Study Questions Homework 3 Introduction to Computational Finance Spring 2023

These are study questions. You are not required to submit solutions (even though the problems are worded like graded assignment problems).

Written Study Exercises

Problem 3.1

Recall when defining an interpolatory cubic spline $s(t)$ in terms of the parameters s''_i for $0 \le i \le n$, we have the underdetermined $(n-1) \times (n+1)$ linear system

$$\begin{pmatrix} \mu_1 & 2 & \lambda_1 & 0 & \dots & 0 \\ \mu_2 & 2 & \lambda_2 & \ddots & \vdots & \\ 0 & \ddots & \ddots & \ddots & 0 & \\ \vdots & \ddots & \mu_{n-2} & 2 & \lambda_{n-2} & \\ 0 & \dots & 0 & \mu_{n-1} & 2 & \lambda_{n-1} \end{pmatrix} \begin{pmatrix} s''_0 \\ s''_1 \\ \vdots \\ s''_{n-1} \\ s''_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}.$$

This requires two boundary conditions to get the system that defines the unique interpolatory cubic spline.

For example, when the boundary conditions $s''_0 = c_0$ and $s''_n = c_n$ are specified, where c_0 and c_n are given constants and we have the $(n-1) \times (n-1)$ linear system

$$\begin{pmatrix} 2 & \lambda_1 & 0 & \dots & 0 \\ \mu_2 & 2 & \lambda_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \mu_{n-2} & 2 & \lambda_{n-2} \\ 0 & \dots & 0 & \mu_{n-1} & 2 \end{pmatrix} \begin{pmatrix} s''_1 \\ \vdots \\ s''_{n-1} \end{pmatrix} = \begin{pmatrix} d_1 - \mu_1 s''_0 \\ \vdots \\ d_{n-1} - \lambda_{n-1} s''_n \end{pmatrix}$$

where

$$h_i = x_i - x_{i-1}, \quad \mu_i = \frac{h_i}{h_i + h_{i+1}}$$

$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \quad d_i = \frac{6}{h_i + h_{i+1}} \left(f[i, i+1] - f[i-1, i] \right).$$

However, when the boundary conditions

$$s'_0 = c_0 \quad \text{and} \quad s'_n = c_n$$

are specified for given constants c_0 and c_n, two additional equations are required to define the $(n+1) \times (n+1)$ linear system that specifies the unique interpolatory cubic spline. Derive these additional equations.

Problem 3.2

Define the following sets of polynomial bases:

$$\mathcal{B}_1 = \{1, x\}$$
$$\mathcal{B}_2 = \{\phi_0(x), \phi_1(x)\} = \{0.5(x+1), 0.5(1-x)\}$$
$$\mathcal{B}_3 = \{T_0(x), T_1(x), T_2(x)\} = \{1, x, 2x^2 - 1\}$$
$$\mathcal{B}_4 = \{\psi_0(x), \psi_1(x), \psi_2(x)\} = \{0.5x(x-1), (1-x)(x+1), 0.5x(x+1)\}$$

Define the following set of mesh points

$$X = \{x_0, x_1, x_2, x_3, x_4\} = \{-1, -0.5, 0, 0.5, 1\}.$$

Defing the following sets of function values on the mesh X

$$Y_1 = \{y_0^{(1)}, y_1^{(1)}, y_2^{(1)}, y_3^{(1)}, y_4^{(1)}\} = \{-2, 1/2, 3, 11/2, 8\}$$

$$Y_2 = \{y_0^{(2)}, y_1^{(2)}, y_2^{(2)}, y_3^{(2)}, y_4^{(2)}\} = \{0, -3/4, -1, -3/4, 0\}$$

$$Y_3 = \{y_0^{(3)}, y_1^{(3)}, y_2^{(3)}, y_3^{(3)}, y_4^{(3)}\} = \{3, 19/8, 1, -3/8, -1\}$$

$$Y_4 = \{y_0^{(4)}, y_1^{(4)}, y_2^{(4)}, y_3^{(4)}, y_4^{(4)}\} = \{-1, -1/32, 0, 1/32, 1\}$$

(3.2.a) For each polynomial basis compute the matrix F and $M = F^T F$ used for polynomial regression using the mesh X.

(3.2.b) For each polynomial basis, determine the coefficients for the optimal polynomial regression of the appropriate degree for each set of datapoints defined by the mesh X and function values Y_k, i.e., $\{(x_0, y_0^{(k)}), (x_1, y_1^{(k)}), (x_2, y_2^{(k)}), (x_3, y_3^{(k)}), (x_4, y_4^{(k)})\}$. Specifically, in your solutions for each combination of basis and datapoints:

- Give the linear system solved to get the coefficients.
- Express the optimal polynomial in terms of the polynomials in the basis and in terms of the powers of x.
- Evaluate the norm of the residual, i.e., $\sum_{i=0}^{4}(y_i^{(k)} - p(x_i))^2$ where $p(x)$ is the optimal regression polynomial for the basis.

You are encouraged to write a simple program (you need not turn it in since this is a study question to compute the solutions for this problem. However, note that all of these matrices and vectors can be computed "by-hand" to get an exact answer since all of the entries are rational numbers. You are encouraged to verify your computed values "by-hand" for some or all of the combinations.

Problem 3.3

Recall, Simpson's (First) Rule uses $x_0 = a$, $x_1 = (a+b)/2$, and $x_2 = b$ to define a quadratic interpolating polynomial, $p_2(x)$, that is integrated to approximate the definite integral of $f(x)$, i.e.,

$$\int_a^b f(x)dx \approx \int_a^b p_2(x)dx = h_2 \left[\frac{1}{3} f(x_0) + \frac{4}{3} f(x_1) + \frac{1}{3} f(x_2) \right]$$

where $h_2 = (b-a)/2$.

Derive this method using the Lagrange form of $p_2(x)$.

Problem 3.4

Consider the definite integral

$$I(f) = \int_a^b f(x)dx$$

and its approximation by the **open Newton-Cotes quadrature** formula that uses 3 points $a < x_0 < x_1 < x_2 < b$, i.e., $n = 2$ and $h = (b-a)/4$ given by

$$I_2(f) = h\left(\alpha_0 f(x_0) + \alpha_1 f(x_1) + \alpha_2 f(x_2)\right)$$

(3.4.a) Derive the coefficients of this open Newton-Cotes quadrature formula.

(3.4.b) Using Taylor expansions, derive the error formula with the form

$$I(f) - I_2(f) = Ch^k f^{(d)}(a) + O(h^{k+1})$$