 2.3

Let $f(x) = e^{8x}$ on $0 \leq x \leq 1$. Suppose $f(x)$ is to be approximated by a **piecewise linear interpolating function**, $g_1(x)$. The accuracy required is

$$\forall 0 \leq x \leq 1, \quad |f(x) - g_1(x)| \leq 10^{-6}$$

Determine a bound on $h = x_i - x_{i-1}$ for uniformly spaced points that satisfies the required accuracy.

## Problem 2.4

Recall that the error in the interpolating polynomial of degree $n$ is $\forall x_0 \leq x \leq x_n$

$$|f(x) - p_n(x)| = \frac{|(x - x_0)(x - x_1) \ldots (x - x_n)|}{(n+1)!} |f^{(n+1)}(\xi(x))| = \frac{|\omega_{n+1}(x)|}{(n+1)!} |f^{(n+1)}(\xi(x))|$$

Note it is sometimes useful to work with a change of variables to analyze the error and related quantities. The interval $x_0 \leq x \leq x_n$ can be changed to $0 \leq s \leq 1, \quad x = x_0 + sh$.

**i**. Assuming a uniform spacing of the $x_i$, bound $\|\omega_2(x)\|_\infty$, i.e., $n = 1$. The norm $\|g(x)\|_\infty$ is defined to be the maximum of magnitude $|g(x)|$ when $x$ is in a given interval. In this case the interval is $x_0 \le x \le x_n$ or $0 \le s \le 1, \quad x = x_0 + sh$.

**ii**. Assuming a uniform spacing of the $x_i$, bound $\|\omega_3(x)\|_\infty$, i.e., $n = 2$.

**iii**. Let $f(x) = e^{8x}$ on $0 \le x \le 1$. Suppose $f(x)$ is to be approximated by an interpolating polynomial $p_2(x)$ of degree 2, i.e., $n = 2$ and three points $x_0$, $x_1$ and $x_2$ are used to define $p_2(x)$. Suppose three points are taken to be $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1$. Bound the error $|f(x) - p_2(x)|$ using the formulas above then determine $p_2(x)$ and compute $|f(x) - p_2(x)|$ for many points in the interval $0 \le x \le 1$ to check the tightness of the bound for absolute and relative error.