Study Questions Homework 2 Foundations of Computational Math 1 Fall 2020

Problem 2.1

Consider the data points

$$(x, y) = \{(0, 2),\ \ (0.5, 5),\ \ (1, 8)\}$$

Write the interpolating polynomial in both Lagrange and Newton form for the given data.

Problem 2.2

Use this divided difference table for this problem.

i	0		1		2		3		4		5
x_i	-1		0		2		4		5		6
f_i	13		2		-14		18		67		91
$f[-,-]$		-11		-8		16		49		24	
$f[-,-,-]$			1		6		11		$-25/2$		
$f[-,-,-,-]$				1		1		$-47/8$			
$f[-,-,-,-,-]$					0		$-55/48$				
$f[-,-,-,-,-,-]$						$-55/336$					

2.2.a

Use the divided difference information about the unknown function $f(x)$ and consider the unique polynomial, denoted $p_{1,5}(x)$, that interpolates the data given by pairs (x_1, f_1), (x_2, f_2), (x_3, f_3), (x_4, f_4) , and (x_5, f_5). Use two different sets of divided differences to express $p_{1,5}(x)$ in two distinct forms.

2.2.b

What is the significance of the value of 0 for $f[x_0, x_1, x_2, x_3, x_4]$?

2.2.c

Denote by $p_{0,4}(x)$, the unique polynomial, that interpolates the data given by pairs (x_0, f_0), (x_1, f_1), (x_2, f_2), (x_3, f_3), and (x_4, f_4) and recall the definition of $p_{1,5}(x)$ from part (a). Use the divided difference information about the unknown function $f(x)$ to derive error estimates for $f(x) - p_{1,5}(x)$ and $f(x) - p_{0,4}(x)$ for any $x_0 \leq x \leq x_5$.

Problem 2.3

Assume you are given distinct points $x_0, \ldots, x_n$ and, $p_n(x)$, the interpolating polynomial defined by those points for a function f.

2.3.a. If $p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$ is the Lagrange form show that

$$\sum_{i=0}^{n} \ell_i(x) = 1$$

2.3.b. Assume $x \neq x_i$ for $0 \leq i \leq n$ and show that the divided difference $f[x_0, \ldots, x_n, x]$ satisfies

$$f[x_0, \ldots, x_n, x] = \sum_{i=0}^{n} \frac{f[x, x_i]}{\prod_{j=0, j\neq i}^{n}(x_i - x_j)}$$

2.3.c. Show that

$$y[x_0, \ldots, x_n] = \sum_{i=0}^{n} \frac{y_i}{\omega'_{n+1}(x_i)}, \quad \text{where } \omega_{k+1} = (x - x_0)\ldots(x - x_k)$$

Problem 2.4

Text exercise 8.10.1 on page 375

Problem 2.5

Text exercise 8.10.8 on page 376

Problem 2.6

Text exercise 8.10.4 on page 376

Problem 2.7

Consider a polynomial

$$p_n(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

$p_n(\gamma)$ can be evaluated using Horner's rule (written here with the dependence on the formal argument x more explicitly shown)

$c_n(x) = \alpha_n$
for $i = n-1 : -1 : 0$
$\qquad c_i(x) = xc_{i+1}(x) + \alpha_i$
end

$p_n(x) = c_0(x)$

Note that when evaluating $x = \gamma$ the algorithm produces $n+1$ constants $c_0(\gamma), \dots, c_n(\gamma)$ one of which is equal to $p_n(\gamma)$.

2.7.a

Suppose that Horner's rule is applied to evaluate $p_n(\gamma)$ and that the constants $c_0(\gamma), \dots, c_n(\gamma)$ are saved. Show that

$$p_n(x) = (x-\gamma)q(x) + p_n(\gamma)$$
$$q(x) = c_1(\gamma) + c_2(\gamma)x + \cdots + c_n(\gamma)x^{n-1}$$

2.7.b

Suppose that Horner's rule, with labeling modified appropriately, is applied to evaluate $p_n(\gamma)$ and that the constants $c_0^{(1)}(\gamma), \dots, c_n^{(1)}(\gamma)$ are saved to define $p_n(\gamma) - c_0^{(1)}(\gamma)$ and $q_{(1)}(x) = c_1^{(1)}(\gamma) + c_2^{(1)}(\gamma)x + \cdots + c_n^{(1)}(\gamma)x^{n-1}$. Suppose further that Horner's rule is applied to evaluate $q_{(1)}(\gamma)$ and that the constants $c_1^{(2)}(\gamma), \dots, c_n^{(2)}(\gamma)$ are saved to define $q_{(1)}(\gamma) = c_1^{(2)}(\gamma)$ and $q_{(2)}(x) = c_2^{(2)}(\gamma) + c_3^{(2)}(\gamma)x + \cdots + c_n^{(2)}(\gamma)x^{n-2}$. This can continue until Horner's rule is applied to evaluate $q_{(n)}(\gamma) = c_n^{(n)}(\gamma)$ and $q_{(n+1)}(x) = 0$, i.e., there are no constants other than $c_n^{(n)}(\gamma)$ produced.

Show that

$$\begin{aligned} q_{(1)}(\gamma) &= p_n'(\gamma) \\ q_{(2)}(\gamma) &= p_n''(\gamma)/2 \\ q_{(3)}(\gamma) &= p_n'''(\gamma)/3! \\ &\vdots \\ q_{(n-1)}(\gamma) &= p_n^{(n-1)}(\gamma)/(n-1)! \\ q_{(n)}(\gamma) &= p_n^{(n)}(\gamma)/n! \end{aligned}$$

and therefore form the coefficients of the Taylor form of $p_n(x)$

$$p_n(x) = p_n(\gamma) + (x-\gamma)p_n'(\gamma) + \frac{(x-\gamma)^2}{2}p_n''(\gamma) + \frac{(x-\gamma)^3}{3!}p_n'''(\gamma) \cdots + \frac{(x-\gamma)^{n-1}}{(n-1)!}p_n^{(n-1)}(\gamma) + \frac{(x-\gamma)^n}{n!}p_n^{(n)}(\gamma)$$

Problem 2.8

The set of square integrable functions

$$\mathcal{L}^2[-1,1] = \{f(x),\ -1 \leq x \leq 1 \mid \int_{-1}^{1} f^2(x)dx < \infty\}$$

is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\ g(x)dx$$

and the associated induced norm. The space of polynomials with degree n or less, $\mathbb{P}_n$, is a finite dimensional subspace of $\mathcal{L}^2[-1,1]$ with basis $\{b_k\} = \{x^k\}$ with $0 \leq k \leq n$.

A basis can be problematic if there is wide variation in the norm of the vectors, $\|b_k\|$ or if the angles between b_k and b_j become small for various pairs of vectors.

2.8.a. Analyze the magnitudes of the monomial basis vectors.

2.8.b. Analyze the angles between the monomial basis vectors.

2.8.c. Discuss the results in terms of the robustness of the basis for representing polynomials.

Problem 2.9

Show that given a set of points

$$x_0,\ \ x_1, \ldots, x_n$$

a Leja ordering can be computed in $O(n^2)$ operations.