Study Questions Homework 4 Introduction to Computational Finance Spring 2023

These are study questions. You are not required to submit solutions (even though the problems are worded like graded assignment problems).

Written Study Exercises

Problem 4.1

Recall Simpson's **second** rule approximates the integral

$$I(f) = \int_a^b f(x)dx$$

by

$$I_{s2r}(f) = h_3 \, \frac{3}{8} \, [f_0 + 3f_1 + 3f_2 + f_3]$$

with error

$$I(f) - I_{s2r}(f) = -\frac{3}{80} \, h_3^5 \, f^{(4)} + O(h_3^6), \quad h_3 = (b-a)/3.$$

This method can be used to define a composite method, I_{cs2}, by using rule I_{s2r} on a set of intervals $[a_i, b_i]$ for $i = 1, \ldots, m$ and summing the values.

(4.1.a) Suppose m intervals are used each of width $H = (b-a)/m$. Determine the expression for the error for the composite Simpson's second rule

$$E_{cs2} = I(f) - I_{cs2}$$

(4.1.b) Use this error bound for the composite Simpson's second rule to determine the number of intervals needed so that the error will be less than or equal to the tolerance 10^{-6} when approximating

$$I = \int_{-1}^1 e^x dx$$

Problem 4.2

4.2.a

Consider the definite integral

$$I(f) = \int_a^b f(x)dx$$

and its approximation by the **open Newton-Cotes quadrature** formula that uses 2 points $a < x_0 < x_1 < b$, i.e., $n = 1$ and $h = (b - a)/3$ given by

$$I_1(f) = h\left(\alpha_0 f(x_0) + \alpha_1 f(x_1)\right)$$

(i) Derive the coefficients of this open Newton-Cotes quadrature formula.

(ii) Using Taylor expansions, derive the error formula with the form

$$I(f) - I_1(f) = Ch^3 f''(a) + O(h^4)$$

4.2.b

Consider the composite rule based on the two-point open Newton-Cotes formula $I_1(f)$ given in the previous part of the problem when it is used as the foundation of a global step refinement quadrature method.

(i) Suppose the refinement rule uses $\alpha = 1/3$, i.e., $h_f = h_c/3$. Does the method completely reuse function evaluations from the coarse mesh quadrature, I_c, when computing the fine mesh quadrature, I_f? Justify your answer.

(ii) Suppose the refinement rule uses $\alpha = 1/2$, i.e., $h_f = h_c/2$. Does the method completely reuse function evaluations from the coarse mesh quadrature, I_c, when computing the fine mesh quadrature, I_f? Justify your answer.

Problem 4.3

4.3.a

Consider the difference operator on a uniform grid, i.e., $x_i - x_{i-1} = h$ and y_i denotes $y(x_i)$, defined by

$$Dy_i = \frac{-y_{i+2} + 8y_{i+1} - 8y_{i-1} + y_{i-2}}{12h}.$$

Determine the order to which this difference operator approximates $y'(x_i)$, i.e., determine C and k in the expression

$$y'(x_i) = Dy_i + Ch^k y^{(k+1)} + O(h^{k+1}).$$

You must Justify your answer. Simply stating a value for k will receive no credit.

4.3.b

Consider applying the the difference operator to the function $y(x) = \sin x$.

(i) Take $x_i = \pi/4$. What value of h must be used to get an approximation of $y'(\pi/4)$ that satisfies

$$|y'(\pi/4) - Dy_i| \leq 10^{-4}?$$

(ii) Apply the difference operator to $y(x_i)$ with $x_i = \pi/4$ using the h you have derived to verify that your error is less than the required bound.

Problem 4.4

4.4.a

Suppose $y_j = y(x_j)$ for a set of points x_j with constant spacing $h = x_j - x_{j-1}$. Consider the following linear difference formula, D:

$$Dy_i = \frac{1}{h}\left(\alpha_2 y_{i+2} + \alpha_1 y_{i+1} + \alpha_0 y_i\right)$$

Determine the coefficients $\alpha_i,\ \ i = 0,\ 1,\ 2$ to maximize the order to which Dy_i approximates $y'(x_i)$ and determine C and k in the resulting error expression

$$y'(x_i) = Dy_i + Ch^k y^{(k+1)} + O(h^{k+1}).$$

4.4.b

Consider applying the the difference operator to the function $y(x) = \sin x$.

(i) Take $x_i = \pi/4$. What value of h must be used to get an approximation of $y'(\pi/4)$ that satisfies

$$|y'(\pi/4) - Dy_i| \leq 10^{-4}?$$

(ii) Apply the difference operator to $y(x_i)$ with $x_i = \pi/4$ using the h you have derived to verify that your error is less than the required bound.